NOTE ON A PAIR OF DUAL TRIGONOMETRIC SERIES

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1. Introduction. It is the purpose of this note to discuss the solution of the pair of series

$$\sum_1^\infty (n-\tfrac{1}{2})a_n \sin(n-\tfrac{1}{2})x = F(x) \qquad (0 < x < c), \tag{1}$$

$$\sum_1^\infty a_n \sin(n-\tfrac{1}{2})x = G(x) \qquad (c < x < \pi), \tag{2}$$

where $F(x)$ and $G(x)$ are given and the coefficients a_n are to be determined.

A pair into which these two can be transformed was originally solved by Tranter [2, 3]. One of the difficulties in his solution is the need to find a certain constant by summing a series. This may be troublesome. Tranter [4] later gave a greatly simplified solution, but there is still the need to find a constant by the same technique.

Srivastav [1] gave an alternative solution which avoided this difficulty, but this was partly in error. This solution is reproduced in Sneddon's recent book [5] and in view of the wider circulation which will now be given to the work of Srivastav it would seem to be desirable to point out the error and to see how it can be circumvented.

2. Srivastav's solution. The solution is divided into two parts, in the first of which [problem (*a*)] $G(x)$ is put equal to zero, and in the second [problem (*b*)] $F(x)$ is put equal to zero. Then the complete solution is the sum of the two thus obtained.

According to Srivastav [1], problem (*a*) has the solution

$$a_1 = \frac{1}{\sqrt{2}}\int_0^c h_1(t)\{1-P_1(\cos t)\}\,dt, \tag{3}$$

$$a_n = \frac{1}{\sqrt{2}}\int_0^c h_1(t)\{P_{n-2}(\cos t)-P_n(\cos t)\}\,dt \qquad (n = 2, 3, \ldots), \tag{4}$$

where

$$h_1(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t \frac{dx}{\sqrt{(\cos x-\cos t)}}\int_0^x F(u)\,du - \frac{\sqrt{2}\,.\,y_1}{\pi}\frac{d}{dt}\int_0^t \frac{1-\cos\frac{1}{2}x}{\sqrt{(\cos x-\cos t)}}\,dx \tag{5}$$

and

$$y_1 = \int_0^c h_1(u)\,du. \tag{6}$$

The solution of problem (*b*) given by Srivastav is

$$a_1 = \frac{1}{\sqrt{2}}\int_c^\pi h_2(t)\{1+P_1(\cos t)\}\,dt, \tag{7}$$

$$a_n = \frac{1}{\sqrt{2}} \int_c^{\pi} h_2(t) \{P_n(\cos t) - P_{n-2}(\cos t)\} \, dt \qquad (n = 2, 3, \ldots), \tag{8}$$

where

$$h_2(t) = \frac{1}{\pi} \frac{d}{dt} \int_t^{\pi} \frac{G(x) \, dx}{\sqrt{(\cos t - \cos x)}}. \tag{9}$$

This is part of the solution which is in error in general. $h_2(t)$ was found by solving the equation

$$\int_x^{\pi} \frac{h_2(t) \, dt}{\sqrt{(\cos x - \cos t)}} = \operatorname{cosec} x \, \{\sqrt{2} \, . \, y_2 \sin \tfrac{1}{2}x - G(x)\}, \tag{10}$$

where

$$y_2 = \int_c^{\pi} h_2(u) \, du. \tag{11}$$

However, $h_2(t)$ as found by (9) is not in general the solution of (10). Suppose for instance that $G(x) = \sin x$. Then (9) gives

$$h_2(t) = -\frac{\sqrt{2}}{\pi} \sin \tfrac{1}{2}t.$$

Substituting this in (10) we find that

$$-1 = \frac{2 \cos \tfrac{1}{2}c}{\pi \cos \tfrac{1}{2}x} - 1,$$

which is manifestly untrue.

It is the right-hand side of equation (10) which causes the trouble. It will be noticed that in general this right-hand side tends to infinity as x tends to π. This cannot be permitted, and we must arrange matters so that the term in brackets on the right-hand side tends to zero as x tends to π.

A line of approach which suggests itself, using generalised function theory, is to write (10) as

$$\int_x^{\pi} \frac{h_2(t) \, dt}{\sqrt{(\cos x - \cos t)}} = \operatorname{cosec} x \, \{\sin \tfrac{1}{2}x G(\pi) - G(x)\} + \operatorname{cosec} x \, \{\sqrt{2} \, . \, y_2 - G(\pi)\} \sin \tfrac{1}{2}x.$$

The first part of the right-hand side is of the required form and the solution of this part can be written down as in (9). The other part may be written, on assuming that $h_2(t) = 0$ for $t > \pi$,

$$\int_{-\infty}^{\infty} \frac{h_2(t) H(t-x) \, dt}{\sqrt{(\cos x - \cos t)}} = \operatorname{cosec} x \, \{\sqrt{2} \, . \, y_2 - G(\pi)\} \sin \tfrac{1}{2}x H(\pi - x)$$

and a solution of this part, as may be directly verified, is

$$h_2(t) = \left\{ y_2 - \frac{G(\pi)}{\sqrt{2}} \right\} \delta(\pi - t).$$

Here H is Heaviside's unit function and δ is Dirac's delta function. However no further

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progress seems possible on these lines. We therefore complete the solution by adding an extra term to each series in the manner now to be described.

3. The corrected solution. We write (1) and (2) in the forms

$$\tfrac{1}{2}(\sqrt{2}\,.\,y_3+a_1)\sin\tfrac{1}{2}x+\sum_{2}^{\infty}(n-\tfrac{1}{2})a_n\sin(n-\tfrac{1}{2})x=\frac{1}{\sqrt{2}}y_3\sin\tfrac{1}{2}x+F(x), \tag{12}$$

$$(\sqrt{2}\,.\,y_3+a_1)\sin\tfrac{1}{2}x+\sum_{2}^{\infty}a_n\sin(n-\tfrac{1}{2})x=\sqrt{2}\,.\,y_3\sin\tfrac{1}{2}x+G(x), \tag{13}$$

where y_3 is an unknown constant to be determined.

The value of $h_1(t)$ for problem (*a*) can be written down as equation (5). For problem (*b*) equation (10) now becomes

$$\int_x^{\pi}\frac{h_2(t)\,dt}{\sqrt{(\cos x-\cos t)}}=\operatorname{cosec}x\,\{\sqrt{2}\,.\,y_2\sin\tfrac{1}{2}x-G(x)-\sqrt{2}\,.\,y_3\sin\tfrac{1}{2}x\}. \tag{14}$$

All that we now need to do is to choose y_3 so that the expression in brackets on the right-hand side tends to zero as x tends to π. Hence we must have

$$\sqrt{2}\,.\,y_2-G(\pi)-\sqrt{2}\,.\,y_3=0. \tag{15}$$

The solution of (14) is now given correctly by (9). In addition we have three equations, namely (6), (11) and (15) connecting y_1, y_2 and y_3 and so they may be determined and the solution completed.

4. An example. Tranter [2] solved equations which can be transformed into (1) and (2) with $F(x)=0$ and $G(x)=1$. His solution when transformed is

$$a_n=\frac{2}{2n-1}\,\frac{P_{n-1}(\cos c)}{K(\sin\tfrac{1}{2}c)}, \tag{16}$$

where K is the complete elliptic integral of the first kind. We shall use this as a check on the present solution.

Equation (5) now gives by (A2) (see Appendix)

$$h_1(t)=\frac{2}{\pi}(y_3-y_1)\frac{d}{dt}K(\sin\tfrac{1}{2}t), \tag{17}$$

whilst by (6)

$$y_1=\frac{2}{\pi}(y_3-y_1)(K-\tfrac{1}{2}\pi), \tag{18}$$

where we write

$$K=K(\sin\tfrac{1}{2}c).$$

We shall also write for brevity

$$K(\sin\tfrac{1}{2}t)=K_t,\quad K(\cos\tfrac{1}{2}t)=K_t',\quad K(\cos\tfrac{1}{2}c)=K'.$$

The solution of problem (*b*), provided that (15) holds, is, by (9) and (A1),

$$h_2(t) = \frac{\sqrt{2}}{\pi}\frac{dK_t'}{dt} \tag{19}$$

and (11) gives

$$y_2 = \frac{\sqrt{2}}{\pi}(\tfrac{1}{2}\pi - K'). \tag{20}$$

y_1, y_2 and y_3 are found from (15), (18) and (20), and have values

$$y_1 = \frac{\sqrt{2}\,.\,K'(\tfrac{1}{2}\pi - K)}{\pi K}, \quad y_2 = \frac{\sqrt{2}}{\pi}(\tfrac{1}{2}\pi - K'), \quad y_3 = -\frac{\sqrt{2}\,.\,K'}{\pi}. \tag{21}$$

Hence we find that

$$h_1 = -\frac{\sqrt{2}}{\pi}\frac{K'}{\pi}\frac{dK_t}{dt}, \tag{22}$$

$$h_2 = \frac{\sqrt{2}}{\pi}\frac{dK_t'}{dt} \tag{23}$$

and the coefficients a_n will be the sum of the right-hand sides of (4) and (8).

This gives the result (16) but the analysis is tedious. It will be found in the appendix.

5. Conclusion. The aim here has been to correct the solution of Srivastav as quoted by Sneddon [5]. As to which method is used in any given case, it depends considerably on the problem. One can only say that it is convenient to have three methods at one's disposal. One can then choose the particular one of these which gives the easiest analysis.

APPENDIX

We quote first a few results which will be of use in the sequel:

$$\int_t^\pi \frac{dx}{\sqrt{(\cos t - \cos x)}} = \sqrt{2}\,.\,K_t', \tag{A1}$$

$$\int_0^t \frac{dx}{\sqrt{(\cos x - \cos t)}} = \sqrt{2}\,.\,K_t. \tag{A2}$$

We also have, with $k = \sin\frac{1}{2}c$, $k' = \cos\frac{1}{2}c$,

$$K_t = \tfrac{1}{2}\pi P_{-\frac{1}{2}}(\cos t), \tag{A3}$$

by the equation following 3.14 (5) in Erdélyi [6]. Standard relations in the usual notation for elliptic integrals are

$$Bk^2 = E - k'^2K, \quad B'k^2 = E' - k^2K', \tag{A4}$$

$$KE' + K'E - KK' = \tfrac{1}{2}\pi. \tag{A5}$$

In addition, by (A3) and Erdélyi [6, equation 13.8 (27)], we have

$$\frac{d}{d(\cos c)}P_{-\frac{1}{2}}(\cos c)=\frac{2}{\pi}\frac{dK}{d(\cos c)}=-\frac{B}{2\pi k'^2}. \tag{A6}$$

We also have

$$P_{n-1}(-x)=(-1)^{n-1}P_{n-1}(x), \tag{A7}$$

$$P_n'(x)-P_{n-2}'(x)=(2n-1)P_{n-1}(x), \tag{A8}$$

$$\int P_{-\frac{1}{2}}(x)P_{n-1}(x)\,dx=\frac{4(1-x^2)}{(2n-1)^2}\{P_{n-1}(x)P_{-\frac{1}{2}}'(x)-P_{-\frac{1}{2}}(x)P_{n-1}'(x)\}. \tag{A9}$$

The last equation comes from Erdélyi [6, equations 3.12 (3) and 3.6.1 (6)]. Consider the integral

$$I=\int_0^c \frac{dK_t}{dt}\{P_n(\cos t)-P_{n-2}(\cos t)\}\,dt.$$

We have, integrating by parts, and then using (A8),

$$I=K\{P_n(\cos c)-P_{n-2}(\cos c)\}+(2n-1)J,$$

where

$$J=\int_0^c K_t P_{n-1}(\cos t)\sin t\,dt=-\tfrac{1}{2}\pi\int_1^{\cos c} P_{-\frac{1}{2}}(x)P_{n-1}(x)\,dx$$

by (A3). Hence

$$J=\frac{4\sin^2 c}{(2n-1)^2}\left\{\frac{B}{2k'^2}P_{n-1}(\cos c)-K\frac{d}{d(\cos c)}P_{n-1}(\cos c)\right\},$$

by (A9) and (A6). Similarly

$$I'=\int_c^\pi \frac{dK_t'}{dt}\{P_n(\cos t)-P_{n-2}(\cos t)\}\,dt=-K'\{P_n(\cos c)-P_{n-2}(\cos c)\}-(2n-1)J',$$

where

$$J'=-\tfrac{1}{2}\pi(-1)^{n-1}\int_{-1}^{-\cos c} P_{-\frac{1}{2}}(x)P_{n-1}(x)\,dx$$

and so J' is the same as J with k', K and B replaced by k, K' and B' and $d/d(\cos c)$ changed to $d/d(-\cos c)$, whilst $P_{n-1}(\cos c)$ is not changed.

Now, in the problem under consideration, for $n \geqq 2$ the coefficient a_n is the sum of the right-hand sides of (4) and (8) with h_1 and h_2 given by (22) and (23). We find that

$$a_n=\frac{\sin^2 c}{(2n-1)K\pi}P_{n-1}(\cos c)\left\{\frac{K'B}{k^2}+\frac{KB'}{k'^2}\right\}.$$

The other terms cancel. Hence finally, using (A4) and (A5), we have

$$a_n=\frac{2}{2n-1}\frac{P_{n-1}(\cos c)}{K(\sin\frac{1}{2}c)} \qquad (n \geqq 2).$$

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A separate and simpler computation for the modified a_1 using (3) and (7) gives

$$\sqrt{2}\,.\,y_3+a_1=\frac{1}{\pi}\left[\frac{2\pi}{K}-2K'\right],$$

and so, by (21),

$$a_1=\frac{2}{K}.$$

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