STRING VIBRATING AT FINITE AMPLITUDE

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Summary

Equations of motion of a vibrating string are established in terms of the transverse and longitudinal displacements. These equations contain the terms of lowest order which are neglected in the classical treatment with vanishing amplitude. These extra terms lead to the natural modes being dependent on amplitude. By a simple procedure a solution of these equations is obtained which separates, as in the classical theory. The familiar circular functions are replaced by a Mathiew Function of position and a Jacobi elliptic function of time. Agreement with a previous study is shown.

1. Introduction

This is a study of the plane motion of a length of stretched string, the end of which are fixed in the plane of motion.

A consistent account is taken of the lowest order terms which involve the dependence of the modes of oscillation on the amplitude. These terms are the largest terms neglected in the classical treatment when the assumption is made that the amplitude is small.

The string is stretched and is assumed to be elastic. Since its ends are fixed, in order to bow, the string must stretch further. In so doing it must experience an increase in tension. This increase in tension varies with both position and time since the longitudinal motion is coupled to the transverse motion. Thus we must take account of both the longitudinal displacement and the transverse displacement, and introduce a parameter to describe the increase of tension with length of the string.

Carrier [1, 2] has studied the motion of a string vibrating at finite amplitude. Some aspects of his work are considered in the discussion.

The main result of this investigation is a demonstration that the displacements may be separated into products of functions of position and of time, with amplitude as a further parameter in each case.

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2. Analysis

2.1. Preamble

This analysis falls into three steps. Firstly we define dimensionless variables and in terms of these we establish the equations of motion. These equations are partial differential equations in the transverse displacement (y) and the longitudinal displacement (u), which are functions of time (τ) and the position (x) along the string, measured in the quiescent state.

These equations ((5) and (6)) contain order terms in powers of the amplitude (A) of the transverse displacement. These terms correspond to the lowest order terms neglected when the new equations of motion are deduced. These equations also contain a parameter e which is the strain in the string in its quiescent stretched state referred to its unstretched length.

Secondly we proceed to examine the equations of motion and substitute in the higher order terms approximations based on the form of the solution sought at vanishingly small amplitude. For example, we use in a modified form of equation (5) the approximation

$$\frac{\partial}{\partial x}(y_x \cos 2x) = y_{\tau\tau}(2+3\cos 2x)$$

which is exact if $y = A \sin x \cos \tau$. We substitute thus for the left side in a term in equation (5) which has been shown to be $O(A^3)$ when terms $O(A^5)$ are being neglected.

This procedure is directed at and eventually leads to a reduction of equation (5) to the form

$$G(x)y_{\tau\tau} = F(\tau)y_{xx}+O(A^5).$$

If terms $O(A^5)$ are neglected, the solution of this equation separates. That is

$$y = \theta(x)\phi(\tau),$$

where $\theta(x)$ and $\phi(\tau)$ are solutions of the differential equations

$$\theta_{xx}+K^2G(x)\theta = 0,$$
$$\phi_{\tau\tau}+K^2F(\tau)\phi = 0.$$

Here, K^2 is the separation constant.

Thirdly we complete the solution of these two differential equations. The solution of the linear differential equation in the position function, $\theta(x)$, which satisfies the boundary condition is a periodic Mathiew Function.

$$\theta(x) = se_1(x) = \sin x - \frac{3A^2}{128(1+e)}\sin 3x+O(A^4).$$

The separation constant K^2 is also determined in the process of ensuring that the boundary conditions

$$y(0, \tau) = y(\pi, \tau) = 0 \text{ for all } \tau$$

are satisfied.

The solution of this equation in $\theta(x)$ is straight forward as the function $G(x)$ is determined in the course of the analysis in the second step. However, the function $F(\tau)$ is not determined, but contains a function of time, $f(\tau)$ which arises in an indefinite integration with respect to x of a modified form of equation (6). This unknown function is found by substituting $y = se_1(x)\phi(\tau)$ in a modified form of equation (6).

The non-linear differential equation in $\phi(\tau)$ can then be solved. The solution satisfying the boundary conditions

$$\phi(0) = A, \quad \frac{\partial \phi}{\partial \tau} = 0 \text{ at } \tau = 0$$

is a Jacobi elliptic function

$$\phi(\tau) = A \operatorname{cn} \lambda\tau.$$

This elliptic function is expanded into the leading terms of its Fourier representation to obtain the complete solution

$$\begin{aligned} y &= A\left[\sin x - \frac{3A^2}{128(1+e)} \sin 3x\right] \\ &\quad \times \left[\left(1 - \frac{A^2}{128e}\right) \cos w\tau + \frac{A^2}{128e} \cos 3\, w\tau + O(A^5)\right] \\ u &= -\frac{A^2}{16} \sin 2x \left(\frac{1}{1+e} + \cos 2w\tau\right) + O(A^4), \end{aligned}$$

where

$$w = 1 + \frac{3A^2}{32e} + \frac{A^2}{32(1+e)} + O(A^4).$$

2.2. Procedure

(i) Equations of motion

If we are to take into account the lowest order terms neglected in the classical treatment of the vibrating string two additional features must be recognised. Firstly we must allow for the variation of the tension in the string, with both time and position. Secondly we must consider the longitudinal displacement.

Thus the equations of motion which express the inertial equilibrium of the element of string which originates from the element dX in the quiescent state are (see Carrier [1, 2])

$$\frac{\partial}{\partial x}(T \sin \theta) = \rho \frac{\partial^2 Y}{\partial t^2} \tag{1}$$

and

$$\frac{\partial}{\partial x}(T \cos \theta) = \rho \frac{\partial^2 U}{\partial t^2}, \tag{2}$$

where $T = T(x, t)$ is the tension in the element from dX at time t. The transverse displacement of the element is $Y = Y(X, t)$ and its longitudinal displacement is $U = U(X, t)$. With ρ the density of the stretched string in its quiescent state, the mass of the element is ρdX. θ is the inclination of the element, and it can be shown that

$$\tan \theta = \frac{\partial Y}{\partial X} \Big/ \left(1 + \frac{\partial U}{\partial X}\right).$$

The length of the element is SdX, where

$$S^2 = \left(1 + \frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial Y}{\partial X}\right)^2. \tag{3}$$

In its quiescent state the string is stretched to tension T_0, and to length l between $X = 0$ and $X = l$. The resulting strain is e, referred to the unstretched length. If the string is elastic, and obeys Hooke's law, then the tension in the element dX is given by

$$T(X, t) = T_0 \left[1 + \frac{1+e}{e}(S-1)\right]. \tag{4}$$

We now introduce dimensionless parameters to simplify subsequent manipulation (see Figure 1).

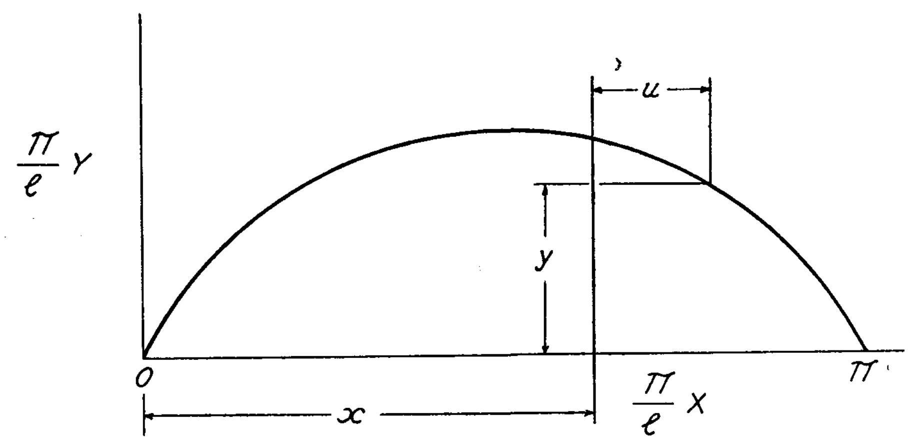

Fig. 1. Transverse and longitudinal displacements of the string in an intermediate bowed position.

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For time: $\tau = \pi/l\sqrt{T_0/\rho}$; and for position, displacements: $(x; y, u) = \pi/l$ $(X; Y, U)$.

Using these parameters in the equations of motion, and substituting for T by equation (4) with S given by equation (3), we find on performing the differentiations with respect to x and collecting terms to the order terms indicated

$$y_{\tau\tau} = y_{xx} + \frac{1}{e}\frac{\partial}{\partial x}[y_x(u_x+\tfrac{1}{2}y_x^2)]+O(A^5), \tag{5}$$

and

$$eu_{\tau\tau}-(1+e)u_{xx} = y_x y_{xx}+O(A^4). \tag{6}$$

Here we have assumed that $u = O(A^2)$. This is an a posteriori assumption though there is some justification for making it at this stage. There is a simple solution of equations (5) and (6) which satisfies the boundary conditions

$$u(o, \tau) = u(\pi, \tau) = y(o, \tau) = y(\pi, \tau) = 0,$$

namely $y = 0$

$$u = u_L = \sum B_n \sin nx \cos\left(n\sqrt{\frac{e}{1+e}}\cdot\tau+\varepsilon_n\right),$$

the natural longitudinal oscillations.

We assume that any such longitudinal vibration is damped out, and will subsequently be neglected. The the "particular integral" of equation (6) would be such that $u = O(A^2)$ where A is the amplitude of the dimensionless transverse displacement y.

(ii) Reduction of equation (5) to the form

$$G(x)y_{\tau\tau} = F(\tau)y_{xx}.$$

To solve equations (5) and (6) for the transverse mode we first eliminate u from equation (5) by using an integrated form of equation (6).

If $y = A \sin x \cos \tau$, the solution of equation (5) for $A \to 0$ in the form we seek, then we can solve equation (6) to obtain

$$u = -\frac{A^2}{16}\sin 2x\left(\frac{1}{1+e}+\cos 2\tau\right)+u_L.$$

Assuming the natural longitudinal motion to be damped out, $u_L = 0$, and

$$e(u_{\tau\tau}-u_{xx}) = -\frac{A^2}{4}\cdot\frac{e}{1+e}\cdot\sin 2x. \tag{7}$$

We find a posteriori that this equation is exact if an order term $O(A^4)$

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is added to the right side and so it can be substituted into equation (6) in the form

$$\text{(6)} \qquad \frac{\partial}{\partial x}\left(u_x+\tfrac{1}{2}y_x^2\right) = e(u_{\tau\tau}-u_{xx})$$

$$\text{(8)} \qquad = -\frac{A^2}{4}\cdot\frac{e}{1+e}\cdot\sin 2x+O(A^4).$$

We integrate with respect to x to obtain

$$\text{(9)} \qquad u_x+\tfrac{1}{2}y_x^2 = \frac{A^2}{8}\cdot\frac{e}{1+e}\cdot\cos 2x+f(\tau)+O(A^4),$$

where $f(\tau)$ is an as yet undetermined function of τ arising in this integration.

Substituting thus for $u_x+\frac{1}{2}y_x^2$ in equation (5), we obtain

$$\text{(10)} \qquad y_{\tau\tau} = y_{xx}\left\{1+\frac{1}{e}f(\tau)\right\}+\frac{A^2}{8(1+e)}\frac{\partial}{\partial x}(y_x\cos 2x).$$

Next this equation is reduced to a form which has a separated solution. Now, if $y = A\sin x\cos\tau$,

$$\text{(11)} \qquad \frac{A^2}{8(1+e)}\frac{\partial}{\partial x}(y_x\cos 2x) = \frac{A^2}{8(1+e)}y_{\tau\tau}(2+3\cos 2x).$$

It is found a posteriori that this equation is exact if an order term $O(A^5)$ is added to the right side. Consequently we can substitute equation (11) into equation (10) to obtain

$$\text{(12)} \qquad y_{\tau\tau}\left\{1-\frac{A^2}{4(1+e)}-\frac{3}{16(1+e)}\cdot 2\cos 2x\right\} = y_{xx}\left\{1+\frac{1}{e}f(\tau)\right\}+O(A^5).$$

If the order term is neglected, the solution of this partial differential equation separates, giving

$$\text{(13)} \qquad y = \theta(x)\phi(\tau)$$

where

$$\text{(14)} \qquad \frac{d^2\theta}{dx^2}+K^2\left[1-\frac{A^2}{4(1+e)}-\frac{3A^2}{16(1+e)}\cdot 2\cos 2x\right] = 0$$

and

$$\text{(15)} \qquad \frac{d^2\phi}{d\tau^2}+K^2\phi\left[1+\frac{1}{e}f(\tau)\right] = 0.$$

Here K^2 is the separation constant.

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(iii) Solution of differential equations in $\theta(x)$ and $\phi(\tau)$. Equation (14) is the Mathiew equation

$$\frac{d^2\theta}{dx^2}+\theta(a-2q\cos 2x)=0,$$

with

$$a=K^2\left[1-\frac{A^2}{4(1+e)}\right],$$

$$q=\frac{K^2\cdot 3A^2}{16(1+e)}.$$

The solution satisfying the boundary conditions $y=0$ when $x=0,\ \pi$ is [1] the periodic Mathiew function (of integral order)

(16) $$\theta=se_1(x,q)=\sin x-\frac{q}{8}\sin 3x+O(q^2)$$

obtaining when $a=1-q+O(q^2)$.

This condition determines the separation constant K^2. Thus

(17) $$K^2=1+\frac{A^2}{16(1+e)}+O(A^4),$$

whence

$$q=\frac{3A^2}{16(1+e)}+O(A^4)$$

and

(18) $$\theta(x)=\sin x-\frac{A^2}{128(1+e)}\sin 3x+O(A^4).$$

We can now determine the unknown function $f(\tau)$ in equation (12). Using equations (13), (18) to substitute for y in equation (8), and noting that $\phi(\tau)=O(A)$,

$$u_{xx}=\tfrac{1}{2}\sin 2x\phi^2(\tau)-\frac{A^2}{4}\cdot\frac{e}{1+e}\cdot\sin 2x+O(A^4).$$

On integrating with respect to x

(19) $$u_x=-\tfrac{1}{4}\cos 2x\phi^2(\tau)+\frac{A^2}{8}\cdot\frac{e}{1+e}\cdot\cos 2x+h(\tau)+O(A^4)$$

and integrating again

$$u=-\tfrac{1}{8}\sin 2x\,\phi^2(\tau)+\frac{A^2}{16}\frac{e}{1+e}\sin 2x+xh(\tau)+j(\tau)+O(A^4).$$

[1] See McLachlan [3] pages 13 and 14.

The boundary conditions

$$u(0, \tau) = u(\pi, \tau) = 0 \text{ for all } \tau \text{ give}$$
$$h(\tau) = 0, j(\tau) = 0.$$

Hence equation (19) can be written (after some manipulation)

$$u_x + \tfrac{1}{2} y_x^2 = \frac{A^2}{8} \cdot \frac{e}{1+e} \cdot \cos 2x + \tfrac{1}{4}\phi^2(\tau) + O(A^4). \tag{20}$$

Comparison of equation (20) with equation (9) gives

$$f(\tau) = \tfrac{1}{4}\phi^2(\tau). \tag{21}$$

Substituting for $f(\tau)$ using equation (21) and for K^2 using equation (17) in the differential equation for $\phi(\tau)$, equation (15), we thus obtain

$$\frac{d^2\phi}{d\tau^2} + \left(1 + \frac{A^2}{16(1+e)}\right) \phi \left(1 + \frac{1}{4e}\phi^2\right) = 0. \tag{22}$$

Now the Jacobi elliptic function $cn(u, k)$ has the property

$$\frac{d^2 cn\, u}{du^2} + cn\, u(1 - 2k^2 + 2k^2 cn^2 u) = 0.$$

(This has been obtained from Byrd and Friedman [4], page 25, equation 128.01:

$$\left[\frac{d}{du}(cn\, u)\right]^2 = (1 - cn^2 u)(1 - k^2 + k^2 cn^2 u)$$

by differentiation with respect to $cn\, u$.)

It follows that the solution of equation (22) satisfying the boundary conditions

$$\phi = A, \frac{d\phi}{dt} = 0 \text{ when } \tau = 0,$$

is

$$\phi = A\ cn\ (\lambda\tau, k)$$

where

$$k^2 = \frac{A^2/8e}{1 + A^2/4e},$$

and

$$\lambda^2 = \left(1 + \frac{A^2}{4e}\right)\left(1 + \frac{A^2}{16(1+e)}\right).$$

It is convenient to represent $cn\ (\lambda\tau, k)$ by the leading terms of its Fourier expansion.

From Byrd and Friedman [4], page 304, equation 908.02,

$$cn\, u = \frac{2q^{\frac{1}{2}}}{kK} \sum_{m=0}^{m=\infty} \frac{q^m}{1+q^{2m+1}} \cos\,(2m+1)\,\frac{\pi u}{2K},$$

where, from the same source, page 297, equation 900.00,

$$K = \frac{\pi}{2}\left[1+\frac{k^2}{4}+O(k^4)\right]$$

and from page 299, equation 901.01,

$$q = \frac{k^2}{16}\left[1+\frac{k^2}{2}+O(k^4)\right].$$

It follows that

$$\frac{2\pi q^{\frac{1}{2}}}{kK} = 1+O(k^4)$$

and

$$\frac{\pi u}{2K} = w\tau,$$

where

$$w = \frac{\pi\lambda}{2K}$$

so

$$\phi = A\left[\left(1-\frac{k^2}{16}\right)\cos w\tau + \frac{k^2}{16}\cos 3w\tau + O(k^4)\right] \tag{23}$$

where

$$w = 1+\frac{3A^2}{32e}+\frac{A^2}{32(1+e)}+O(A^4). \tag{24}$$

The complete solution can then be written

$$\begin{aligned} y = A\left\{\sin x - \frac{3A^2}{128(1+e)}\sin 3x\right\}\Bigg\{&\left(1-\frac{A^2}{128e}\right)\cos w\tau \\ &+\frac{A^2}{128e}\cos 3w\tau\Bigg\} + O(A^5) \end{aligned} \tag{25}$$

$$u = -\frac{A^2}{16}\sin 2x\left\{\frac{1}{1+e}+\cos 2w\tau\right\}+O(A^4). \tag{26}$$

3. Discussion

3.1. The nature of the separated solution

In a sense, the separation of the transverse displacement y amounts to the non-appearance in y of a term $O(A^3)$ of the form $A^3 \sin 3x \cos 3w\tau$.

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The result of this paper could thus be summarised: *There is no term of the form* $A^3 \sin 3x \cos w\tau$ *in* $y(x, \tau)$.

The separated solution

$$y = \phi(\tau, A)\theta(x, A)$$

means that, at a given amplitude, the form of the y displacement is described by $\theta(x, A)$, with a magnitude which varies periodically with time. The form of $\theta(x, A)$ changes with amplitude as does the nature of the periodic variation $\phi(\tau, A)$ of the magnitude of the transverse displacement. The departure of these functions from simple circular functions is $O(A^2)$ in the case of the position function and $O(A^2/e)$, for the time function.

3.2. The behaviour of the tension

From equations (3) and (4),

$$T = T_0\left\{1+\frac{1+e}{e}\,(u_x+\tfrac{1}{2}y_x^2)+O(A^4)\right\},$$

whence from equations (20) and (23),

$$T = T_0\left\{1+\frac{(1+e)A^2}{4e}\cos w\tau+\frac{A^2}{8}\cos 2x+O(A^4)\right\}. \tag{27}$$

Now, for most conditions met in practice the strain e is much smaller than 1 ($e \ll 1$), and terms $O(A^2)$ would be negligible compared with terms $O(A^2/e)$. If we ignore terms $O(A^2)$ but account for terms $O(A^2/e)$,

$$T' = T_0\left\{1+\left(\frac{1+e}{e}\right)\cdot\frac{A^2}{8}+\frac{1+e}{e}\cdot\frac{A^2}{8}\cos 2w\tau\right\} \tag{28}$$

where the prime indicates the less precise analysis. Then T' is independent of position but varies periodically in phase with the bowing of the string away from its quiescent position about an elevated mean value,

$$\overline{T'} = T_0\left\{1+\frac{1+e}{e}\cdot\frac{A^2}{8}\right\}.$$

3.3. A simpler solution which collects terms $O(A^2/e)$

If we concern ourselves only with terms $O(A^2/e)$ but choose to ignore terms $O(A^2)$, equation (28) shows that the tension is a function only of time. So we can write the equation of transverse motion

$$y'_{\tau\tau} = \frac{T'}{T_0}\,y'_{xx}. \tag{29}$$

Now let

(30) $$T' = T_0\left\{1+\frac{1+e}{e}\cdot\frac{\Delta l}{l}\right\}$$

where Δl is the increase in the length of the string over l at time τ.

If $y' = A \sin x \cos w\tau$, where w is as yet unknown,

$$\Delta l = \frac{l}{\pi}\int_0^{\pi}\{(1+y_x^2)^{\frac{1}{2}}-1\}dx$$
$$\doteqdot \frac{lA^2}{8}(1+\cos 2w\tau),$$

and

(31) $$\frac{T'}{T_0} = 1+\frac{1+e}{e}\cdot\frac{A^2}{8}+\frac{1+e}{e}\cdot\frac{A^2}{16}\cdot 2\cos w\tau.$$

Thus the equation of motion, equation (29), becomes

$$y'_{\tau\tau} = y'_{xx}\left\{1+\frac{1+e}{e}\cdot\frac{A^2}{8}+\frac{1+e}{e}\frac{A^2}{16}\cdot 2\cos 2w\tau\right\}$$

which has the separated solution

$$y' = \theta'(x)\phi'(\tau).$$

Here

(32) $$\theta'_{xx}+K^2\theta' = 0$$

and

(33) $$\phi'_{\tau\tau}+K^2\phi'\left\{1+\frac{1+e}{e}\cdot\frac{A^2}{16}\cdot 2\cos 2w\tau\right\} = 0.$$

The solution of equation (32) which satisfies the boundary condition $\theta'(0) = \theta'(\pi) = 0$, and corresponds to the fundamental mode is $\theta'(x) = \sin x$, with

$$K^2 = 1.$$

If we write into equation (33) $v = w\tau$, this equation becomes

$$\phi'_{vv}+\frac{1}{w^2}\phi'\left\{1+\frac{1+e}{e}\frac{A^2}{8}+2\cdot\frac{1+e}{e}\cdot\frac{A^2}{16}\cos 2v\right\} = 0.$$

This is in the form of the Mathiew equation. The relevant periodic solution with boundary conditions $\phi'(0) = A$, $\phi'_v(0) = 0$ is [2]

[2] McLachlan [3], page 3, equations (16) and (17).

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$$\phi' = \frac{A}{1-q/8}\left\{\cos v - \frac{q}{8}\cos 3v + O(q^2)\right\}$$

where

$$a = \frac{1}{w^2}\left(1+\frac{1+e}{e}\frac{A^2}{8}\right)$$

and

$$q = (-)\,\frac{1}{w^2}\cdot\frac{1+e}{e}\cdot\frac{A^2}{16}.$$

The condition that the solution be periodic,

$$a = 1+q+O(q^2)$$

gives

$$w^2 = 1+\frac{1+e}{e}\cdot\tfrac{3}{16}\cdot A^2$$

or

$$w = 1+\tfrac{3}{32}\cdot\frac{A^2}{e}+O(A^2 \text{ etc.}).$$

This technique is useful when investigating the effect of amplitude on resonant frequency when the boundary conditions are more complicated (see Stuart (5)).

3.4. Comparison with previous work

G. F. Carrier [1, 2] has studied the motion of a string vibrating at finite amplitude, using the slope θ and a measure of the tension increment τ as the descriptive functions. He forms the exact equations of motion

$$\frac{\partial^2}{\partial\xi^2}\{(1+\tau)e^{i\theta}\} = \frac{\partial^2}{\partial\eta^2}\{(1+\alpha^2\tau)e^{i\theta}\}$$

with boundary conditions

$$\int_0^\pi (1+\alpha^2\tau)e^{i\theta}\,d\xi = \pi.$$

These equations correspond to the ends of the string, $\xi = 0$ and $\xi = \eta$ being fixed both longitudinally and transversely. α^2 is the strain of the string in the quiescent state referred to the stretched length, and ξ and η are dimensionless position and time coordinates. He puts

$$\theta = \alpha w$$

and proceeds, with α^2 as the perturbation parameter, to find the coefficient of $\cos\xi$ in w which satisfies the equations of motion and the boundary conditions with terms of order $O(\alpha^4)$ neglected.

In the present treatment the definition of elasticity is different to that of Carrier. It can be shown that the two treatments are consistent in regard to this aspect if we make the following transformations. With α^2, ε^2 as defined in the above papers and A and e as in this paper,

$$\alpha^2 = \frac{e}{1+e},$$

$$\varepsilon^2 = \frac{A^2}{4e}(1+e).$$

Using then the notation of this paper, Carrier's result may be written, to the order to which we can make comparisons,

$$\tan^{-1} y_x = A \cos x\phi(\tau),$$

where

$$\frac{d^2\phi}{d\tau^2} + \left\{1 + \frac{13A^2}{16}\right\}\phi\left\{1 + \left(\frac{A^2}{4e} - A^2\right)\phi^2\right\} = 0$$

with

$$\phi(0) = 1;\ \phi_\tau(0) = 0.$$

Thus

$$\phi = cn\ (\lambda\tau, k)$$

where

$$\lambda = 1 + \frac{A^2}{8e} - \frac{3A^2}{32} + \text{higher order terms}$$

and

$$k^2 = \frac{A^2}{8e} - \frac{A^2}{2} + \text{higher order terms.}$$

The frequency of the function $cn\ (\lambda\tau, k)$,

$$w = \frac{\pi\lambda}{2K}$$

$$= 1 + \frac{3A^2}{32e} + \frac{A^2}{32} + \text{higher order terms,}$$

which is that given by equation (24).

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