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Minimizing movements for forced anisotropic mean curvature flow of partitions with mobilities

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Under suitable assumptions on the family of anisotropies, we prove the existence of a weak global $1/(n+1)$-Hölder continuous in time mean curvature flow with mobilities of a bounded anisotropic partition in any dimension using the method of minimizing movements. The result is extended to the case when suitable driving forces are present. We improve the Hölder exponent to $1/2$ in the case of partitions with the same anisotropy and the same mobility and provide a weak comparison result in this setting for a weak anisotropic mean curvature flow of a partition and an anisotropic mean curvature two-phase flow.

Keywords: mean curvature flow; partitions; minimizing movements; forcing; anisotropy; mobility

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1. Introduction

Many processes in material sciences such as phase transformation, crystal growth, grain growth, stress-driven rearrangement instabilities, etc., can be modelled as geometric interface motions, in which surface tensions act as a principal driving force (see e.g., [**15**, **40**, **49**, **51**] and references therein). A typical example of such a motion is anisotropic mean curvature flow: given a norm ϕ on $\mathbb{R}^n$ (called anisotropy), the equation for the anisotropic mean curvature flow of hypersurfaces parametrized as Γ_t reads as

$$\beta(\nu)V = -\mathrm{div}_{\Gamma_t}[\nabla\phi(\nu)] \quad \text{on } \Gamma_t, \tag{1.1}$$

where V denotes the normal velocity of Γ_t in the direction of the unit outer normal ν of Γ_t and β is the mobility, a positive kinetic coefficient [**29**]. Anisotropic mean curvature flow is called crystalline provided the boundary of the Wulff shape $W_\phi := \{\phi \leqslant 1\}$ lies on finitely many hyperplanes; in this quite interesting case, equation (1.1) must be properly interpreted, due to the nondifferentiability of ϕ; see for instance [**2**, **5**–**7**, **9**, **17**, **19**, **20**, **27**, **28**, **31**, **32**, **53**]. Equation (1.1) (sometimes referred to as the two-phase evolution) can be generalized to the case of networks in the plane, and more generally to the case of partitions of space (sometimes called the multiphase case): here the evolving sets are intrinsically nonsmooth, since the presence of triple junctions (in the plane), or multiple lines, quadruple points (in space), etc., during the flow is unavoidable. It must be stressed that evolutions of partitions received recently a lot of attention from the mathematical community [**11**, **13**, **23**, **25**, **26**, **38**, **39**, **45**, **50**, **52**] both as a natural generalization of the case of two phases, and because they model a variety of physical phenomena, such as grain growth and evolution of multicrystals [**8**, **40**].

The presence of singularities at finite time is a common feature of mean curvature flow type motions, both in the two-phase case [**33**–**36**, **44**], and in the multiphase case (see for instance [**45**]). This phenomenon justifies to introduce and study some notion of weak solution, defined globally in time. This has been done in several different ways: just to quote a few, the Brakke varifold-solution [**15**], the viscosity solution (see [**30**] and references therein), the Ilmanen elliptic regularization [**37**], the level-set theoretic subsolution and the minimal barrier solution (see [**12**] and references therein), the Almgren–Taylor–Wang [**1**] and Luckhaus–Sturzenhecker [**42**] solutions, next included by De Giorgi into his notion of minimizing movement and generalized minimizing movement (GMM) [**21**, **22**]; see also [**16**, **24**, **47**]. Some of those solutions (e.g., the Brakke solution [**15**, **54**], the GMM solution [**14**], the elliptic regularization [**50**]) can be adapted to treat the multiphase case at least in the Euclidean case, especially those that do not rely heavily on the comparison principle. Also, the existence of a distributional solution of mean curvature evolution of partitions on the torus using the time thresholding method introduced in [**46**] has been proved in [**41**]; see also [**39**].

The aim of the present paper is to prove the existence of a GMM for anisotropic mean curvature flow of partitions with no restrictions on the space dimension, in the presence of a set of mobilities and forcing terms, and to point out some qualitative properties of this weak evolution, which are obtained via a comparison argument with a GMM of each single phase considered separately.

Let us recall the definition of GMM for partitions from [**22**] (see definition 2.6 for the notion of bounded partition).

DEFINITION 1.1 *Generalized minimizing movement for partitions*. Let $\mathbb{P}_b(N+1)$ be the set of all bounded $(N+1)$-partitions of $\mathbb{R}^n$ (definition 2.6) endowed with the $L^1(\mathbb{R}^n)$-convergence, and let $\mathfrak{F} : \mathbb{P}_b(N+1) \times \mathbb{P}_b(N+1) \times [1, +\infty) \to [-\infty, +\infty]$ be defined as

$$\mathfrak{F}(\mathcal{A}, \mathcal{B}, \lambda) = \sum_{j=1}^{N+1} P_{\phi_j}(A_j) + \lambda \sum_{j=1}^{N+1} \int_{A_j \Delta B_j} d_{\psi_j}(x, \partial B_j)\,\mathrm{d}x + \sum_{j=1}^{N+1} \int_{A_j} H_j\,\mathrm{d}x,$$

where ϕ_j and ψ_j are norms on $\mathbb{R}^n$, called anisotropies and mobilities, respectively, $H_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $i = 1, \ldots, N$, and $H_{N+1} \in L^1(\mathbb{R}^n)$ are driving forces, $P_{\phi_j}(A_j)$

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is the ϕ_j-anisotropic perimeter, $\mathcal{A} = (A_1, \ldots, A_{N+1})$, $\mathcal{B} = (B_1, \ldots, B_{N+1})$ and $d_{\psi_j}(\cdot, E)$ is the ψ_j-distance function from $E \subseteq \mathbb{R}^n$. We say that a map $\mathcal{M}: [0, +\infty) \to \mathbb{P}_b(N+1)$ is a GMM associated to $\mathfrak{F}$ starting from $\mathcal{G} \in \mathbb{P}_b(N+1)$, and we write $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$, if there exist $\mathcal{L}: [1, +\infty) \times \mathbb{N}_0 \to \mathbb{P}_b(N+1)$ and a diverging sequence $\{\lambda_h\}$ such that

$$\lim_{h\to+\infty} \mathcal{L}(\lambda_h, [\lambda_h t]) = \mathcal{M}(t) \quad \text{in } L^1(\mathbb{R}^n) \text{ for any } t \geqslant 0,$$

where the bounded partitions $\mathcal{L}(\lambda, k)$, $\lambda \geqslant 1$, $k \in \mathbb{N}_0$, are defined inductively as $\mathcal{L}(\lambda, 0) = \mathcal{G}$ and

$$\mathfrak{F}(\mathcal{L}(\lambda, k+1), \mathcal{L}(\lambda, k), \lambda) = \min_{\mathcal{A}\in\mathbb{P}_b(N+1)} \mathfrak{F}(\mathcal{A}, \mathcal{L}(\lambda, k), \lambda) \quad \forall k \geqslant 0.$$

Our first result (see theorems 4.1 and 4.2 for the precise statements) extends the existence results of [**14**] to the case with anisotropies, mobilities and external forces. We also improve the $1/(n+1)$-Hölder regularity in time of GMM proven in [**14**] to $1/2$-Hölder continuity in the two-phase case, without any restriction on the anisotropies.

THEOREM 1.2. *Suppose that the driving forces* $\{H_i\}$ *satisfy* (4.4). *Let* $\mathcal{G} \in \mathbb{P}_b(N+1)$. *The following assertions hold:*

(a) *Let* $N \geqslant 2$. *If* $\{\phi_j\}$ *satisfy* (3.1) *and* (4.3), *then* $GMM(\mathfrak{F}, \mathcal{G})$ *is nonempty. Moreover, any* $\mathcal{M} = (M_1, \ldots, M_{N+1}) \in GMM(\mathfrak{F}, \mathcal{G})$ *is locally* $1/(n+1)$-*Hölder continuous in time and for any* $t \geqslant 0$, $\bigcup_{j=1}^N M_j(t)$ *is contained in the bounded closed convex set related to* $\mathcal{G}$ *and* H_j *(see* (4.6)*).*

(b) *Let* $N = 1$. *Then, with no assumptions on the anisotropies* ϕ_1, ϕ_2 *and the mobilities* ψ_1, ψ_2, $GMM(\mathfrak{F}, \mathcal{G})$ *is nonempty. Moreover, any* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ *is locally* $1/2$-*Hölder continuous in time.*

To prove theorem 1.2 we establish uniform density estimates for minimizers of $\mathfrak{F}$ using the method of cutting out and filling in with balls, an argument of [**42**]. At this level the presence of mobilities does not create any new substantial problem. While in the two-phase case we do not need any assumption on the anisotropies, in the multiphase case assumption (3.1) is needed to get the lower volume density estimate for minimizers which is important in the proof of time-continuity of GMM.

In case of partitions with the same anisotropies and the same mobilities and without forcing, the Hölder exponent of GMM can be improved to $1/2$ (see theorem 5.1). Denoting by $\mathfrak{F}_2$ the restriction of $\mathfrak{F}$ to two-phase case without forcing (see (5.2)), this can be done using the comparison property (theorem 5.2) between the minimizers of $\mathfrak{F}$ and the minimizers of $\mathfrak{F}_2$. This comparison result also enables us to get a weak comparison flow of corresponding multiphase and two-phase flows (theorem 5.5):

THEOREM 1.3. *Assume that* $\phi_j = \phi_i$ *and* $\psi_j = \psi_i$ *for all* $i, j = 1, \ldots, N+1$, *and* $H_i = 0$ *for all* $i = 1, \ldots, N+1$. *Then any* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ *is locally* $1/2$-*Hölder continuous in time and* $\bigcup_{j=1}^N M_j(t)$ *is contained in the closed convex*

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envelope of the union $\bigcup_{j=1}^N G_j$ *of the bounded components of* $\mathcal{G}$ *for any* $t \geqslant 0$. *Moreover:*

(a) *for any* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ *and for any* $i \in \{1, \dots, N+1\}$ *there exists* $L_i \in GMM(\mathfrak{F}_2, G_i)$ *such that*

$$L_i(t) \subseteq M_i(t), \quad t \geqslant 0;$$

(b) *Let* C_i, $i \in \{1, \dots, N\}$, *and* C_{N+1} *be any convex sets such that* $C_i \subset G_i$ *for any* $i \in \{1, \dots, N+1\}$ *and let* $L_i \in MM(\mathfrak{F}_2, C_i)$ *be the unique minimizing movement starting from* C_i. *Then for any* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$,

$$L_i(t) \neq \emptyset \Longrightarrow M_i(t) \neq \emptyset \quad \textit{for any } i \in \{1, \dots, N\}, \tag{1.2}$$

and

$$\mathbb{R}^n \setminus L_{N+1}(t) = \emptyset \Longrightarrow \mathbb{R}^n \setminus M_{N+1}(t) = \emptyset. \tag{1.3}$$

Note that the comparison principle (1.3) implies that any bounded partition will disappear in the long run; moreover, (1.2) allows to estimate the extinction time of the i-th bounded phase (Corollary 5.6).

Finally, let us mention that a natural problem remains open, namely the consistency of GMM with the classical solution, provided the latter exists, at least on a short time interval. Such a result has been proven by Almgren–Taylor–Wang in [**1**] in the two-phase case without mobility; the proof is based on various stability properties of the flow, and using comparison arguments. It has also been proven by Almgren–Taylor [**2**] in the two-phase crystalline case. However, consistency is not known in the case of networks in the plane (and a fortiori for partitions in space), even in the Euclidean case without mobilities and forcing.

The paper is organized as follows. In § 2 we introduce the notation, some results from the theory of sets of finite perimeter, and the definition of a partition. In § 3 we prove the density estimates for almost minimizers. The existence of generalized minimizing movements (theorem 1.2) is established in § 4. In § 5 we improve the Hölder regularity of GMM (theorem 1.3) and provide some weak comparison principles.

2. Notation and preliminaries

In this section, we introduce the notation and collect some important properties of sets of locally finite perimeter. The standard references for BV-functions and sets of finite perimeter are [**4**, **43**].

We use $\mathbb{N}_0$ to denote the set of all nonnegative integers. The symbol $B_r(x)$ stands for the open ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ of radius $r > 0$. The characteristic function of a Lebesgue measurable set F is denoted by χ_F and its Lebesgue measure by $|F|$; we set also $\omega_n := |B_1(0)|$. We denote by E^c the complement of E in $\mathbb{R}^n$.

Given a norm ψ in $\mathbb{R}^n$ and a nonempty set $E \subseteq \mathbb{R}^n$, $d_\psi(\cdot, E)$ stands for the ψ-distance from E, i.e.,

$$d_\psi(x, E) = \inf\{\psi(x - y) : \ y \in E\},$$

and

$$\widetilde{d}_\psi(x, \partial E) = d_\psi(x, E) - d_\psi(x, \mathbb{R}^n \setminus E)$$

is the signed ψ-distance function from ∂E, negative inside E. When ψ is Euclidean for simplicity we drop the dependence on ψ. We also write

$$\mathrm{diam}_\psi E := \sup\{\psi(x - y) : \ x, y \in E\}$$

to denote the ψ-diameter of E.

By $\mathbb{O}(\mathbb{R}^n)$ (resp. $\mathbb{O}_b(\mathbb{R}^n)$) we denote the collection of all open (resp. open and bounded) subsets of $\mathbb{R}^n$. The set of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-functions having locally bounded total variation in $\mathbb{R}^n$ is denoted by $BV_{\mathrm{loc}}(\mathbb{R}^n)$ and the elements of

$$BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\}) := \{E \subseteq \mathbb{R}^n : \ \chi_E \in BV_{\mathrm{loc}}(\mathbb{R}^n)\}$$

are called locally finite perimeter sets. Given a $E \in BV_{\mathrm{loc}}(\mathbb{R}^n, \{0,1\})$ we denote by

(a) $P(E, \Omega) := \int_\Omega |D\chi_E|$ the perimeter of E in $\Omega \in \mathbb{O}(\mathbb{R}^n)$;

(b) ∂E the measure-theoretic boundary of E :

$$\partial E := \{x \in \mathbb{R}^n : \ 0 < |B_\rho \cap E| < |B_\rho| \quad \forall \rho > 0\};$$

(c) $\partial^* E$ the reduced boundary of E;

(d) ν_E the outer generalized unit normal to $\partial^* E$.

For simplicity, we set $P(E) := P(E, \mathbb{R}^n)$ provided $E \in BV(\mathbb{R}^n; \{0,1\})$. Further, given a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ and $\alpha \in [0,1]$ we define

$$E^{(\alpha)} := \left\{ x \in \mathbb{R}^n : \ \lim_{\rho \to 0^+} \frac{|B_\rho(x) \cap E|}{|B_\rho(x)|} = \alpha \right\}.$$

Unless otherwise stated, we always suppose that any locally finite perimeter set E we consider coincides with $E^{(1)}$ (so that by [**43**, equation (15.3)] ∂E coincides with the topological boundary). We recall that $\overline{\partial^* E} = \partial E$ and $D\chi_E = \nu_E d\mathcal{H}^{n-1} \llcorner \partial^* E$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ and $\llcorner$ is the symbol of restriction.

Remark 2.1. Given $E \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{0,1\})$ the map $\Omega \in \mathbb{O}(\mathbb{R}^n) \mapsto P(E, \Omega)$ extends to a Borel measure in $\mathbb{R}^n$ so that $P(E, B) = \mathcal{H}^{n-1}(B \cap \partial^* E)$ for every Borel set $B \subseteq \mathbb{R}^n$. Moreover, by [**4**, theorem 3.61] for every $E \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{0,1\})$

$$\mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup E \cup \partial^* E)) = 0.$$

In particular, $\mathcal{H}^{n-1}(E^{(1/2)} \setminus \partial^* E) = 0$.

THEOREM 2.2. [43, *theorem* 16.3] *If E and F are sets of locally finite perimeter, and we let*

$$\{\nu_E = \nu_F\} = \{x \in \partial^* E \cap \partial^* F : \ \nu_E(x) = \nu_F(x)\},$$
$$\{\nu_E = -\nu_F\} = \{x \in \partial^* E \cap \partial^* F : \ \nu_E(x) = -\nu_F(x)\},$$

then $E \cap F$, $E \setminus F$ and $E \cup F$ are locally finite perimeter sets with

$$\partial^*(E\cap F) \approx \big(F \cap \partial^* E\big) \cup \big(E \cap \partial^* F\big) \cup \big\{\nu_E = \nu_F\big\}, \tag{2.1}$$
$$\partial^*(E\setminus F) \approx \big(F^{(0)} \cap \partial^* E\big) \cup \big(E \cap \partial^* F\big) \cup \big\{\nu_E = -\nu_F\big\}, \tag{2.2}$$
$$\partial^*(E\cup F) \approx \big(F^{(0)} \cap \partial^* E\big) \cup \big(E^{(0)} \cap \partial^* F\big) \cup \big\{\nu_E = \nu_F\big\}, \tag{2.3}$$

where $A \approx B$ means $\mathcal{H}^{n-1}(A\Delta B) = 0$.

The following generalizes the notion of the perimeter.

DEFINITION 2.3 *Anisotropic perimeter*. Let $\phi : \mathbb{R}^n \to [0,\infty)$ be a norm in $\mathbb{R}^n$. Given $\Omega \in \mathbb{O}(\mathbb{R}^n)$ the ϕ-perimeter of $E \in BV(\Omega;\{0,1\})$ is

$$P_\phi(E,\Omega) := \int_{\Omega\cap\partial^* E} \phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1}.$$

When $\Omega = \mathbb{R}^n$, we write $P_\phi(E) := P_\phi(E,\mathbb{R}^n)$, and when ϕ is Euclidean, we write P in place of P_ϕ.

It is well-known that $E \mapsto P_\phi(E;\Omega)$ is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous. Recall also that for every $E, F \in BV_{\mathrm{loc}}(\mathbb{R}^n;\{0,1\})$ and $\Omega \in \mathbb{O}(\mathbb{R}^n)$

$$P_\phi(E\cap F,\Omega) + P_\phi(E\cup F,\Omega) \leqslant P_\phi(E,\Omega) + P_\phi(F,\Omega). \tag{2.4}$$

2.1. Anisotropic partitions

We recall the notions of partition, almost-minimizer and bounded partition, see [14].

DEFINITION 2.4 **Partition**. Given an integer $N \geqslant 2$, an N-tuple $\mathcal{C} = (C_1,\ldots,C_N)$ of subsets of $\mathbb{R}^n$ is called an N-partition of $\mathbb{R}^n$ (a partition, for short) if

(a) $C_i \in BV_{\mathrm{loc}}(\mathbb{R}^n;\{0,1\})$ for every $i = 1,\ldots,N$,

(b) $\sum_{i=1}^N |C_i \cap K| = |K|$ for each compact $K \subset \mathbb{R}^n$.

The collection of all N-partitions of $\mathbb{R}^n$ is denoted by $\mathbb{P}(N)$. Our assumptions $C_i = C_i^{(1)}$ imply $C_i \cap C_j = \emptyset$ for $i \neq j$.

The elements of $\mathbb{P}(N)$ are denoted by calligraphic letters $\mathcal{A},\mathcal{B},\mathcal{C},\ldots$ and the components of $\mathcal{A} \in \mathbb{P}(N)$ by the corresponding roman letters $(A_1,\ldots,A_N)$.

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Let $\phi_1, \ldots, \phi_N$ be norms in $\mathbb{R}^n$ and set $\Phi := \{\phi_1, \ldots, \phi_N\}$. The functional

$$(\mathcal{A}, \Omega) \in \mathbb{P}(N) \times \mathbb{O}(\mathbb{R}^n) \mapsto \mathrm{Per}_\Phi(\mathcal{A}, \Omega) := \sum_{i=1}^N P_{\phi_i}(A_i, \Omega)$$

is called the anisotropic perimeter, or Φ-perimeter of the partition $\mathcal{A}$ in Ω. For simplicity, we write $\mathrm{Per}_\Phi(\mathcal{A}) := \mathrm{Per}_\Phi(\mathcal{A}, \mathbb{R}^n)$. For shortness, we also set $\mathrm{Per}_\Phi = \mathrm{Per}$ when all ϕ_i are Euclidean. Since N is finite, there exist $0 < c_\Phi \leqslant C_\Phi < +\infty$ such that

$$c_\Phi \leqslant \phi_i(\nu) \leqslant C_\Phi \tag{2.5}$$

for any $i = 1, \ldots, N$ and $\nu \in \mathbb{S}^{n-1}$, therefore,

$$c_\Phi \mathrm{Per}(\mathcal{A}, \Omega) \leqslant \mathrm{Per}_\Phi(\mathcal{A}, \Omega) \leqslant C_\Phi \mathrm{Per}(\mathcal{A}, \Omega). \tag{2.6}$$

In view of [**14**, proposition 3.3]

$$\mathrm{Per}_\Phi(\mathcal{A}, \Omega) = \sum_{1 \leqslant i < j \leqslant N} \int_{\Omega \cap \partial^* A_i \cap \partial^* A_j} \big(\phi_i(\nu_{A_i}) + \phi_j(\nu_{A_i})\big)\, \mathrm{d}\mathcal{H}^{n-1},$$

i.e., on a generalized hypersurface $\Sigma_{ij} := \Omega \cap \partial^* A_i \cap \partial^* A_j$ dividing the phase i from the phase j the perimeter contributes

$$\int_{\Sigma_{ij}} (\phi_i(\nu_{\Sigma_{ij}}) + \phi_j(\nu_{\Sigma_{ij}}))\, \mathrm{d}\mathcal{H}^{n-1},$$

where $\nu_{\Sigma_{ij}}$ is the generalized unit normal to Σ_{ij} pointing for instance from A_i to A_j. We set

$$\mathcal{A}\Delta\mathcal{B} := \bigcup_{j=1}^N A_j \Delta B_j \quad \text{and} \quad |\mathcal{A}\Delta\mathcal{B}| := \sum_{j=1}^N |A_j \Delta B_j|, \tag{2.7}$$

where Δ is the symmetric difference of sets, i.e., $E\Delta F = (E \setminus F) \cup (F \setminus E)$.

We say that the sequence $\{\mathcal{A}^{(k)}\} \subseteq \mathbb{P}(N)$ *converges* to $\mathcal{A} \in \mathbb{P}(N)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if

$$|(\mathcal{A}^{(k)} \Delta \mathcal{A}) \cap K| := \sum_{j=1}^N |(A_j^{(k)} \Delta A_j) \cap K| \to 0 \quad \text{as } k \to +\infty$$

for every compact set $K \subset \mathbb{R}^n$. Since $E \in BV_{\mathrm{loc}}(\mathbb{R}^n; \{0,1\}) \mapsto P_{\phi_i}(E, \Omega)$ is $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-lower semicontinuous for any $\Omega \in \mathbb{O}(\mathbb{R}^n)$, so is the map $\mathcal{A} \in \mathbb{P}(N) \mapsto \mathrm{Per}_\Phi(\mathcal{A}, \Omega)$. From (2.6) and [**14**, theorem 3.2] we get

Proposition 2.5 *Compactness. Let $\{\mathcal{A}^{(l)}\} \subset \mathbb{P}(N)$ be a sequence of partitions such that*

$$\sup_{l \geqslant 1} \mathrm{Per}_\Phi(\mathcal{A}^{(l)}, \Omega) < +\infty \quad \forall \Omega \in \mathbb{O}_b(\mathbb{R}^n).$$

Then there exist a partition $\mathcal{A} \in \mathbb{P}(N)$ and a subsequence $\{\mathcal{A}^{(l_k)}\}$ converging to $\mathcal{A}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as $k \to +\infty$.

2.2. Bounded partitions

Definition 2.6 *Bounded partition.* A partition $\mathcal{C} = (C_1, \dots, C_{N+1}) \in \mathbb{P}(N+1)$ is called bounded, and we write $\mathcal{C} \in \mathbb{P}_b(N+1)$, if C_i is bounded for each $i = 1, \dots, N$.

Note that $\mathcal{A}\Delta\mathcal{B} \subset\subset \mathbb{R}^n$ for every $\mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1)$, and therefore,

$$|\mathcal{A}\Delta\mathcal{B}| = \sum_{j=1}^{N+1} |A_j \Delta B_j|$$

is the $L^1(\mathbb{R}^n)$-*distance* in $\mathbb{P}_b(N+1)$.

Given $\mathcal{A} \in \mathbb{P}_b(N+1)$, we denote by $\mathrm{co}(\mathcal{A})$ the closed convex hull of $\bigcup_{i=1}^N A_i$.

In view of (2.6) and [**14**, theorem 3.10] we have the following compactness result.

Proposition 2.7 *Compactness. Let* $\mathcal{A}^{(k)} \in \mathbb{P}_b(N+1)$, $k = 1, 2, \dots,$ *and* $\Omega \in \mathbb{O}_b(\mathbb{R}^n)$ *be such that*

$$\sup_{k \geqslant 1} \mathrm{Per}_\Phi(\mathcal{A}^{(k)}) < +\infty, \quad \mathrm{co}(\mathcal{A}^{(k)}) \subseteq \Omega \quad \forall k \geqslant 1.$$

Then there exist $\mathcal{A} \in \mathbb{P}_b(N+1)$ *and a subsequence* $\{\mathcal{A}^{(k_l)}\}$ *converging to* $\mathcal{A}$ *in* $L^1(\mathbb{R}^n)$ *as* $l \to +\infty$. *Moreover,* $\bigcup_{i=1}^N A_j \subseteq \overline{\Omega}$.

3. Density estimates for almost minimizers

In this section we prove density estimates for almost minimizers (see theorem 3.2). In the two-phase case without mobility, density estimates have been proven in [**1**, **42**] (see also the proof of theorem 4.2 for the case with mobility and forcing) and in the isotropic N-phase case is proven in [**14**]. The proof of theorem 3.2 is similar to [**14**, theorem 3.6], however, some (technical) difficulties arise when two anisotropies differ too much and this is why we need assumption (3.5) for proving the lower-density estimates.

Definition 3.1 *Almost-minimizers.* Given $\Phi = \{\phi_1, \dots, \phi_N\}$, $\Lambda_1, \Lambda_2 \geqslant 0$, $\alpha_1, \alpha_2 > (n-1)/n$ and $r_0 \in (0, +\infty]$, we say that a partition $\mathcal{A} \in \mathbb{P}(N)$ is a $(\Phi, \Lambda_1, \Lambda_2, r_0, \alpha_1, \alpha_2)$-minimizer in $\mathbb{R}^n$ of Per_Φ (a $(\Lambda_1, \Lambda_2, r_0, \alpha_1, \alpha_2)$-minimizer, or also an almost-minimizer for short) if

$$\mathrm{Per}_\Phi(\mathcal{A}, B_r) \leqslant \mathrm{Per}_\Phi(\mathcal{B}, B_r) + \Lambda_1 |\mathcal{A}\Delta\mathcal{B}|^{\alpha_1} + \Lambda_2 |\mathcal{A}\Delta\mathcal{B}|^{\alpha_2}$$

whenever $\mathcal{B} \in \mathbb{P}(N)$, $B_r \subset \mathbb{R}^n$ is a ball of radius $r \in (0, r_0)$ and $\mathcal{A}\Delta\mathcal{B} \subset\subset B_r$.

Define

$$\kappa_N := \min_{1\leqslant i<j\leqslant N} \|\phi_i - \phi_j\|_{L^\infty(\mathbb{S}^{n-1})}, \tag{3.1}$$

$$\beta_1 := \Big(\frac{c_\Phi n\omega_n^{1-\alpha_1}}{2^{1+\alpha_1}\Lambda_1}\Big)^{1/(n\alpha_1-n+1)},$$

$$\beta_2 := \Big(\frac{c_\Phi n\omega_n^{1-\alpha_2}}{2^{1+\alpha_2}\Lambda_2}\Big)^{1/(n\alpha_2-n+1)},$$

$$\gamma_N := \frac{c_\Phi - (N-1)\kappa_N/2}{2c_\Phi + 2(N-1)C_\Phi - (N-1)\kappa_N}.$$

THEOREM 3.2 **Density estimates for almost minimizers**. *Assume that the entries of* Φ *satisfy* (2.5). *Let* $\mathcal{A}\in\mathbb{P}(N)$ *be a* $(\Lambda_1,\Lambda_2,r_0,\alpha_1,\alpha_2)$ *-minimizer and* $i\in\{1,\dots,N\}$. *Then either* $A_i=\emptyset$ *or for any* $x\in\partial A_i$ *and* $r\in(0,\widehat{r}_0]$

$$\frac{|A_i\cap B_r(x)|}{|B_r(x)|} \leqslant 1 - \left(\frac{c_\Phi}{2(c_\Phi + C_\Phi)}\right)^n \tag{3.2}$$

and

$$\frac{P(A_i,B_r(x))}{r^{n-1}} \leqslant \left(\frac{C_\Phi}{c_\Phi} + \frac{1}{2}\right) n\omega_n, \tag{3.3}$$

where

$$\widehat{r}_0 := \min\{r_0,\beta_1,\beta_2\}. \tag{3.4}$$

Moreover, if

$$\kappa_N < \frac{2c_\Phi}{N-1}, \tag{3.5}$$

then for any $r\in(0,\widetilde{r}_0]$

$$\gamma_N^n \leqslant \frac{|A_i\cap B_r(x)|}{|B_r(x)|}, \tag{3.6}$$

and

$$c \leqslant \frac{P(A_i,B_r(x))}{r^{n-1}}, \tag{3.7}$$

where

$$\widetilde{r}_0 := \min\Big\{r_0, \Big(\frac{c_\Phi}{N-1} - \frac{\kappa_N}{2}\Big)^{1/(n\alpha_1-n+1)}\beta_1, \Big(\frac{c_\Phi}{N-1} - \frac{\kappa_N}{2}\Big)^{1/(n\alpha_2-n+1)}\beta_2\Big\} \tag{3.8}$$

and

$$c := c(n,N,c_\Phi,C_\Phi,\kappa_N) := \frac{n\omega_n(2^{1/n}-1)}{2^{1+1/n}}\gamma_N^{n-1}.$$

Proof. Without loss of generality, we assume $i=1$ and $A_i\neq\emptyset$. Since $\overline{\partial^* A_1} = \partial A_1$, it is sufficient to show (3.2), (3.3), (3.6), (3.7) when $x\in\partial^* A_1$. For shortness, we write $B_r := B_r(x)$.

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We start by proving (3.2) and (3.3). Let us show

$$\begin{aligned} c_\Phi P(A_1^{(0)}, B_r) \leqslant & C_\Phi \mathcal{H}^{n-1}(A_1^{(0)} \cap \partial B_r) \\ & + 2^{\alpha_1 - 1}\Lambda_1 |A_1^{(0)} \cap B_r|^{\alpha_1} + 2^{\alpha_2 - 1}\Lambda_2 |A_1^{(0)} \cap B_r|^{\alpha_2} \end{aligned} \tag{3.9}$$

for all $r \in (0, \widehat{r}_0)$ such that

$$\sum_{j=1}^N \mathcal{H}^{n-1}(\partial B_r \cap \partial^* A_j) = 0. \tag{3.10}$$

Indeed, setting

$$\mathcal{B} := (A_1 \cup B_r, A_2 \setminus B_r, \ldots, A_N \setminus B_r),$$

we have $\mathcal{A}\Delta\mathcal{B} \subset\subset B_s$ for every $s \in (r, \widehat{r}_0)$ and thus, by almost minimality, the definition (2.7) of $|\mathcal{A}\Delta\mathcal{B}|$ and the essential disjointness of A_j,

$$\begin{aligned} 0 \leqslant & \mathrm{Per}_\Phi(\mathcal{B}, B_s) - \mathrm{Per}_\Phi(\mathcal{A}, B_s) + \Lambda_1 |\mathcal{A}\Delta\mathcal{B}|^{\alpha_1} + \Lambda_2 |\mathcal{A}\Delta\mathcal{B}|^{\alpha_2} \\ = & P_{\phi_1}(A_1 \cup B_r, B_s) - P_{\phi_1}(A_1, B_s) + \sum_{j=2}^N \big(P_{\phi_j}(A_j \setminus B_r, B_s) - P_{\phi_j}(A_j, B_s)\big) \\ & + 2^{\alpha_1}\Lambda_1 |B_r \cap A_1^{(0)}|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |B_r \cap A_1^{(0)}|^{\alpha_2}, \end{aligned} \tag{3.11}$$

since $B_r \setminus A_1 = B_r \cap A_1^{(0)}$ up to a $\mathcal{L}^n$-negligible set and $|\mathcal{A}\Delta\mathcal{B}| = 2|B_r \setminus A_1| = 2|B_r \cap A_1^{(0)}|$. By (2.3) and (3.10),

$$P_{\phi_1}(A_1 \cup B_r, B_s) = P_{\phi_1}(A_1, B_s \setminus \overline{B_r}) + \int_{A_1^{(0)} \cap \partial B_r} \phi_1(\nu_{B_r}) \mathrm{d}\mathcal{H}^{n-1},$$

and for any $j = 2, \ldots, N$,

$$P_{\phi_j}(A_j \setminus B_r, B_s) = P_{\phi_j}(A_j, B_s \setminus \overline{B_r}) + \int_{A_j \cap \partial B_r} \phi_j(\nu_{B_r}) \mathrm{d}\mathcal{H}^{n-1}. \tag{3.12}$$

Thus by (3.11)

$$\begin{aligned} \sum_{j=1}^N P_{\phi_j}(A_j, B_s) \leqslant & \sum_{j=1}^N P_{\phi_j}(A_j, B_s \setminus \overline{B_r}) \\ & + \int_{A_1^{(0)} \cap \partial B_r} \phi_1(\nu_{B_r}) \mathrm{d}\mathcal{H}^{n-1} + \sum_{j=2}^N \int_{A_j \cap \partial B_r} \phi_j(\nu_{B_r}) \mathrm{d}\mathcal{H}^{n-1} \\ & + 2^{\alpha_1}\Lambda_1 |B_r \cap A_1^{(0)}|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |B_r \cap A_1^{(0)}|^{\alpha_2}. \end{aligned} \tag{3.13}$$

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By (2.5), the essential disjointness of A_j and (3.10) we have

$$\int_{A_1^{(0)}\cap\partial B_r}\phi_1(\nu_{B_r})\mathrm{d}\mathcal{H}^{n-1}+\sum_{j=2}^N\int_{A_j\cap\partial B_r}\phi_j(\nu_{B_r})\mathrm{d}\mathcal{H}^{n-1}$$
$$\leqslant C_\Phi\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r)+C_\Phi\sum_{j=2}^N\mathcal{H}^{n-1}(A_j\cap\partial B_r)=2C_\Phi\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r),$$

thus, (3.13) and (3.10) imply

$$\sum_{j=1}^N P_{\phi_j}(A_j,B_r)\leqslant 2C_\Phi\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r)+2^{\alpha_1}\Lambda_1|B_r\setminus A_1|^{\alpha_1}+2^{\alpha_2}\Lambda_2|B_r\setminus A_1|^{\alpha_2}.$$

By (2.5), (2.4) and the essential disjointness of A_j,

$$\sum_{j=2}^N P_{\phi_j}(A_j,B_r)\geqslant c_\Phi\sum_{j=2}^N P(A_j,B_r)\geqslant c_\Phi P\left(\bigcup_{j=2}^N A_j,B_r\right)=c_\Phi P(A_1^{(0)},B_r),$$

and thus $\sum_{j=1}^N P_{\phi_j}(A_j,B_r)\geqslant 2c_\Phi P(A_1^{(0)},B_r)$ so that (3.9) follows from (3.4).

To prove (3.2) we add $c_\Phi\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r)$ to both sides of (3.9) and using $\mathcal{H}^{n-1}(\partial B_r\cap\partial^*A_1)=0$ we get

$$c_\Phi P(A_1^{(0)}\cap B_r)\leqslant(c_\Phi+C_\Phi)\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r)+2^{\alpha_1-1}\Lambda_1|A_1^{(0)}\cap B_r|^{\alpha_1}$$
$$+2^{\alpha_2-1}\Lambda_2|A_1^{(0)}\cap B_r|^{\alpha_2},$$

hence by the isoperimetric inequality

$$c_\Phi n\omega_n^{1/n}|A_1^{(0)}\cap B_r|^{(n-1/n}\leqslant(c_\Phi+C_\Phi)\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r)$$
$$+2^{\alpha_1-1}\Lambda_1|A_1^{(0)}\cap B_r|^{\alpha_1}+2^{\alpha_2-1}\Lambda_2|A_1^{(0)}\cap B_r|^{\alpha_2}. \tag{3.14}$$

By the choice of $\widehat{r}_0$ in (3.4) we have, for $l=1,2$,

$$2^{\alpha_l-1}\Lambda_l|A_1^{(0)}\cap B_r|^{\alpha_l-((n-1)/n)}\leqslant 2^{\alpha_l-1}\Lambda_l\omega_n^{\alpha_l-((n-1)/n)}\widehat{r}_0^{n\alpha_l-n+1}\leqslant\frac{c_\Phi n\omega_n^{1/n}}{4}. \tag{3.15}$$

Inserting (3.15) in (3.14) we obtain

$$\frac{c_\Phi}{2(c_\Phi+C_\Phi)}n\omega_n^{1/n}|A_1^{(0)}\cap B_r|^{(n-1/n}\leqslant\mathcal{H}^{n-1}(A_1^{(0)}\cap\partial B_r),$$

and whence, repeating for instance the arguments of the proof of [**14**, equation (3.19)], we obtain

$$|A_1^{(0)}\cap B_r|\geqslant\left(\frac{c_\Phi}{2(c_\Phi+C_\Phi)}\right)^n\omega_n r^n,$$

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i.e.,

$$\frac{|A_1 \cap B_r|}{|B_r|} \leqslant 1 - \left(\frac{c_\Phi}{2(c_\Phi + C_\Phi)}\right)^n.$$

From (3.9) and the definition of $\widehat{r}_0$ for all $r \in (0, \widehat{r}_0]$ we get

$$\begin{aligned} P(A_1, B_r) &\leqslant \frac{C_\Phi}{c_\Phi}\mathcal{H}^{n-1}(\partial B_r) + \frac{2^{\alpha_1 - 1}\Lambda_1}{c_\Phi}|B_r|^{\alpha_1} + \frac{2^{\alpha_2-1}\Lambda_2}{c_\Phi}|B_r|^{\alpha_2} \\ &\leqslant \left(\frac{C_\Phi}{c_\Phi} + \frac{1}{2}\right) n\omega_n r^{n-1}. \end{aligned}$$

Now we prove (3.6) and (3.7). Note that assumption (3.5) implies $\widetilde{r}_0, \gamma_N > 0$. Let us show

$$\begin{aligned} \Big(\frac{c_\Phi}{N-1} - \frac{\kappa_N}{2}\Big) P(A_1, B_r) \leqslant & C_\Phi \mathcal{H}^{n-1}(A_1 \cap \partial B_r) \\ & + 2^{\alpha_1-1}\Lambda_1 |A_1 \cap B_r|^{\alpha_1} + 2^{\alpha_2-1}\Lambda_2|A_1 \cap B_r|^{\alpha_2} \end{aligned} \tag{3.16}$$

for all $r \in (0, \widetilde{r}_0)$ such that

$$\sum_{j=1}^N \mathcal{H}^{n-1}(\partial^* A_j \cap \partial B_r) = 0. \tag{3.17}$$

Set

$$I_1 := \{j \in \{2, \ldots, N\} : \ \mathcal{H}^{n-1}(B_{\widetilde{r}_0} \cap \partial^* A_1 \cap \partial^* A_j) > 0\}.$$

Since $x \in \partial A_1$, $I_1 \neq \emptyset$. Fix $r \in (0, \widetilde{r}_0)$; for every $j \in I_1$ consider the competitor

$$\mathcal{B}^{(j)} := (A_1 \setminus B_r, A_2, \ldots, A_{j-1}, A_j \cup (A_1 \cap B_r), A_{j+1}, \ldots, A_N).$$

Since $\mathcal{B}^{(j)} \Delta \mathcal{A} \subset\subset B_s$ for every $s \in (r, \widetilde{r}_0)$, by the almost minimality of $\mathcal{A}$ (recall that $\widetilde{r}_0 \leqslant r_0$) and the equality $|\mathcal{A}\Delta\mathcal{B}^{(j)}| = 2|A_1 \cap B_r|$ one has

$$\begin{aligned} P_{\phi_1}(A_1, B_s) + P_{\phi_j}(A_j, B_s) \leqslant & P_{\phi_1}(A_1 \setminus B_r, B_s) + P_{\phi_j}(A_j \cup (A_1 \cap B_r), B_s) \\ & + 2^{\alpha_1}\Lambda_1|A_1 \cap B_r|^{\alpha_1} + 2^{\alpha_2}\Lambda_2 |A_1 \cap B_r|^{\alpha_2}. \end{aligned} \tag{3.18}$$

Using the equality

$$\begin{aligned} P_{\phi_j}(A_j \cup (A_1 \cap B_r), B_s) = & P_{\phi_j}(A_j, B_s) + P_{\phi_j}(A_1, B_r) + \int_{A_1 \cap \partial B_r} \phi_j(\nu_{B_r}) \mathrm{d}\mathcal{H}^{n-1} \\ & - \int_{B_r \cap \partial^* A_1 \cap \partial^* A_j} \big(\phi_j(\nu_{A_1}) + \phi_j(\nu_{A_j})\big) \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

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the analogue of (3.12) with $j=1$ and also (3.17) in (3.18) we establish

$$\begin{aligned} P_{\phi_1}(A_1,B_s)+P_{\phi_j}(A_j,B_s) \leqslant & P_{\phi_j}(A_j,B_s)+P_{\phi_j}(A_1,B_r)+\int_{A_1\cap\partial B_r}\phi_j(\nu_{B_r})\mathrm{d}\mathcal{H}^{n-1} \\ & -\int_{B_r\cap\partial^* A_1\cap\partial^* A_j}\big(\phi_j(\nu_{A_1})+\phi_j(\nu_{A_j})\big)\mathrm{d}\mathcal{H}^{n-1} \\ & +P_{\phi_1}(A_1,B_s\setminus B_r)+\int_{A_1\cap\partial B_r}\phi_1(\nu_{B_r})\mathrm{d}\mathcal{H}^{n-1} \\ & +2^{\alpha_1}\Lambda_1|A_1\cap B_r|^{\alpha_1}+2^{\alpha_2}\Lambda_2|A_1\cap B_r|^{\alpha_2}. \end{aligned}$$

Hence using $\phi_j(\nu_{A_1})=\phi_j(\nu_{A_j})$,

$$\begin{aligned} & 2\int_{B_r\cap\partial^* A_1\cap\partial^* A_j}\phi_j(\nu_{A_1})\mathrm{d}\mathcal{H}^{n-1}\leqslant P_{\phi_j}(A_1,B_r)-P_{\phi_1}(A_1,B_r) \\ & +\int_{A_1\cap\partial B_r}\Big(\phi_1(\nu_{B_r})+\phi_j(\nu_{B_r})\Big)\mathrm{d}\mathcal{H}^{n-1}+2^{\alpha_1}\Lambda_1|A_1\cap B_r|^{\alpha_1}+2^{\alpha_2}\Lambda_2|A_1\cap B_r|^{\alpha_2}. \end{aligned}$$

Summing these inequalities in $j\in I_1$ and using (2.5) we get

$$\begin{aligned} 2c_\Phi\sum_{j=2}^{N}\mathcal{H}^{n-1}(B_r\cap\partial^* A_1\cap\partial^* A_j)\leqslant & \sum_{j\in I_1}\big(P_{\phi_j}(A_1,B_r)-P_{\phi_1}(A_1,B_r)\big) \\ & +\sum_{i\in I_1}\int_{A_1\cap\partial B_r}\Big(\phi_1(\nu_{B_r})+\phi_j(\nu_{B_r})\Big)\mathrm{d}\mathcal{H}^{n-1} \\ & +|I_1|(2^{\alpha_1}\Lambda_1|A_1\cap B_r|^{\alpha_1}+2^{\alpha_2}\Lambda_2|A_1\cap B_r|^{\alpha_2}), \end{aligned} \tag{3.19}$$

where $|I_1|$ is the number of elements of I_1. By the definition of I_1,

$$\sum_{j\in I_1}\mathcal{H}^{n-1}(B_r\cap\partial^* A_1\cap\partial^* A_j)=P(A_1,B_r),$$

by the definition of κ_N in (3.1)

$$\sum_{j\in I_1}\big(P_{\phi_j}(A_1,B_r)-P_{\phi_1}(A_1,B_r)\big)\leqslant\kappa_N|I_1|\,P(A_1,B_r),$$

and by (2.5)

$$\sum_{i\in I_1}\int_{A_1\cap\partial B_r}\Big(\phi_1(\nu_{B_r})+\phi_j(\nu_{B_r})\Big)\mathrm{d}\mathcal{H}^{n-1}\leqslant 2C_\Phi|I_1|\mathcal{H}^{n-1}(A_1\cap\partial B_r).$$

Therefore, from (3.19) we obtain

$$\begin{aligned} \left(\frac{c_\Phi}{|I_1|}-\frac{\kappa_N}{2}\right)P(A_1,B_r)\leqslant & C_\Phi\mathcal{H}^{n-1}(A_1\cap\partial B_r)+2^{\alpha_1-1}\Lambda_1|A_1\cap B_r|^{\alpha_1} \\ & +2^{\alpha_2-1}\Lambda_2|A_1\cap B_r|^{\alpha_2}. \end{aligned}$$

Since $|I_1|\leqslant N-1$, inequality (3.16) follows.

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To prove (3.6) we add $(c_\Phi/N-1-\kappa_N/2)\mathcal{H}^{n-1}(A_1\cap\partial B_r)$ to both sides of (3.16) and get

$$\Big(\frac{c_\Phi}{N-1}-\frac{\kappa_N}{2}\Big)P(A_1\cap B_r)\leqslant\Big(\frac{c_\Phi}{N-1}-\frac{\kappa_N}{2}+C_\Phi\Big)\mathcal{H}^{n-1}(A_1\cap\partial B_r) + 2^{\alpha_1-1}\Lambda_1|A_1\cap B_r|^{\alpha_1}+2^{\alpha_2-1}\Lambda_2|A_1\cap B_r|^{\alpha_2}, \tag{3.20}$$

By the definition (3.8) of $\widetilde{r}_0$ we have $r\leqslant\widetilde{r}_0\leqslant(c_\Phi/N-1-\kappa_N/2)^{1/n\alpha_l-n+1}\beta_l$ for $l=1,2$ and therefore

$$2^{\alpha_l-1}\Lambda_l|A_1\cap B_r|^{\alpha_l-n-1/n}\leqslant 2^{\alpha_l-1}\Lambda_l|B_{\widetilde{r}_0}|^{\alpha_l-n-1/n} = \Big(\frac{c_\Phi}{N-1}-\frac{\kappa_N}{2}\Big)\frac{n\omega_n^{1/n}}{4},\quad l=1,2,$$

and thus, by (3.20) and the isoperimetric inequality,

$$\frac{c_\Phi-(N-1)\kappa_N/2}{2c_\Phi+2(N-1)C_\Phi-(N-1)\kappa_N}\,n\omega_n^{1/n}\,|A_1\cap B_r|^{n-1/n}\leqslant\mathcal{H}^{n-1}(A_1\cap\partial B_r).$$

Now integrating we get

$$\gamma_N^n\omega_n r^n\leqslant|A_1\cap B_r|$$

and (3.6) follows. Finally, since

$$\frac{c_\Phi}{2c_\Phi+2C_\Phi}>\gamma_N,$$

from (3.2), (3.6) and the relative isoperimetric inequality we deduce (3.7). □

The following volume–distance comparison appeared in a similar form also in [**1**, **14**, **42**] and will be used in the proof of the existence of GMM.

Proposition 3.3. *Given $\theta,r_0>0$, let $A\in BV(\mathbb{R}^n;\{0,1\})$ be such that*

$$\theta r^{n-1}\leqslant P(A,B_r(x)),\quad r\in(0,r_0], \tag{3.21}$$

whenever $x\in\partial A$. Then for any $\ell>0$ and $B\in BV(\mathbb{R}^n;\{0,1\})$ one has

$$|B\Delta A|\leqslant\frac{5^n\omega_n}{\theta}\max\Big\{1,\Big(\frac{\ell}{r_0}\Big)^{n-1}\Big\}P(A)\,\ell+\frac{1}{\ell}\int_{A\Delta B}d(x,\partial A)\,\mathrm{d}x. \tag{3.22}$$

Proof. We follow [**14**, proposition 4.5] with minor modifications and we give the details for the convenience of the reader. Define

$$E:=\{x\in B\Delta A:\ d(x,\partial A)\leqslant\ell\},\quad F:=\{x\in B\Delta A:\ d(x,\partial A_j)\geqslant\ell\}.$$

By the Chebyshev inequality,

$$|F|\leqslant\frac{1}{\ell}\int_F d(x,\partial A)\,\mathrm{d}x\leqslant\frac{1}{\ell}\int_{A\Delta B}d(x,\partial A)\,\mathrm{d}x.$$

Let us estimate $|E|$. By a covering argument, one can find a finite family of disjoint balls $\{B_\ell(x_k)\}$ $x_k\in\partial A$, such that E is covered by the family $\{B_{5\ell}(x_k)\}_{k=1}^m$. If

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$\ell \geqslant r_0$, by (3.21) and the disjointness of $\{B_\ell(x_k)\}$ (and hence of $\{B_{r_0}(x_k)\}$),

$$\begin{aligned} |E| &\leqslant \sum_{k=1}^m \omega_n (5\ell)^n = \frac{5^n\omega_n\ell^n}{\theta r_0^{n-1}} \sum_{k=1}^m \theta r_0^{n-1} \leqslant \frac{5^n\omega_n\ell^n}{\theta r_0^{n-1}} \sum_{k=1}^m P(A_j, B_{r_0}(x_k)) \\ &\leqslant \frac{5^n\omega_n\ell^n}{\theta r_0^{n-1}} P\Big(A_j, \bigcup_{k=1}^m B_{r_0}(x_k)\Big) \leqslant \frac{5^n\omega_n}{\theta}\Big(\frac{\ell}{r_0}\Big)^{n-1} P(A_j)\,\ell. \end{aligned}$$

Analogously, if $\ell < r_0$, then

$$|E| \leqslant \frac{5^n\omega_n}{\theta} P(A_j)\,\ell.$$

Now (3.22) follows from the inequality $|B\Delta A| \leqslant |E| + |F|$ and estimates for $|A|$ and $|B|$. □

4. Existence of GMM for bounded partitions

Given a norm ψ in $\mathbb{R}^n$ and $E, F \subseteq \mathbb{R}^n$ set

$$\bar\sigma_\psi(E,F) := \int_{E\Delta F} d_\psi(x, \partial F)\,\mathrm{d}x.$$

Note that $\bar\sigma_\psi(E,F) = 0$ if $|E\Delta F| = 0$ whereas $\bar\sigma_\psi(E,F) = +\infty$ if $\partial F = \emptyset$ and $|E\Delta F| > 0$. Moreover, $X, Y \subseteq \mathbb{R}^n$ are measurable and $\partial Y \neq \emptyset$,

$$\begin{aligned} \int_{X\Delta Y} d_\psi(x,\partial Y)\,\mathrm{d}x &= \int_X \widetilde{d}_\psi(x,\partial Y)\,\mathrm{d}x \\ &\quad - \int_Y \widetilde{d}_\psi(x,\partial Y)\,\mathrm{d}x \quad \text{if } X\cap Y \text{ is bounded,} \\ \int_{X\Delta Y} d_\psi(x,\partial Y)\,\mathrm{d}x &= \int_{Y^c} \widetilde{d}_\psi(x,\partial Y)\,\mathrm{d}x \\ &\quad - \int_{X^c} \widetilde{d}_\psi(x,\partial Y)\,\mathrm{d}x \quad \text{if } X^c\cap Y^c \text{ is bounded.} \end{aligned}$$

Given a family $\Psi := \{\psi_1, \dots, \psi_{N+1}\}$ of norms ψ_i in $\mathbb{R}^n$, and $\mathcal{A}, \mathcal{B} \in \mathbb{P}_b(N+1)$, we set

$$\sigma_\Psi(\mathcal{A},\mathcal{B}) := \sum_{i=1}^{N+1} \bar\sigma_{\psi_i}(A_i, B_i),$$

where $N+1 \geqslant 2$. In the literature Ψ is called the set of mobilities. Since N is finite, there exist $0 < c_\Psi \leqslant C_\Psi < +\infty$ such that

$$c_\Psi \leqslant \psi_i(\nu) \leqslant C_\Psi, \quad i = 1,\dots,N+1, \quad \nu \in \mathbb{S}^{n-1}. \tag{4.1}$$

Observe that for every $\mathcal{B} \in \mathbb{P}_b(N+1)$ the map $\sigma_\Psi(\cdot,\mathcal{B})$ is $L^1(\mathbb{R}^n)$-lower semicontinuous in $\mathbb{P}_b(N+1)$.

Given families $\Phi := \{\phi_1, \dots, \phi_{N+1}\}$ of anisotropies and $\Psi := \{\psi_1, \dots, \psi_{N+1}\}$ of mobilities, and $\mathbf{H} := (H_1, \dots, H_{N+1})$ of functions $H_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $i =$

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$1,\dots,N$, and $H_{N+1}\in L^1(\mathbb{R}^n)$, consider the functional $\mathfrak{F}:\mathbb{P}_b(N+1)\times\mathbb{P}_b(N+1)\times[1,\infty)\to[-\infty,\infty]$,

$$\mathfrak{F}(\mathcal{E},\mathcal{F},\lambda)=\mathrm{Per}_\Phi(\mathcal{E})+\Upsilon(\mathcal{E})+\lambda\sigma_\Psi(\mathcal{E},\mathcal{F}),$$

where

$$\Upsilon(\mathcal{E}):=\sum_{i=1}^{N+1}\int_{E_i}H_i\,\mathrm{d}x.$$

Note that $\mathfrak{F}(\cdot,\mathcal{F};\lambda)$ is well-defined and $L^1(\mathbb{R}^n)$-lower semicontinuous in $\mathbb{P}_b(N+1)$. Notice that Υ can also be represented as

$$\Upsilon(\mathcal{E})=\sum_{j=1}^{N}\int_{E_j}(H_j-H_{N+1})\,\mathrm{d}x+\int_{\mathbb{R}^n}H_{N+1}\,\mathrm{d}x. \tag{4.2}$$

The functional $\mathfrak{F}$ is a generalization of the Almgren–Taylor–Wang functional [1] to the case of partitions [14, 22] in presence of anisotropies, mobilities and external forces.

The main result of this section is the following, which generalizes [14, theorems 4.9 and 5.1] to the anisotropic case with mobilities; recall that κ_{N+1} is defined in (3.1).

Theorem 4.1 **Existence of** *GMM*. *Let* $\Phi=\{\phi_1,\dots,\phi_{N+1}\}$ *and* $\Psi=\{\psi_1,\dots,\psi_{N+1}\}$ *be families of anisotropies and mobilities, respectively. Suppose that*

$$\kappa_{N+1}<\frac{2c_\Phi}{N}, \tag{4.3}$$

and $\boldsymbol{H}=(H_1,\dots,H_{N+1})$ *satisfies*

$$\begin{cases} H_i\in L^p_{\mathrm{loc}}(\mathbb{R}^n),\ i=1,\dots,N+1,\ \textit{for some}\ p>n\ \textit{and}\ H_{N+1}\in L^1(\mathbb{R}^n);\\ [1mm]\exists R>0\ \textit{s.t.}\ H_i\geqslant H_{N+1}\ \textit{a.e. in}\ \mathbb{R}^n\setminus B_R(0)\ \textit{for}\ i=1,\dots,N. \end{cases} \tag{4.4}$$

Then for every $\mathcal{G}\in\mathbb{P}_b(N+1)$, $GMM(\mathfrak{F},\mathcal{G})$ *is nonempty. Moreover, there exists a constant* $C=C(N,n,\Phi,\Psi,\boldsymbol{H},\mathcal{G})>0$ *such that for any* $\mathcal{M}\in GMM(\mathfrak{F},\mathcal{G})$,

$$|\mathcal{M}(t)\Delta\mathcal{M}(t')|\leqslant C\,|t-t'|^{\frac{1}{n+1}},\quad t,t'>0,\ |t-t'|<1 \tag{4.5}$$

and

$$\bigcup_{j=1}^{N}M_j(t)\subseteq D:=\textit{closed convex hull of}\ \mathrm{co}(\mathcal{G})\cup B_R\qquad\forall t\geqslant 0 \tag{4.6}$$

and B_R *is not present in* (4.6) *if* $\boldsymbol{H}\equiv 0$. *In addition, if* $\sum_{j=1}^{N+1}|\overline{G_j}\setminus G_j|=0$, *then* (4.5) *holds for any* $t,t'\geqslant 0$ *with* $|t-t'|<1$.

Proof. We give only few details of the proof since it can be done following the arguments of the proofs of [14, theorems 4.9 and 5.1].

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Step 1: Existence of minimizers. Given $\mathcal{A} \in \mathbb{P}_b(N+1)$ and $\lambda \geqslant 1$, the problem

$$\inf_{\mathcal{B}\in\mathbb{P}_b(N+1)} \mathfrak{F}(\mathcal{B}, \mathcal{A}; \lambda)$$

has a solution. Moreover, every minimizer $\mathcal{A}(\lambda) = (A_1(\lambda), \ldots, A_{N+1}(\lambda))$ satisfies the bound

$$\bigcup_{i=1}^{N} A_i(\lambda) \subseteq \text{ closed convex hull of } \mathrm{co}(\mathcal{A}) \cup B_R(0).$$

We omit the proof since it is proven along the same lines as [**14**, theorem 4.2] using the anisotropic comparison theorem with convex sets [1] and the inequality $d_\psi(\cdot, E_0) > 0$ in any $F \subset \mathbb{R}^n \setminus \overline{E_0}$.

Step 2: Density estimates for minimizers. Let $\mathcal{A} \in \mathbb{P}_b(N+1)$ satisfy $\mathrm{co}(\mathcal{A}) \subset D$ and set

$$\Lambda_1 := \lambda \max_{1\leqslant j\leqslant N+1} (\mathrm{diam}_{\psi_j} D + 2), \quad \Lambda_2 := N^{1/p} \max_{1\leqslant j\leqslant N} \|H_j - H_{N+1}\|_{L^p(D_1)}, \tag{4.7}$$

where $D_1 := \{x \in \mathbb{R}^n : \ d(x, D) \leqslant 1\}$. Let $\lambda \geqslant 1$ and $\mathcal{A}(\lambda) \in \mathbb{P}_b(N+1)$ be a minimizer of $\mathfrak{F}(\cdot, \mathcal{A}; \lambda)$. Then for every $i \in \{1, \ldots, N+1\}$ either $\partial A_i(\lambda)$ is empty or for any $x \in \partial A_i(\lambda)$ and

$$r \in \left(0, \min\left\{1, \left(\frac{c_\Phi}{N} - \frac{\kappa_{N+1}}{2}\right)\frac{n}{4\Lambda_1}, \left[\left(\frac{c_\Phi}{N} - \frac{\kappa_{N+1}}{2}\right)\frac{n\omega_n^{1/p}}{2^{2-1/p}\Lambda_2}\right]^{p/(p-n)}\right\}\right] \tag{4.8}$$

one has

$$\left(\frac{2c_\Phi - N\kappa_{N+1}}{2c_\Phi + 2NC_\Phi - N\kappa_{N+1}}\right)^n \leqslant \frac{|A_i(\lambda) \cap B_r(x)|}{|B_r(x)|} \leqslant 1 - \left(\frac{c_\Phi}{2(c_\Phi + C_\Phi)}\right)^n \tag{4.9}$$

and

$$c^\Phi \leqslant \frac{P(A_i(\lambda), B_r(x))}{r^{n-1}} \leqslant \left(\frac{C_\Phi}{c_\Phi} + \frac{1}{2}\right) n\omega_n, \tag{4.10}$$

where κ_{N+1} is given by (3.1) and

$$c^\Phi = c^\Phi(N, n) := \frac{n\omega_n(2^{1/n} - 1)}{2^{1+1/n}} \left(\frac{2c_\Phi - N\kappa_{N+1}}{2c_\Phi + 2NC_\Phi - N\kappa_{N+1}}\right)^{n-1}.$$

The proof is analogous to the the proof of [**14**, theorem 4.6]: we only show that

$$\mathcal{A}(\lambda) \text{ is a } (\Phi, \Lambda_1, \Lambda_2, 1, 1 - 1/p)\text{-minimizer,}$$

[1] If $E \in BV(\mathbb{R}^n; \{0, 1\})$, then $P_\phi(E) \geqslant P_\phi(E \cap C)$ for every anisotropy ϕ and every closed convex set $C \subset \mathbb{R}^n$.

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and hence (4.9)–(4.10) follow from theorem 3.2. Let $\mathcal{C} \in \mathbb{P}_b(N+1)$ be such that $\mathcal{C}\Delta\mathcal{A}(\lambda) \subset\subset B_\rho(x)$ with $\rho \in (0,1)$. By the minimality of $\mathcal{A}(\lambda)$,

$$\begin{aligned}\mathrm{Per}_\Phi(\mathcal{A}(\lambda), B_\rho(x)) \leqslant \mathrm{Per}_\Phi(\mathcal{C}, B_\rho(x)) + \lambda \sum_{j=1}^{N+1} \int_{C_j \Delta A_j(\lambda)} d_{\psi_j}(x, \partial A_j)\, \mathrm{d}x \\ + \sum_{j=1}^{N} \int_{C_j\Delta A_j(\lambda)} |H_j - H_{N+1}|\, \mathrm{d}x.\end{aligned}$$

By step 1, $\mathrm{co}(\mathcal{A}(\lambda)) \subseteq D$, thus

$$d_{\psi_j}(z, \partial A_j) \leqslant \mathrm{diam}_{\psi_j} D + 2\rho \quad \text{for all } j = 1, \dots, N+1 \text{ and } z \in \mathcal{C}\Delta\mathcal{A}(\lambda),$$

where diam_{ψ_j} is the ψ_j-diameter of a set. Then, since $\mathcal{C}\Delta\mathcal{A}(\lambda) \subset D_1$,

$$\sum_{j=1}^{N+1} \int_{C_j\Delta A_j(\lambda)} d_{\psi_j}(x, \partial A_j)\, \mathrm{d}x \leqslant \max_{1\leqslant j\leqslant N+1} (\mathrm{diam}_{\psi_j} D + 2)\, |\mathcal{C}\Delta\mathcal{A}(\lambda)|$$

and

$$\begin{aligned}\sum_{j=1}^{N} \int_{C_j\Delta A_j(\lambda)} |H_j - H_{N+1}|\, \mathrm{d}x \leqslant& \sum_{j=1}^{N} |C_j \Delta A_j(\lambda)|^{1-1/p} \|H_j - H_{N+1}\|_{L^p(D_1)} \\ \leqslant& N^{1/p} \max_{1\leqslant j \leqslant N} \|H_j - H_{N+1}\|_{L^p(D_1)}\, |\mathcal{C}\Delta\mathcal{A}(\lambda)|^{1-1/p}.\end{aligned}$$

Thus,

$$\mathrm{Per}_\Phi(\mathcal{A}(\lambda), B_\rho(x)) \leqslant \mathrm{Per}_\Phi(\mathcal{C}, B_\rho(x)) + \Lambda_1 |\mathcal{C}\Delta\mathcal{A}(\lambda)| + \Lambda_2 |\mathcal{C}\Delta\mathcal{A}(\lambda)|^{1-1/p}.$$

Step 3: Existence of GMM. Given $\lambda \geqslant 1$ and $k \in \mathbb{N}_0$ we define $\mathcal{G}(\lambda,k)$ recursively as: $\mathcal{G}(\lambda,0) = \mathcal{G}$ and

$$\mathfrak{F}(\mathcal{G}(\lambda,k), \mathcal{G}(\lambda,k-1), \lambda) = \min_{\mathcal{A}\in\mathbb{P}_b(N+1)} \mathfrak{F}(\mathcal{A}, \mathcal{G}(\lambda,k-1),\lambda).$$

Since $\mathfrak{F}(\mathcal{G}(\lambda,k), \mathcal{G}(\lambda,k-1), \lambda) \leqslant \mathfrak{F}(\mathcal{G}(\lambda,k-1), \mathcal{G}(\lambda,k-1), \lambda)$, we have

$$\begin{aligned}\mathrm{Per}_\Phi(\mathcal{G}(\lambda,k)) + \Upsilon(\mathcal{G}(\lambda,k)) + \lambda\sigma_\Psi(\mathcal{G}(\lambda,k), \mathcal{G}(\lambda,k-1)) \\ \leqslant \mathrm{Per}_\Phi(\mathcal{G}(\lambda,k-1)) + \Upsilon(\mathcal{G}(\lambda,k-1)).\end{aligned} \tag{4.11}$$

Thus, the map $k \in \mathbb{N}_0 \mapsto \mathrm{Per}_\Phi(\mathcal{G}(\lambda,k)) + \Upsilon(\mathcal{G}(\lambda,k))$ is nonincreasing for any $\lambda \geqslant 1$. In particular,

$$\begin{aligned}\mathrm{Per}_\Phi(\mathcal{G}(\lambda,k)) \leqslant& \mathrm{Per}_\Phi(\mathcal{G}(\lambda,0)) + \sum_{j=1}^{N} \int_{G_j(\lambda,k)\Delta G_j(\lambda,0)} |H_j - H_{N+1}| \mathrm{d}x \\ \leqslant& \mathrm{Per}_\Phi(\mathcal{G}) + \sum_{j=1}^{N} \|H_j - H_{N+1}\|_{L^1(D)} =: \mu_0, \quad k \geqslant 0\end{aligned} \tag{4.12}$$

and

$$\bigcup_{j=1}^{N} G_j(\lambda, k) \subseteq D \quad \text{for all } \lambda \geqslant 1, \text{ and } k \geqslant 0. \tag{4.13}$$

Fix $t, t' > 0$ with $0 < t - t' < 1$. Let $\lambda > 1$ be so large (depending on t, t', n, N, $\mathbf{H}$, Ψ and Φ) that setting $k_0 = [\lambda t']$, $m_0 = [\lambda t]$, one has $m_0 \geqslant k_0 + 3 \geqslant 4$ and

$$r_\lambda := \min\Big\{1, \Big(\frac{c_\Phi}{N} - \frac{\kappa_{N+1}}{2}\Big)\frac{n}{4\Lambda_1}, \Big[\Big(\frac{c_\Phi}{N} - \frac{\kappa_{N+1}}{2}\Big)\frac{n\omega_n^{1/p}}{2^{2-1/p}\Lambda_2}\Big]^{p/(p-n)}\Big\} = \frac{\gamma}{\lambda},$$

where Λ_1 and Λ_2 are given in (4.7), and recalling (4.3),

$$\gamma := \Big(\frac{c_\Phi}{N} - \frac{\kappa_{N+1}}{2}\Big)\frac{n}{4 \max\limits_{1\leqslant j\leqslant N+1}(\mathrm{diam}_{\psi_j} D + 2)} > 0.$$

By (4.10) for such λ and for any $k \geqslant 1$ any minimizer $\mathcal{G}(\lambda, k)$ satisfies

$$P(G_j(\lambda, k), B_r(x)) \geqslant c^\Phi r^{n-1}$$

for any $x \in \partial G_j(\lambda, k)$ and $r \in (0, r_\lambda)$ provided $\partial G_j(\lambda, k)$ is nonempty. Therefore, by proposition 3.3 applied with $r_0 = r_\lambda$, $\theta = c^\Phi$, $A = G_j(\lambda, k-1)$, $B = G_j(\lambda, k)$ and $\ell = r_\lambda |t - t'|^{-1/(n+1)} > r_\lambda$, for any $j \in \{1, \dots, N+1\}$ and $k \in \{k_0 + 1, \dots, m_0\}$, we have

$$\begin{aligned}|\mathcal{G}(\lambda, k-1)\Delta\mathcal{G}(\lambda, k)| \leqslant &\frac{5^n\omega_n\gamma}{\lambda c^\Phi |t-t'|^{n/n+1}}\mathrm{Per}(\mathcal{G}(\lambda, k-1))\\ &+ \frac{\lambda|t-t'|^{1/n+1}}{\gamma}\,\sigma(\mathcal{G}(\lambda, k), \mathcal{G}(\lambda, k-1)).\end{aligned}$$

Now the bounds (2.5), (4.1) and (4.11) imply

$$\begin{aligned}&|\mathcal{G}(\lambda, k-1)\Delta\mathcal{G}(\lambda, k)| \leqslant \frac{5^n\omega_n\gamma}{\lambda c_\Phi c^\Phi |t-t'|^{n/n+1}}\mathrm{Per}_\Phi(\mathcal{G}(\lambda, k-1))\\ &+ \frac{|t-t'|^{1/n+1}}{\gamma c_\Psi}\,(\mathrm{Per}_\Phi(\mathcal{G}(\lambda, k-1)) + \Upsilon(\mathcal{G}(\lambda, k-1)) - \mathrm{Per}_\Phi(\mathcal{G}(\lambda, k) - \Upsilon(\mathcal{G}(\lambda, k))).\end{aligned}$$

Summing this inequality in $k \in \{k_0 + 1, \dots, m_0\}$, we obtain

$$\begin{aligned}&|\mathcal{G}(\lambda, [\lambda t])\Delta\mathcal{G}(\lambda, [\lambda t'])| \leqslant \sum_{k=k_0+1}^{m_0} |\mathcal{G}(\lambda, k-1)\Delta\mathcal{G}(\lambda, k)|\\ &\leqslant \frac{5^n\omega_n\gamma}{\lambda c_\Phi c^\Phi |t-t'|^{\frac{n}{n+1}}} \sum_{k=k_0+1}^{m_0} \mathrm{Per}_\Phi(\mathcal{G}(\lambda, k-1))\\ &\quad + \frac{|t-t'|^{\frac{1}{n+1}}}{\gamma c_\Psi}\Big(\mathrm{Per}_\Phi(\mathcal{G}(\lambda, k_0)) + \Upsilon(\mathcal{G}(\lambda, k_0)) - \mathrm{Per}_\Phi(\mathcal{G}(\lambda, m_0) - \Upsilon(\mathcal{G}(\lambda, m_0))\Big).\end{aligned} \tag{4.14}$$

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By (4.12)

$$\sum_{k=k_0+1}^{m_0} \mathrm{Per}_\Phi(\mathcal{G}(\lambda, k-1)) \leqslant \mu_0(m_0-k_0) \leqslant \lambda\mu_0\Big(|t-t'|+\frac{1}{\lambda}\Big)$$

and

$$\mathrm{Per}_\Phi(\mathcal{G}(\lambda,k_0)) + \Upsilon(\mathcal{G}(\lambda,k_0)) - \mathrm{Per}_\Phi(\mathcal{G}(\lambda,m_0)) - \Upsilon(\mathcal{G}(\lambda,m_0))$$
$$\leqslant \mathrm{Per}_\Phi(\mathcal{G}(\lambda,k_0)) + \sum_{j=1}^{N} \|H_j - H_{N+1}\|_{L^1(D)} \leqslant 2\mu_0.$$

Thus, from (4.14) we get

$$|\mathcal{G}(\lambda,[\lambda t])\Delta\mathcal{G}(\lambda,[\lambda t'])| \leqslant C|t-t'|^{\frac{1}{n+1}} + \widetilde{C}|t-t'|^{-\frac{n}{n+1}}\lambda^{-1}, \tag{4.15}$$

where

$$C := \frac{5^n\omega_n\gamma\mu_0}{c_\Phi c^\Phi} + \frac{2\mu_0}{\gamma c_\Psi} \quad \text{and} \quad \widetilde{C} := \frac{5^n\omega_n\gamma\mu_0}{c_\Phi c^\Phi}.$$

The remaining part of the proof is as the proofs of [**14**, theorems 4.9 and 5.1]. We note here that if $\mathcal{G}\in\mathbb{P}_b(N+1)$ satisfies $\sum_{j=1}^{N+1}|\overline{G_j}\setminus G_j|=0$, then $1/(n+1)$-Hölderianity of GMM at time $t=0$ follows from the relations

$$\lim_{\lambda\to+\infty} |\mathcal{G}(\lambda,1)\Delta\mathcal{G}|=0, \qquad \lim_{\lambda\to+\infty} \mathrm{Per}_\Phi(\mathcal{G}(\lambda,1)) = \mathrm{Per}_\Phi(\mathcal{G}),$$

$$\lim_{\lambda\to+\infty} \lambda\sigma_\Psi(\mathcal{G}(\lambda,1),\mathcal{G})=0, \qquad \lim_{\lambda\to+\infty} \Upsilon(\mathcal{G}(\lambda,1)) = \Upsilon(\mathcal{G})$$

whose proofs can be done following the arguments of [**14**, proposition 4.5]. $\square$

4.1. Two-phase case

When $N=1$, repeating the arguments of [**42**] in our more general setting, we can improve the Hölder exponent of GMM to $1/2$ without any restriction on the anisotropies.

THEOREM 4.2. *Let $\Phi=(\phi_1,\phi_2)$ and $\Psi=(\psi_1,\psi_2)$, and assume that $\boldsymbol{H}=(H_1, H_2)$ satisfies (4.4) with $N=1$. Then for every $\mathcal{G}\in\mathbb{P}_b(2)$, $GMM(\mathfrak{F},\mathcal{G})$ is nonempty. Moreover, there exists a constant $C=C(n,\Phi,\Psi,\boldsymbol{H},\mathrm{Per}_\Phi(\mathcal{G}))>0$ such that for any $\mathcal{N}\in GMM(\mathfrak{F},\mathcal{G})$,*

$$|\mathcal{N}(t)\Delta\mathcal{N}(t')| \leqslant C\,|t-t'|^{1/2}, \quad t,t'>0,\ |t-t'|<1 \tag{4.16}$$

and

$$N_1(t) \subseteq \textit{ closed convex hull of } G_1\cup B_R \quad \forall t\geqslant 0.$$

In addition, if $|\overline{G_1}\setminus G_1|=0$, then (4.16) holds for any $t,t'\geqslant 0$ with $|t-t'|<1$.

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The proof runs along the same lines of theorem 4.1 with however an improved bound for the radii in the proof of the density estimates, see (4.30) below. We need to make a detailed proof since this will be used in the proof of theorem 5.1.

Proof. Letting $\phi := \phi_1 + \phi_2$, $H := H_1 - H_2$, and

$$d_\psi^E(\cdot) := d_{\psi_1}(\cdot, \partial E) + d_{\psi_2}(\cdot, \partial E), \quad \widetilde{d}_\psi^E(\cdot) := \widetilde{d}_{\psi_1}(\cdot, \partial E) + \widetilde{d}_{\psi_2}(\cdot, \partial E),$$

we have

$$\begin{aligned} \mathfrak{F}(\mathcal{A}, \mathcal{B}, \lambda) - \int_{\mathbb{R}^n} H_2 \,\mathrm{d}x = P_\phi(A_1) + \int_{A_1} H \,\mathrm{d}x + \lambda \int_{A_1 \Delta B_1} d_\psi^{B_1} \,\mathrm{d}x \\ = P_\phi(A_1) + \int_{A_1} H \,\mathrm{d}x + \lambda \int_{A_1} \widetilde{d}_\psi^{B_1} \,\mathrm{d}x - \lambda \int_{B_1} \widetilde{d}_\psi^{B_1} \,\mathrm{d}x =: \mathfrak{F}_2(A_1, B_1, \lambda). \end{aligned} \tag{4.17}$$

Therefore, it suffices to show that for any bounded $G \in BV(\mathbb{R}^n; \{0,1\})$, $GMM(\mathfrak{F}_2, G)$ is nonempty and there exists $C_0 := C_0(n, \Phi, \Psi, H, P_\phi(G))$ such that for any $L \in GMM(\mathfrak{F}_2, G)$

$$|L(t) \Delta L(s)| \leqslant C_0 \, |t - t'|^{1/2}, \quad t, t' > 0, \quad |t - t'| < 1 \tag{4.18}$$

and

$$L(t) \subseteq D := \mathrm{co}(G \cup B_R), \quad t \geqslant 0. \tag{4.19}$$

Note that, except for the presence of $\int_{A_1} H \mathrm{d}x$, $\mathfrak{F}_2$ is of the form of the Almgren–Taylor–Wang functional.

We divide the proof into five steps.

Step 1: Existence of minimizers. Let $E_0 \in BV(\mathbb{R}^n; \{0,1\})$ be such that $E_0 \subset D$. Since ϕ is a norm, $\widetilde{d}_\psi^{E_0} \geqslant 0$ in $\mathbb{R}^n \setminus E_0$ and $H \geqslant 0$ in $\mathbb{R}^n \setminus D$, as in the Euclidean two-phase case (see e.g. [**3**]) we can use the comparison theorem with the convex set D to establish the existence of a minimizer of $\mathfrak{F}_2(\cdot, E_0, \lambda)$ and also that every minimizer E_λ satisfies $E_\lambda \subseteq D$.

Step 2: Unconstrained density estimates for minimizers. Let E_λ minimize $\mathfrak{F}_2(\cdot, E_0, \lambda)$ and $x_0 \in E_\lambda \Delta E_0$ be such that $d(x_0, \partial E_0) \geqslant r_1$ for some $r_1 > 0$ satisfying

$$w_n^{1/n - 1/p} r_1^{1 - n/p} \Big(\int_D |H|^p \,\mathrm{d}x \Big)^{1/p} < c_\Phi n \omega_n^{1/n}. \tag{4.20}$$

Notice that there are no restrictions on $r_1 > 0$; in addition x_0 need not to be on ∂E_λ. Let us show that

(a) if $x_0 \in E_\lambda \setminus E_0$, then

$$\frac{|E_\lambda \cap B_r(x_0)|}{|B_r(x_0)|} \geqslant \Big(\frac{c_\Phi}{4 C_\Phi} \Big)^n \quad \text{for any } r \in (0, r_1)\,;$$

(b) if $x_0 \in E_0 \setminus E_\lambda$, then

$$\frac{|B_r(x_0) \setminus E_\lambda|}{|B_r(x_0)|} \geqslant \Big(\frac{c_\Phi}{4 C_\Phi} \Big)^n \quad \text{for any } r \in (0, r_1)\,.$$

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We prove only (a), since the proof of (b) is similar. For shortness we write $B_r := B_r(x_0)$. Fix any $r \in (0, r_1)$ such that

$$\mathcal{H}^{n-1}(\partial^* E_\lambda \cap \partial B_r) = 0. \tag{4.21}$$

By the minimality of E_λ we have $\mathfrak{F}_2(E_\lambda, E_0, \lambda) \leqslant \mathfrak{F}_2(E_\lambda \setminus B_r, E_0, \lambda)$ so that

$$P_\phi(E_\lambda, B_s) + \lambda \int_{E_\lambda \cap B_r} \widetilde{d}_\psi^{E_0}\, \mathrm{d}x \leqslant P_\phi(E_\lambda \setminus B_r, B_s) + \int_{E_\lambda \cap B_r} |H|\, \mathrm{d}x \tag{4.22}$$

for any $s > r$. The choice of x_0 and the definition of $\widetilde{d}_\psi^{E_0}$ imply $\widetilde{d}_\psi^{E_0} \geqslant 0$ in B_{r_1} and hence using (4.21) and (3.12) (applied with $\phi_j = \phi$ and $A_j = E_\lambda$) and the inclusion $(E_\lambda \setminus B_r)\Delta E_\lambda \subset B_r$, from (4.22) we get

$$P_\phi(E_\lambda \cap B_r) \leqslant 2 \int_{E_\lambda \cap \partial B_r} \phi(\nu_{B_r})\, \mathrm{d}\mathcal{H}^{n-1} + \int_{E_\lambda \cap B_r} |H|\, \mathrm{d}x. \tag{4.23}$$

The definition of ϕ, (2.5), the isoperimetric inequality and the Hölder inequality yield

$$\begin{aligned} 2c_\Phi n\omega_n^{1/n} |E_\lambda \cap B_r|^{(n-1/n} \leqslant & 4C_\Phi \mathcal{H}^{n-1}(E_\lambda \cap \partial B_r) \\ & + |E_\lambda \cap B_r|^{1-\frac{1}{p}} \Big(\int_{E_\lambda \cap B_r} |H|^p\, \mathrm{d}x \Big)^{1/p}. \end{aligned} \tag{4.24}$$

Recall by step 1 that $E_\lambda \subseteq D$. Thus from the inequality

$$|E_\lambda \cap B_r|^{1-1/p} \leqslant |E_\lambda \cap B_r|^{(n-1/n} |B_r|^{1/n-1/p} = \omega_n^{1/n-1/p} r^{1-n/p} |E_\lambda \cap B_r|^{(n-1/n}$$

and (4.20), it follows that

$$|E_\lambda \cap B_r|^{1-1/p} \left(\int_{E_\lambda \cap B_r} |H|^p\, \mathrm{d}x \right)^{1/p} \leqslant c_\Phi n \omega_n^{1/n} |E_\lambda \cap B_r|^{(n-1)/n},$$

and therefore, from (4.24) we deduce

$$c_\Phi n\omega_n^{1/n} |E_\lambda \cap B_r|^{(n-1)/n} \leqslant 4C_\Phi \mathcal{H}^{n-1}(E_\lambda \cap \partial B_r).$$

Now integrating we get

$$\frac{|E_\lambda \cap B_r|}{|B_r|} \geqslant \Big(\frac{c_\Phi}{4C_\Phi} \Big)^n$$

for any $r \in (0, r_1)$.

The next step is valid in the two-phase case. We miss the proof of a similar statement in the multiphase case because we are not able to prove the analogue of step 2 [2] .

[2] In the multiphase case we miss the analogue of (4.23), that was obtained neglecting the term $\int_{E_\lambda \cap B_r} \widetilde{d}_\psi^{E_0}\, \mathrm{d}x$ in (4.22). For instance, in the planar 4 -phase, at a triple junction involving ϕ_1, ϕ_2, ϕ_3 and surrounded by the fourth phase having ϕ_4 as surface tension, it is conceivable that, if ϕ_1, ϕ_2, ϕ_3 are quite large compared to ϕ_4 , then around the triple point, the fourth phase appears after one minimization step.

We essentially follow the arguments of [42, 48]. Let

$$C_1 = C_1(n, \Phi, \Psi) := 8C_\Psi \left(\frac{(4C_\Phi)^{n+1} n}{2c_\Psi c_\Phi^n} \right)^{1/2}$$

and

$$C_2 = C_2(n, \Phi, \Psi, H, p) := (nc_\Phi)^{2p/(n-p)} \Big(\frac{C_1}{2C_\Psi} \Big)^2 \Big(\frac{1}{\omega_n} \int_D |H|^p \mathrm{d}x \Big)^{2/(p-n)}.$$

Step 3: L^∞ -bound for minimizers. For any $\lambda > C_2$, if E_λ minimizes $\mathfrak{F}_2(\cdot, E_0, \lambda)$ then

$$\sup_{x \in E_\lambda \Delta E_0} d_\psi^{E_0}(x) \leqslant C_1 \lambda^{-1/2}.$$

Assume by contradiction that there exist $\lambda > C_2$ and $x_0 \in E_\lambda \Delta E_0$ such that $d_\psi^{E_0}(x_0) > C_1 \lambda^{-1/2}$. Then from (4.1) we get $d(x_0, \partial E_0) > C_1/2C_\Psi\, \lambda^{-1/2}$. Since $\lambda > C_2$, we can choose $\epsilon > 0$ such that $r_1 := 2r = d(x_0, \partial E_0) > (C_1/2C_\Psi + \epsilon)\lambda^{-1/2}$ satisfies (4.20), where for shortness we drop the dependence of r on n, λ, ϵ, Φ and Ψ. Setting $B_r := B_r(x_0)$, without loss of generality we also suppose that (4.21) holds. First we assume $x_0 \in E_\lambda \setminus E_0$. Then the minimality of E_λ implies $\mathfrak{F}_2(E_\lambda, E_0, \lambda) \leqslant \mathfrak{F}_2(E_\lambda \setminus B_r, E_0, \lambda)$ so that, similarly to (4.23),

$$P_\phi(E_\lambda \cap B_r) + \lambda \int_{E_\lambda \cap B_r} \widetilde{d}_\psi^{E_0} \,\mathrm{d}x \leqslant 2 \int_{E_\lambda \cap \partial B_r} \phi(\nu_{B_r}) \,\mathrm{d}\mathcal{H}^{n-1} + \int_{E_\lambda \cap B_r} |H| \,\mathrm{d}x. \tag{4.25}$$

By the Hölder inequality, the inclusion $E_\lambda \subset D$ and (4.20),

$$\begin{aligned} \int_{E_\lambda \cap B_r} |H| \,\mathrm{d}x \leqslant & |E_\lambda \cap B_r|^{1-1/p} \Big(\int_{E_\lambda \cap B_r} |H|^p \,\mathrm{d}x \Big)^{1/p} \\ \leqslant & \omega_n^{1/n - 1/p} r^{1-n/p} \Big(\int_D |H|^p \,\mathrm{d}x \Big)^{1/p} |E_\lambda \cap B_r|^{(n-1)/n}, \\ < & c_\Phi n \omega_n^{1/n} |E_\lambda \cap B_r|^{(n-1)/n}, \end{aligned} \tag{4.26}$$

therefore, by (2.5) and the isoperimetric inequality,

$$P_\phi(E_\lambda \cap B_r) > \int_{E_\lambda \cap B_r} |H| \,\mathrm{d}x.$$

This and (4.25) imply

$$\lambda \int_{E_\lambda \cap B_r} d_\psi^{E_0} \,\mathrm{d}x < 2 \int_{E_\lambda \cap \partial B_r} \phi(\nu_{B_r}) \,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.27}$$

By (4.1), the choice of x_0 and the definition of r one has $d_\psi^{E_0} \geqslant c_\Psi d(\cdot, \partial E_0) \geqslant 2c_\Psi r$ in B_r. Thus, from (4.27) and (2.5) we get

$$2c_\Psi \lambda r |E_\lambda \cap B_r| < 4C_\Phi \mathcal{H}^{n-1}(E_\lambda \cap \partial B_r).$$

This, the inequality $\mathcal{H}^{n-1}(E_\lambda \cap \partial B_r) \leqslant n\omega_n r^{n-1}$ and step 2 (a) imply

$$2c_\Psi \lambda \omega_n r^{n+1} \Big(\frac{c_\Phi}{4C_\Phi}\Big)^n < 4C_\Phi n\omega_n r^{n-1}.$$

Therefore, by the definition of C_1 and r,

$$\Big(\frac{C_1}{8C_\Psi}\Big)^2 = \frac{(4C_\Phi)^{n+1}n}{2c_\Psi c_\Phi^n} > \lambda r^2 > \Big(\frac{C_1}{8C_\Psi} + \frac{\epsilon}{4}\Big)^2,$$

a contradiction.

If $x_0 \in E_0 \setminus E_\lambda$, then we use $\mathfrak{F}_2(E_\lambda, E_0, \lambda) \leqslant \mathfrak{F}_2(E_\lambda \cup B_r, E_0, \lambda)$ and repeat a similar argument.

Before passing to the next step let us define

$$C_3 := C_3(n, \Phi, \Psi) = \frac{2nc_\Phi}{C_1 + \sqrt{C_1^2 + 4nc_\Phi C_\Psi}},$$

$$C_4 := C_4(n, \Phi) = \frac{n\omega_n(2^{1/n} - 1)}{2^{n+1/n}} \Big(\frac{c_\Phi}{4C_\Phi}\Big)^{n-1}$$

and

$$C_5 = C_5(n, \Phi, \Psi, H, p) := \max\Big\{C_2, C_3^2\Big(\frac{nc_\Phi}{2}\Big)^{2p/(n-p)}\Big(\frac{1}{\omega_n}\int_D |H|^p \,\mathrm{d}x\Big)^{2/(p-n)}\Big\}.$$

Step 4: Uniform density estimates for minimizers. Given $\lambda > C_5$ and a minimizer E_λ of $\mathfrak{F}_2(\cdot, E_0, \lambda)$, following arguments of [**42**, **48**] let us show that

$$\Big(\frac{c_\Phi}{4C_\Phi}\Big)^n \leqslant \frac{|E_\lambda \cap B_r(x)|}{|B_r(x)|} \leqslant 1 - \Big(\frac{c_\Phi}{4C_\Phi}\Big)^n \tag{4.28}$$

and

$$C_4 \leqslant \frac{P(E_\lambda, B_r(x))}{r^{n-1}} \leqslant \frac{2C_\Phi + c_\Phi}{2c_\Phi} n\omega_n \tag{4.29}$$

for any $x \in \partial E_\lambda$ and

$$r \in (0, C_3\lambda^{-1/2}). \tag{4.30}$$

Since $E_\lambda^{(1)} = E_\lambda$ and $\overline{\partial^* E_\lambda} = \partial E_\lambda$, we can suppose $x \in \partial^* E_\lambda$. For any r as in (4.30) and $y \in B_r(x)$ one has

$$d_\psi^{E_0}(y) \leqslant d_\psi^{E_\lambda}(y) + \sup_{z \in E_\lambda \Delta E_0} d_\psi^{E_0}(z) \leqslant 2C_\Psi r + \sup_{z \in E_\lambda \Delta E_0} d_\psi^{E_0}(z)$$

so that by step 3

$$d_\psi^{E_0}(y) \leqslant (2C_\Psi C_3 + C_1)\lambda^{-1/2}. \tag{4.31}$$

Let us prove the lower volume density estimate in (4.28). For shortness set $B_r := B_r(x)$. Let $r \in (0, C_3\lambda^{-1/2})$ be such that (4.21) holds. As in the proof of step 2,

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from the inequality $\mathfrak{F}_2(E_\lambda, E_0, \lambda) \leqslant \mathfrak{F}_2(E_\lambda \setminus B_r, E_0, \lambda)$ we get

$$P_\phi(E_\lambda, B_r) \leqslant \int_{E_\lambda\cap\partial B_r} \phi(\nu_{B_r})\,\mathrm{d}\mathcal{H}^{n-1} + \lambda \int_{E_\lambda\cap B_r} d_\psi^{E_0}(y)\,\mathrm{d}y + \int_{E_\lambda\cap B_r} |H|\,\mathrm{d}y. \tag{4.32}$$

By (4.31), the choice of r and the equality

$$(2C_\Psi C_3 + C_1)C_3 = \frac{nc_\Phi}{2},$$

we have

$$\begin{aligned}\lambda \int_{E_\lambda\cap B_r} d_\psi^{E_0}(y)\,\mathrm{d}y &\leqslant (2C_\Psi C_3 + C_1)\omega_n^{1/n}\lambda^{1/2} r |E_\lambda \cap B_r|^{(n-1)/n}\\ &\leqslant \omega_n^{1/n}(2C_\Psi C_3 + C_1)C_3 |E_\lambda \cap B_r|^{(n-1)/n}\\ &= \frac{c_\Phi n\omega_n^{1/n}}{2}\,|E_\lambda \cap B_r|^{(n-1)/n}.\end{aligned}$$

Furthermore, using $\lambda > C_5$, as in (4.26)

$$\begin{aligned}\int_{E_\lambda\cap B_r} |H|\,\mathrm{d}x \leqslant& \omega_n^{1/n-1/p} r^{1-n/p}\Big(\int_{E_\lambda\cap B_{r_1}} |H|^p\,\mathrm{d}x\Big)^{1/p} |E_\lambda\cap B_r|^{(n-1)/n}\\ \leqslant& \omega_n^{1/n-1/p}(C_3\lambda^{-1/2})^{1-n/p}\Big(\int_{E_\lambda\cap B_{r_1}} |H|^p\,\mathrm{d}x\Big)^{1/p} |E_\lambda\cap B_r|^{(n-1)/n}\\ \leqslant& \frac{c_\Phi n\omega_n^{1/n}}{2}\,|E_\lambda\cap B_r|^{(n-1)/n}.\end{aligned}$$

Therefore, from (4.32) it follows that

$$P_\phi(E_\lambda, B_r) \leqslant \int_{E_\lambda\cap\partial B_r} \phi(\nu_{B_r})\,\mathrm{d}\mathcal{H}^{n-1} + c_\Phi n\omega_n^{1/n}\,|E_\lambda\cap B_r|^{(n-1)/n}. \tag{4.33}$$

Adding $\int_{E_\lambda\cap\partial B_r} \phi(\nu_{B_r})\,\mathrm{d}\mathcal{H}^{n-1}$ to both sides of (4.33), using (2.5) and the isoperimetric inequality we get

$$c_\Phi n\omega_n^{1/n}|E_\lambda\cap B_r|^{(n-1)/n} \leqslant 4C_\Phi\mathcal{H}^{n-1}(E_\lambda\cap\partial B_r).$$

Integrating this over r we get the lower volume density estimate in (4.28).

To get the upper volume density estimate in (4.28) we use $\mathfrak{F}_2(E_\lambda, E_0, \lambda) \leqslant \mathfrak{F}_2(E_\lambda \cup B_r, E_0, \lambda)$ and proceed as above.

For what concerns the upper perimeter density estimate in (4.29) we observe that from (4.33) and (2.5) it follows that

$$2c_\Phi P(E_\lambda, B_r) \leqslant (2C_\Phi + c_\Phi)n\omega_n r^{n-1}$$

for a.e. $r \in (0, C_3\lambda^{-1/2})$. Since $r \mapsto P(E_\lambda, B_r)$ is nondecreasing and left-continuous, this inequality holds for all r. Finally the lower perimeter density estimate follows from (4.28) and the relative isoperimetric inequality for the ball.

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Step 5: Existence of GMM starting from G. We follow the arguments of [**42**, **48**]. Let $\{G(\lambda,k)\}_{\lambda>C_5,k\in\mathbb{N}_0}$ be defined as follows: $G(\lambda,0)=G$ and

$$G(\lambda,k)\in \mathrm{argmin}\ \mathfrak{F}_2(\cdot,G(\lambda,k-1),\lambda),\quad k\geqslant 1.$$

By step 1 $G(\lambda,k)$ is well-defined and

$$G(\lambda,k)\subseteq D \tag{4.34}$$

for all $\lambda>C_5$ and $k\geqslant 0$. Notice also that

$$k\in\mathbb{N}_0\mapsto P_\phi(G(\lambda,k))+\int_{G(\lambda,k)} H\,\mathrm{d}x\quad\text{is nonincreasing.} \tag{4.35}$$

Given $t>s>0$ with $t-s<1$, let $\lambda>\max\{C_5,5+C_3^{-2}/t-s,5/s\}$ so that $[\lambda t]-[\lambda s]\geqslant 4$, $[\lambda s]\geqslant 5$ and $1/\lambda|t-s|^{1/2}<C_3\lambda^{-1/2}$. By proposition 3.3 applied with $A=G(\lambda,k-1)$, $r_0=C_3\lambda^{-1/2}$, $\theta:=C_4$, $\ell:=1/\lambda|t-s|^{1/2}$ and $B=G(\lambda,k)$, and using the bounds (2.5) and (4.1) for anisotropies and mobilities, for any $k\in\{[\lambda s]+1,\dots,[\lambda t]\}$ we get

$$\begin{aligned}|G(\lambda,k-1)\Delta G(\lambda,k)|\leqslant&\frac{5^n\omega_n}{2C_4c_\Phi\lambda|t-s|^{1/2}}\,P_\phi(G(\lambda,k-1))\\&+\frac{\lambda|t-s|^{1/2}}{2c_\Psi}\int_{G(\lambda,k-1)\Delta G(\lambda,k)} d_\psi^{G(\lambda,k-1)}\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}|G(\lambda,[\lambda s])\Delta G(\lambda,[\lambda t])|\leqslant&\sum_{k=[\lambda s]+1}^{[\lambda t]}|G(\lambda,k-1)\Delta G(\lambda,k)|\\ \leqslant&\frac{5^n\omega_n}{2C_4c_\Phi\lambda|t-s|^{1/2}}\sum_{k=[\lambda s]+1}^{[\lambda t]}P_\phi(G(\lambda,k-1))\\&+\frac{\lambda|t-s|^{1/2}}{2c_\Psi}\sum_{k=[\lambda s]+1}^{[\lambda t]}\int_{G(\lambda,k-1)\Delta G(\lambda,k)} d_\psi^{G(\lambda,k-1)}\,\mathrm{d}x.\end{aligned} \tag{4.36}$$

By (4.35),

$$\begin{aligned}&\sum_{k=[\lambda s]+1}^{[\lambda t]}P_\phi(G(\lambda,k-1))\\&\leqslant\sum_{k=[\lambda s]+1}^{[\lambda t]}\Big(P_\phi(G(\lambda,k-1))+\int_{G(\lambda,k-1)}H\,\mathrm{d}x+\int_{G(\lambda,k-1)}|H|\mathrm{d}x\Big)\\&\leqslant\Big(P_\phi(G)+\int_G H\,\mathrm{d}x+\int_D|H|\,\mathrm{d}x\Big)\Big([\lambda t]-[\lambda s]\Big)\\&\leqslant\Big(P_\phi(G)+2\int_D|H|\,\mathrm{d}x\Big)\Big(\lambda(t-s)+1\Big)\end{aligned}$$

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and

$$\lambda \sum_{k=[\lambda s]+1}^{[\lambda t]} \int_{G(\lambda,k-1)\Delta G(\lambda,k)} d_{\psi}^{G(\lambda,k-1)}\,\mathrm{d}x$$
$$\leqslant P_\phi(G(\lambda,[\lambda s])) + \int_{G(\lambda,[\lambda s])} H\,\mathrm{d}x - P_\phi(G(\lambda,[\lambda t])) - \int_{G(\lambda,[\lambda t])} H\,\mathrm{d}x$$
$$\leqslant P_\phi(G) + 2\int_D |H|\,\mathrm{d}x,$$

therefore, from (4.36) we get

$$|G(\lambda,[\lambda s])\Delta G(\lambda,[\lambda t])| \leqslant \left(C_6\,|t-s|^{1/2} + \frac{C_6 - 1/2c_\Psi}{\lambda|t-s|^{1/2}}\right)\left(P_\phi(G) + 2\int_D |H|\,\mathrm{d}x\right), \tag{4.37}$$

where

$$C_6 := \frac{5^n\omega_n}{2C_4c_\Phi} + \frac{1}{2c_\Psi}. \tag{4.38}$$

Now (4.18) and (4.19) follow from (4.37) and (4.34), respectively. □

We will use (4.29), (4.30) and (4.37) in the proof of theorem 5.1.

5. Improved time Hölder regularity

In this section we show that when $\phi_i = \phi$ and $\psi_i = \psi$ for any $i = 1, \ldots, N+1$, the time Hölder continuity exponent of GMM for partitions can be improved to $1/2$. The result follows from the generalization of [**42**] in the previous section (theorem 4.2) combined with a comparison (theorem 5.2 below) between a multiphase flow and a two-phase flow starting from just one of the phases and its complement. Arguments from our main continuity result (in theorem 4.1) are needed to reconnect both flows in the limit.

THEOREM 5.1. *Let $\Phi = \{\phi, \ldots, \phi\}$ and $\Psi = \{\psi, \ldots, \psi\}$ for some norms ϕ and ψ on $\mathbb{R}^n$, and $\boldsymbol{H} \equiv 0$. Then for any $\mathcal{G} \in \mathbb{P}_b(N+1)$ and $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$*

$$|\mathcal{M}(t)\Delta\mathcal{M}(t')| \leqslant C_6\,\mathrm{Per}_\Phi(\mathcal{G})\,|t-t'|^{1/2}, \quad t,t' > 0,\ |t-t'| < 1, \tag{5.1}$$

where C_6 is given in (4.38). In addition, if $\sum_{j=1}^{N+1} |\overline{G_j} \setminus G_j| = 0$, then (5.1) holds for any $t, t' \geqslant 0$ with $|t-t'| < 1$.

Recall that, by theorem 4.1, for any $\mathcal{G} \in \mathbb{P}_b(N+1)$, $GMM(\mathfrak{F}, \mathcal{G})$ is nonempty, each $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ is locally $1/(n+1)$-Hölder continuous and

$$\bigcup_{i=1}^N M_i(t) \subseteq \mathrm{co}(\mathcal{G}) \quad \text{for any } t \geqslant 0\,.$$

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Besides $\mathfrak{F}$ we need to consider also the functional $\mathfrak{F}_2$ defined (up to constants) in (4.17) with $H=0$, i.e.,

$$\mathfrak{F}_2(G,E,\lambda) := P_\phi(G) + \lambda \int_{G\Delta E} d_\psi(x,\partial E)\,\mathrm{d}x. \tag{5.2}$$

We start with a comparison result: this is the key point of the proof of theorem 5.1 since it allows to compare the evolution of a single phase with the multiphase case.

THEOREM 5.2 *Discrete comparison multiphase-phase. Let* $g_1,\dots,g_{N+1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and suppose that* $\mathcal{A} \in \mathbb{P}_b(N+1)$ *minimize*

$$\mathcal{E} \in \mathbb{P}_b(N+1) \mapsto \sum_{i=1}^{N+1} P_\phi(E_i) + \sum_{i=1}^{N} \int_{E_i} g_i\,\mathrm{d}x - \int_{E^c_{N+1}} g_{N+1}\,\mathrm{d}x.$$

Suppose that for $i \in \{1,\dots,N\}$ *and* $g'_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *, there exists a bounded minimizer* F_i *of*

$$F \in BV(\mathbb{R}^n;\{0,1\}) \mapsto P_\phi(F) + \int_F g'_i\,\mathrm{d}x,$$

and suppose that, given $g'_{N+1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *there exists a bounded minimizer of*

$$G \in BV(\mathbb{R}^n;\{0,1\}) \mapsto P_\phi(G) - \int_G g'_{N+1}\,\mathrm{d}x,$$

the complement of which we denote by F_{N+1}. *If* $2g'_i - g_i + g_j > 0$ *a.e. in* $\mathbb{R}^n$ *for all* $i,j \in \{1,\dots,N+1\}$, $i \neq j$, *then*

$$F_i \subseteq A_i, \qquad i \in \{1,\dots,N+1\}.$$

Proof. Let $i \in \{1,\dots,N\}$. By minimality,

$$\begin{aligned}
&\sum_{j=1}^{N+1} P_\phi(A_j) + \sum_{j=1}^{N} \int_{A_j} g_j\,\mathrm{d}x - \int_{A^c_{N+1}} g_{N+1}\,\mathrm{d}x \leqslant P_\phi(A_i \cup F_i) \\
&\quad + \sum_{j=1, j\neq i}^{N+1} P_\phi(A_j \setminus F_i) + \int_{A_i \cup F_i} g_i\,\mathrm{d}x + \sum_{j=1,\, j\neq i}^{N} \int_{A_j \setminus F_i} g_j\,\mathrm{d}x - \int_{A^c_{N+1}\cup F_i} g_{N+1}\,\mathrm{d}x
\end{aligned} \tag{5.3}$$

and

$$P_\phi(F_i) + \int_{F_i} g'_i\,\mathrm{d}x \leqslant P_\phi(F_i \cap A_i) + \int_{F_i \cap A_i} g'_i\,\mathrm{d}x. \tag{5.4}$$

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Summing (5.3) and twice (5.4), we obtain

$$
\begin{aligned}
&P_\phi(F_i) + P_\phi(A_i) + P_\phi(F_i) + \sum_{j=1, j\neq i}^{N+1} P_\phi(A_j) \\
&\qquad + \int_{F_i \setminus A_i} (g_i' - g_i)\,\mathrm{d}x + \sum_{j=1,j\neq i}^{N+1} \int_{A_j \cap F_i} g_j\,\mathrm{d}x + \int_{F_i\setminus A_i} g_i'\,\mathrm{d}x \\
&\qquad \leqslant P_\phi(F_i \cup A_i) + P_\phi(F_i \cap A_i) + \sum_{j=1,j\neq i}^{N+1} P_\phi(A_j \setminus F_i) + P_\phi(F_i \cap A_i).
\end{aligned}
\tag{5.5}
$$

Let us show that for any $E \in BV(\mathbb{R}^n;\{0,1\})$, $\mathcal{G} \in \mathbb{P}_b(N+1)$ and $i \in \{1,\ldots, N+1\}$,

$$
\sum_{j=1,j\neq i}^{N+1} P_\phi(G_j \setminus E) + P_\phi(G_i \cap E) \leqslant P_\phi(E) + \sum_{j=1,j\neq i}^{N+1} P_\phi(G_j). \tag{5.6}
$$

First assume that $\mathcal{H}^{n-1}(\partial^* E \cap \bigcup_{j=1}^{N+1} \partial^* G_j) = 0$. In this case by (2.1) and (2.2), as well as the inclusion $\partial^* G_i \subset \bigcup_{j=1,j\neq i}^{N+1} \partial^* G_j$, we obtain

$$
P_\phi(G_j \setminus E) = \int_{E^{(0)} \cap \partial^* G_j} \phi(\nu_{G_j})\mathrm{d}\mathcal{H}^{n-1} + \int_{G_j \cap \partial^* E} \phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1}
$$

and

$$
\begin{aligned}
P_\phi(G_i \cap E) &= \int_{E\cap \partial^* G_i} \phi(\nu_{G_i})\mathrm{d}\mathcal{H}^{n-1} + \int_{G_i \cap \partial^* E} \phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1} \\
&= \sum_{j=1,j\neq i}^{N+1} \int_{E \cap \partial^* G_j \cap \partial^* G_i} \phi(\nu_{G_j})\mathrm{d}\mathcal{H}^{n-1} + \int_{G_i\cap \partial^* E} \phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1},
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&\sum_{j=1,j\neq i}^{N+1} P_\phi(G_j \setminus E) + P_\phi(G_i \cap E) \\
&\quad = \sum_{j=1,j\neq i}^{N+1} \left(\int_{E^{(0)}\cap \partial^* G_j} \phi(\nu_{G_j})\mathrm{d}\mathcal{H}^{n-1} + \int_{E \cap \partial^* G_j \cap \partial^* G_i} \phi(\nu_{G_j})\mathrm{d}\mathcal{H}^{n-1} \right) \\
&\qquad + \sum_{j=1}^{N+1} \int_{G_j \cap \partial^* E} \phi(\nu_E)\mathrm{d}\mathcal{H}^{n-1} \leqslant \sum_{j=1,j\neq i}^{N+1} P_\phi(G_j) + P_\phi(E).
\end{aligned}
$$

In the general case we choose a sequence $\{\xi_k\} \subset \mathbb{R}^n$ such that $|\xi_k| \to 0$ and $\mathcal{H}^{n-1}(\partial^*(E+\xi_k) \cap \bigcup_{j=1}^{N+1} \partial^* G_j) = 0$, where $E + \xi_k := \{x \in \mathbb{R}^n :\ x - \xi_k \in$

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$E\}$. By the previous case,

$$\sum_{j=1,\neq i}^{N+1} P_\phi(G_j \setminus (E+\xi_k)) + P_\phi(G_i \cap (E+\xi_k)) \leqslant \sum_{j=1,j\neq i}^{N+1} P_\phi(G_j) + P_\phi(E+\xi_k). \tag{5.7}$$

Since $P_\phi(E+\xi_k) = P_\phi(E)$ and $\lim_{k\to+\infty} |(E+\xi_k)\Delta E| \to 0$, letting $k\to+\infty$ in (5.7) and using the $L^1(\mathbb{R}^n)$-lower semicontinuity of the ϕ-perimeter we get (5.6).

Inserting (5.6) with $\mathcal{G} = \mathcal{A}$ and $E = F_i$ in (5.5) and using (2.4) we get

$$\int_{F_i\setminus A_i} (g_i' - g_i)\,\mathrm{d}x + \sum_{j=1,j\neq i}^{N+1} \int_{A_j\cap F_i} g_j\,\mathrm{d}x + \int_{F_i\setminus A_i} g_i'\,\mathrm{d}x \leqslant 0.$$

Recall that $F_i \setminus A_i = \bigcup_{j=1,j\neq i}^{N+1} F_i \cap A_j$ up to a negligible set, thus,

$$\sum_{j=1,j\neq i}^{N+1} \int_{A_j\cap F_i} (2g_i' - g_i + g_j)\,\mathrm{d}x \leqslant 0.$$

By assumption $2g_i' - g_i + g_j > 0$ a.e., and hence $F_i \subseteq A_i$ up to a negligible set. The case $i = N+1$ is similar. □

Lemma 5.3. *Let $\mathcal{G} \in \mathbb{P}_b(N+1)$ and set*

$$g_j(\cdot) := \widetilde{d}_\psi(\cdot, \partial G_j), \quad j \in \{1,\dots,N+1\}.$$

For $E \subseteq \mathbb{R}^n$ define $e(\cdot) = \widetilde{d}_\psi(\cdot,\partial E)$. If either $E \subseteq G_i$ for some $i \in \{1,\dots,N\}$ or $G_{N+1}^c \subseteq E^c$, then $2e - g_i + g_j \geqslant 0$ a.e. in $\mathbb{R}^n$ for any $j \in \{1,\dots,N+1\}$, $j \neq i$. Similarly, if either $E \subset\subset G_i$ for some $i \in \{1,\dots,N\}$ or $G_{N+1}^c \subset\subset E^c$, then $2e - g_i + g_j \geqslant 0$ a.e. in $\mathbb{R}^n$ for any $j \in \{1,\dots,N+1\}$, $j \neq i$.

Proof. Since $E_i^c \subseteq G_i^c \cup G_j^c$, the assertion follows from the relation

$$A \subseteq B \Longrightarrow \widetilde{d}_\psi(\cdot,\partial A) \geqslant \widetilde{d}_\psi(\cdot,\partial B) \text{ a.e. in } \mathbb{R}^n.$$

□

Lemma 5.4. *Given $\mathcal{A} \in \mathbb{P}_b(N+1)$, let $\mathcal{A}(\lambda)$ minimize $\mathfrak{F}(\cdot,\mathcal{A},\lambda)$ with $\boldsymbol{H} = 0$. For $i \in \{1,\dots,N+1\}$ let $E \in BV(\mathbb{R}^n;\{0,1\})$ be such that $E \subseteq A_i$; in case $i = N+1$ we assume also that E^c is bounded. Then there exists a minimizer $E_i(\lambda)$ of $\mathfrak{F}_2(\cdot,E,\lambda)$ such that $E_i(\lambda) \subseteq A_i(\lambda)$.*

Proof. First we assume that $E = A_i$. Let $E_1 \subset\subset E_2 \subset\subset \ldots \subset\subset A_i$ be sets of finite perimeter such that $A_i = \bigcup_k E_k$ and $\widetilde{d}(\cdot,\partial E_k) \to \widetilde{d}(\cdot,\partial A_i)$ a.e. as $k\to+\infty$. Let $E_k(\lambda)$ be a minimizer of $\mathfrak{F}_2(\cdot,E_k,\lambda)$. By [**18**], $E_1(\lambda) \subseteq E_2(\lambda)\ldots$ and $E(\lambda)_* := \bigcup_k E_k(\lambda)$ is the minimal minimizer of $\mathfrak{F}_2(\cdot,A_i,\lambda)$. Since $E_k \subset\subset A_i$, by lemma 5.3,

$$2\widetilde{d}(\cdot,\partial E_k) - \widetilde{d}(\cdot,\partial A_i) + \widetilde{d}(\cdot,\partial A_j) > 0 \quad \text{a.e. in } \mathbb{R}^n \text{ for all } j\neq i.$$

Thus, by theorem 5.2, $E_k(\lambda) \subseteq A_i(\lambda)$. Hence, we get $E(\lambda)_* \subseteq A_i(\lambda)$.

In the general case, we consider the minimal minimizer $E(\lambda)_*$ of $\mathfrak{F}_2(\cdot, E, \lambda)$ and the minimal minimizer $A_i(\lambda)_*$ of $\mathfrak{F}_2(\cdot, A_i, \lambda)$. Since $E \subseteq A_i$, by [**18**], $E(\lambda)_* \subseteq A_i(\lambda)_*$. Hence, $E_i(\lambda) = E(\lambda)_*$ satisfies the assertion of the lemma. $\square$

Proof of theorem 5.1. Given $\mathcal{G} \in \mathbb{P}_b(N+1)$ define $\{\mathcal{G}(\lambda, k)\}_{\lambda \geqslant 1, k \in \mathbb{N}_0}$ as follows: $\mathcal{G}(\lambda, 0) = \mathcal{G}$ and

$$\mathcal{G}(\lambda, k) \in \operatorname{argmin} \ \mathfrak{F}(\cdot, \mathcal{G}(\lambda, k-1), \lambda), \quad k \geqslant 1.$$

Note that the map $k \in \mathbb{N}_0 \mapsto \mathrm{Per}_\Phi(\mathcal{G}(\lambda, k))$ is nonincreasing. In particular,

$$\mathrm{Per}_\Phi(\mathcal{G}(\lambda, k)) \leqslant \mathrm{Per}_\Phi(\mathcal{G}). \tag{5.8}$$

For any $i \in \{1, \ldots, N+1\}$ and $k \geqslant 0$, let $\{F_i^k(\lambda, l)\}_{l \geqslant k}$ be defined as follows: $F_i^k(\lambda, k) := G_i(\lambda, k)$ and $F_i^k(\lambda, l)$ is the minimal minimizer of $\mathfrak{F}_2(\cdot, F_i^k(\lambda, l-1), \lambda)$ for $l > k$. Notice that, according to step 2 of the proof of theorem 4.1, our actual initial set $F_i^k(\lambda, k) = G_i(\lambda, k)$ satisfies the density estimates (4.9) and (4.10) for all radii $r \leqslant O(1/\lambda)$ and, according to the proof of step 4 of theorem 4.2, all $F_i^k(\lambda, l)$, $l > k$, satisfy the density estimates (4.28) and (4.29) for all radii $r \leqslant O(1/\lambda^{1/2})$. Moreover, since the initial set $F_i^k(\lambda, k)$ also depends on λ, we cannot use the arguments of the $1/(n+1)$-Hölder continuity up to time 0 in the proof of theorem 4.1.

For shortness we call $\{F_i^k(\lambda, l)\}_{l \geqslant k}$ a discrete solution starting from $F_i^k(\lambda, k) = G_i(\lambda, k)$. Applying lemma 5.4 inductively one can show that

$$F_i^k(\lambda, l) \subseteq G_i(\lambda, l), \quad l \geqslant k.$$

In particular,

$$G_i(\lambda, l) = \Big(\bigcup_{j \neq i} G_j(\lambda, l)\Big)^c = \bigcap_{j \neq i} G_j(\lambda, l)^c \subseteq \bigcap_{j \neq i} F_j^k(\lambda, l)^c.$$

Hence, using $F_i^k(\lambda, k) := G_i(\lambda, k)$ for all $i = 1, \ldots, N+1$, we get

$$G_i(\lambda, l) \setminus G_i(\lambda, k) \subseteq \Big(\bigcap_{j \neq i} F_j^k(\lambda, l)^c\Big) \cap \Big(\bigcup_{j \neq i} F_j^k(\lambda, l)\Big) \subseteq \bigcup_{j \neq i} \big(F_j^k(\lambda, k) \setminus F_j^k(\lambda, l)\big).$$

On the other hand,

$$G_i(\lambda, k) \setminus G_i(\lambda, l) = F_i^k(\lambda, k) \setminus G_i(\lambda, l) \subseteq F_i^k(\lambda, k) \setminus F_i^k(\lambda, l),$$

hence,

$$|\mathcal{G}(\lambda, k) \Delta \mathcal{G}(\lambda, l)| \leqslant \sum_{i=1}^{N+1} |F_i^k(\lambda, k) \setminus F_i^k(\lambda, l)|, \tag{5.9}$$

which is the inequality that will allow us to get the $1/2$-Hölderianity of GMM.

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Fix $i \in \{1,\dots,N+1\}$ and choose arbitrary $t > s' > s > 0$. Let $\{F_i^{[\lambda s]}(\lambda,l)\}_{l\geqslant[\lambda s]}$ be a discrete solution starting from $F_i^{[\lambda s]}(\lambda,[\lambda s]) = G_i(\lambda,[\lambda s])$. Then for any $\lambda > (5/s'-s)+(5/t-s')+(5/s)$ we have

$$|F_i^{[\lambda s]}(\lambda,[\lambda s]))\setminus F_i^{[\lambda s]}(\lambda,[\lambda t])| \leqslant \sum_{l=[\lambda s]+1}^{[\lambda s']} |F_i^{[\lambda s]}(\lambda,l)\Delta F_i^{[\lambda s]}(\lambda,l-1)| \\ + \sum_{l=[\lambda s']+1}^{[\lambda t]} |F_i^{[\lambda s]}(\lambda,l)\Delta F_i^{[\lambda s]}(\lambda,l-1)| =: I_1 + I_2. \tag{5.10}$$

Note that by the choice of λ, we have $[\lambda s'] - [\lambda s] \geqslant 4$, $[\lambda t] - [\lambda s'] \geqslant 4$ and $[\lambda s] \geqslant 4$. According to step 4 of the proof of theorem 4.2, $F_i^{[\lambda s]}(\lambda,l)$, $l \geqslant [\lambda s] \geqslant 4$ satisfies the uniform lower perimeter density estimate

$$C_4 \leqslant \frac{P(F_i^{[\lambda s]}(\lambda,l),B_r(x))}{r^{n-1}}, \quad x \in \partial F_i^{[\lambda s]}(\lambda,l), \quad r \in (0, C_3\lambda^{-1/2}),$$

provided $\lambda > C_5$. Hence, from (4.37),

$$I_2 \leqslant \left(C_6|t-s'|^{1/2} + \frac{C_6 - 1/2c_\Psi}{\lambda|t-s'|^{1/2}} \right) P_\phi(G_i(\lambda,[\lambda s])).$$

Since $\mathcal{G}(\lambda,[\lambda s])$ minimizes $\mathfrak{F}(\cdot,\mathcal{G}(\lambda,[\lambda s]-1),\lambda)$, by step 3 of the proof of theorem 4.1, see in particular (4.8) and (4.10),

$$c^\Phi(N,n) \leqslant \frac{P(G_i(\lambda,[\lambda s]),B_r(x))}{r^{n-1}}, \quad x \in \partial G_i(\lambda,[\lambda s]), \quad r \in \left(0, \frac{C(n,N,p,\Phi,\Psi)}{\lambda}\right). \tag{5.11}$$

Because of the presence of $1/\lambda$ (instead of $1/\lambda^{1/2}$) in (5.11), in general we cannot use (4.37). To estimate I_1 we proceed as in the proof of (4.15) and get

$$I_1 \leqslant \left(\mathrm{C}\,|s'-s|^{1/n+1} + \frac{\widetilde{\mathrm{C}}}{\lambda|s'-s|^{n/n+1}} \right) P_\phi(G_i(\lambda,[\lambda s])).$$

From the estimates for I_1 and I_2, and (5.8), (5.9) and (5.10) we obtain

$$\begin{aligned} &|\mathcal{G}(\lambda,[\lambda s])\Delta\mathcal{G}(\lambda,[\lambda t])| \\ &\quad\leqslant \left(\mathrm{C}\,|s'-s|^{1/n+1} + \frac{\widetilde{\mathrm{C}}}{\lambda|s'-s|^{n/n+1}} \right) \mathrm{Per}_\Phi(\mathcal{G}(\lambda,[\lambda s])) \\ &\qquad + \left(C_6|t-s'|^{1/2} + \frac{C_6 - 1/2c_\Psi}{\lambda|t-s'|^{1/2}} \right) \mathrm{Per}_\Phi(\mathcal{G}(\lambda,[\lambda s])) \\ &\quad\leqslant \left(C\,|s'-s|^{1/n+1} + \frac{\widetilde{\mathrm{C}}}{\lambda|s'-s|^{n/n+1}} + C_6|t-s'|^{1/2} + \frac{C_6 - 1/2c_\Psi}{\lambda|t-s'|^{1/2}} \right) \mathrm{Per}_\Phi(\mathcal{G}). \end{aligned} \tag{5.12}$$

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Now if $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$, there exists $\lambda_k \to +\infty$ for which

$$\lim_{k\to\infty} |\mathcal{G}(\lambda_k, [\lambda_k t])\Delta\mathcal{M}(t)| = 0 \quad \text{for any } t \geqslant 0.$$

Thus, from (5.12) we get

$$|\mathcal{M}(s)\Delta\mathcal{M}(t)| \leqslant (\mathrm{C}\,|s' - s|^{1/n+1} + C_6|t - s'|^{1/2})\mathrm{Per}_\Phi(\mathcal{G}).$$

Since $s' \in (s, t)$ is arbitrary, letting $s' \searrow s$ we get (5.1) with $C := C_6$.

Finally, if $\sum_{j=1}^{N+1} |\overline{G_j} \setminus G_j| = 0$, then for any $t > s > 0$ and $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ we have

$$|\mathcal{M}(t)\Delta\mathcal{G}| \leqslant |\mathcal{M}(t)\Delta\mathcal{M}(s)| + |\mathcal{M}(s)\Delta\mathcal{G}| \leqslant C_6\mathrm{Per}_\Phi(\mathcal{G})|t - s|^{1/2} + \mathrm{C}s^{1/n+1},$$

where in the second inequality we used (5.1) and (4.5). Now letting $s \searrow 0$ we get

$$|\mathcal{M}(t)\Delta\mathcal{G}| \leqslant C_6\mathrm{Per}_\Phi(\mathcal{G})\, t^{1/2}.$$

□

From lemma 5.4 we get the following weak comparison property of GMM.

THEOREM 5.5 *Comparison. Let* $\Phi = \{\phi, \dots, \phi\}$ *and* $\Psi = \{\psi, \dots, \psi\}$ *for some norms* ϕ *and* ψ *on* $\mathbb{R}^n$, *and* $\boldsymbol{H} \equiv 0$. *Given* $\mathcal{G} \in \mathbb{P}_b(N+1)$, *let* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ *and given* $i \in \{1, \dots, N+1\}$, *let* $C \in BV(\mathbb{R}^n; \{0,1\})$ *be such that* $C \subseteq G_i$; *in case* $i = N+1$ *we assume also that* C^c *is bounded. Then there exists* $N \in GMM(\mathfrak{F}_2, C)$ *such that* $N(t) \subseteq M_i(t)$ *for all* $t \geqslant 0$.

Proof. Let $\lambda_h \to +\infty$ be such that

$$\lim_{h\to\infty} |\mathcal{G}(\lambda_h, [\lambda_h t])\Delta\mathcal{M}(t)| = 0 \quad \text{for all } t \geqslant 0, \tag{5.13}$$

where for any h the sequence $\{\mathcal{G}(\lambda_h, k)\}_{k\in\mathbb{N}_0}$ is defined as: $\mathcal{G}(\lambda_h, 0) = \mathcal{G}$ and

$$\mathcal{G}(\lambda_h, k) \in \mathrm{argmin}\ \mathfrak{F}(\cdot, \mathcal{G}(\lambda_h, k-1), \lambda_h), \quad k \geqslant 1.$$

Let $i \in \{1, \dots, N+1\}$ and $C \in BV(\mathbb{R}^n; \{0,1\})$ be as in the statement. For any h let $\{G(\lambda_h, k)\}_{k\in\mathbb{N}_0}$ be defined as $G(\lambda_h, 0) = C$ and $G(\lambda_h, k)$ is the minimal minimizer of $\mathfrak{F}_2(\cdot, G(\lambda_h, k-1), \lambda)$, $k \geqslant 1$ (see the proof of lemma 5.4 for the definition). Applying lemma 5.4 inductively we get

$$G(\lambda_h, k) \subseteq G_i(\lambda_h, k) \quad \text{for all } h \geqslant 1 \text{ and } k \geqslant 0. \tag{5.14}$$

Passing to a further (not relabelled) subsequence if necessary, we assume that there exists $N \in GMM(\mathfrak{F}_2, C)$ such that

$$\lim_{h\to\infty} |G(\lambda_h, [\lambda_h t])\Delta N(t)| = 0 \quad \text{for all } t \geqslant 0. \tag{5.15}$$

By (5.13) we have

$$\lim_{h\to\infty} |G_i(\lambda_h, [\lambda_h t])\Delta M_i(t)| = 0 \qquad \text{for all } t \geqslant 0.$$

Now (5.14) and (5.15) imply that $N(t) \subseteq M_i(t)$ for all $t \geqslant 0$ up to a negligible set. □

COROLLARY 5.6. *Under the assumptions of theorem 5.5 let* $\mathcal{G} \in \mathbb{P}_b(N+1)$ *and* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$. *Let* $C_i \subseteq G_i$, $i \in \{1, \ldots, N\}$, *and* $C_{N+1}^c \supseteq \mathrm{co}(\mathcal{G})$ *be convex sets and let* $L_i \in GMM(\mathfrak{F}_2, C_i)$, $i \in \{1, \ldots, N+1\}$. *Then for any* $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$

$$L_i(t) \neq \emptyset \quad \Longrightarrow \quad M_i(t) \neq \emptyset, \quad i = 1, \ldots, N, \tag{5.16}$$

and

$$L_{N+1}(t) = \emptyset \quad \Longrightarrow \quad M_{N+1}(t) = \emptyset. \tag{5.17}$$

Proof. Recall that anisotropic mean curvature flow with a mobility starting from a bounded convex set C is uniquely defined [**10**], coincides with the GMM starting from C and becomes extinct at a finite time $t_C > 0$. By theorem 5.5, the i-th phase M_i of any $\mathcal{M} \in GMM(\mathfrak{F}, \mathcal{G})$ starting from the i-th phase G_i of $\mathcal{G}$ does not disappear in the time-interval $(0, t_{C_i})$ for any $i \in \{1, \ldots, N\}$. Analogously, theorem 5.5 implies that $(N+1)$-th phase of $\mathcal{M}$ becomes empty, i.e., $\mathbb{R}^n \setminus M_{N+1}(t) = \emptyset$ if $t \geqslant t_{C_{N+1}}$. ☐

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