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ESSENTIALLY CONVEXOID OPERATORS

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Let H be a separable complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. Let π be the quotient mapping from $B(H)$ onto the Calkin algebra $B(H)/K(H)$, where $K(H)$ denotes all compact operators on $B(H)$. An operator $T \in B(H)$ is said to be *convexoid* [**14**] if the closure $\overline{W(T)}$ of its numerical range $W(T)$ coincides with the convex hull co $\sigma(T)$ of its spectrum $\sigma(T)$. $T \in B(H)$ is said to be *essentially normal*, *essentially* G_1, or *essentially convexoid* if $\pi(T)$ is normal, G_1 or convexoid in $B(H)/K(H)$ respectively.

In this paper we introduce some essentially generalized growth conditions associated with unitary ρ-dilations defined by B.Sz.-Nagy and C. Foias [**27**] [**28**] and we improve the result of [**28**]. Subsequently we give some characterizations of essentially convexoid operators, one of which improves the result of [**23**]. As some applications of these characterizations we introduce a new subclass of the class of essentially convexoid operators and we show some characterizations of operators belonging to this new subclass. Moreover we consider operators implying a slight "humble" spectral mapping theorems closely related to [**1**] and we show some theorems based on the results of [**11**] [**12**], one of which generalizes the result of [**22**]. Finally we consider essentially ρ-convexoid operators as an extension of ordinary essentially convexoid operators.

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1. Introduction. For an operator $T \in B(H)$, let $\sigma_e(T)$ denote the essential spectrum of T, that is, $\sigma_e(T)$ is the set of all complex numbers λ such that $T - \lambda$ is not a Fredholm operator, i.e., $\pi(T) - \lambda$ is not invertible in the Calkin algebra $B(H)/K(H)$ by Atkinson's result and the essential spectral radius $r_e(T)$ denotes the spectral radius of $\sigma_e(T)$.

The Weyl spectrum $\omega(T)$ of T is defined by the set of all complex numbers λ such that $T - \lambda$ is not a Fredholm operator of index zero. The essential numerical range $W_e(T)$ of T is defined as follows:

$$W_e(T) = \{\phi(\pi(T)) : \phi \text{ varies over the state space of } B(H)/K(H)\}.$$

It is well known [**2**] [**26**] that

$$\sigma_e(T) \subset \omega(T) \subset \sigma(T),$$

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$W_e(T)$ is the compact convex set containing $\sigma_e(T)$,

$$W_e(T) = \cap_K \bar{W}(T + K) \quad \text{and}$$
$$\omega(T) = \cap_K \sigma(T + K)$$

where the intersection is taken over all compact operators K. $T \in B(H)$ is said to be essentially convexoid [**18**] if $W_e(T) = \text{co}\, \sigma_e(T)$ where co S means the convex hull of a set S. Let $e(C)$ denote the set of all essentially convexoid operators. Similarly $e(G_1)$ denotes the set of all essentially G_1 operators. It is known that the Calkin algebra $B(H)/K(H)$ is a C^*-algebra and hence there exists a Hilbert space H_0 such that $B(H)/K(H)$ is isometrically isomorphic to a closed self adjoint subalgebra of $B(H_0)$, i.e., that $\nu: B(H)/K(H) \to B(H_0)$ be an isometric isomorphism. It is also known that $W_e(T) = \bar{W}(\nu\pi(T))$ and $\sigma_e(T) = \sigma(\nu\pi(T))$ [**4**] [**23**].

The class C_ρ $(\rho > 0)$ denotes the set of all operators with unitary ρ-dilation [**27**] [**28**]: there exist a Hilbert space K containing H as a subspace and unitary operator U on K such that

$$T^n h = \rho P U^n h \quad \text{for all } h \in H \ (n = 1, 2, \ldots)$$

where P is the orthogonal projection of K onto H.

Sz.-Nagy and Foias have characterized $C_\rho (\rho > 0)$; for example,

THEOREM A [**27**]. *$T \in C_\rho$ if and only if T has its spectrum in the closed unit disk and the following* (I_ρ) *holds:*

$$(\mathrm{I}_\rho) \qquad ||(\mu I - T)^{-1}|| \leqq \frac{1}{|\mu| - 1} \quad \begin{cases} \textit{for } 1 < |\mu| < \infty & \textit{if } \rho = 2 \\ \textit{for } 1 < |\mu| \leqq \dfrac{\rho - 1}{\rho - 2} & \textit{if } \rho > 2. \end{cases}$$

THEOREM B [**28**]. *$T \in C_\rho$ $(0 < \rho < 2, \rho \neq 1)$ if and only if the following* (I_ρ') *holds:*

$$(\mathrm{I}_\rho') \qquad ||(\mu I - T)h|| \leqq \frac{|\mu|}{|\rho - 1|} ||h|| \qquad \textit{for } \left|\frac{\rho - 1}{\rho - 2}\right| \leqq |\mu| < \infty \quad \textit{and} \quad h \in H.$$

An operator radius $w_\rho(T)$ of T is defined as follows [**15**]:

$$(1.1) \qquad w_\rho(T) = \inf \{u : u > 0, u^{-1}T \in C_\rho\}.$$

$w_\rho(T)$ is a non-increasing function of ρ; in particular, $w_1(T) = \|T\|$, $w_2(T) = w(T)$, the numerical radius of T, and $w_\infty(T) = r(T)$, the spectral radius of T respectively [**15**]. Moreover in [**15**] C_ρ is characterized by

$$(1.2) \qquad C_\rho = \{T : w_\rho(T) \leqq 1\}.$$

An operator T is said to be *ρ-oid* [**8**] [**9**] if $w_\rho(T^k) = (w_\rho(T))^k$ $(k = 1, 2, \ldots)$ and for each $\rho \geqq 1$, T is ρ-oid if and only if $w_\rho(T) = r(T)$.

Clearly 1-oid is normaloid and 2-oid is spectraloid [**14**] (recall that T is normaloid if $\|T\| = r(T)$ and spectraloid if $w(T) = r(T)$). T is said to be *essentially ρ-oid*, *essentially spectraloid* and *essentially normaloid if* $\pi(T)$ is ρ-oid, spectraloid and normaloid respectively.

We shall define essentially generalized growth conditions associated with unitary ρ-dilations as follows.

Definition 1.1. An operator T is said to satisfy the condition $e(\rho - G_1)$ for (M, N) (in symbols, $T \in e(\rho - G_1)$ for (M, N)) if T satisfies the following inequality:

(1.3) $\quad w_\rho(\pi(T - \mu)^{-1}) \leqq 1/d(\mu, M)$ for all complex $\mu \notin N$

where M and N are two closed bounded sets such that $N \supset M \supset \sigma_e(T)$ and $w_\rho(\pi(T))$ is defined by

(1.4) $\quad w_\rho(\pi(T)) = \inf_K w_\rho(T + K)$

where the infimum is taken over all compact operators K, that is, the numerical radius $w_\rho(\pi(T))$ is defined for $\pi(T)$ in the Calkin algebra. $T \in e(\rho - G_1)$ for M denotes $T \in e(\rho - G_1)$ for (M, M).

Similarly to Definition 1.1, we define $T \in (\rho - G_1)$ for (M, N) if

$$w_\rho((T - \mu)^{-1}) \leqq 1/d(\mu, M) \quad \text{for all complex } \mu \notin N$$

where M and N are two closed bounded sets such that $N \supset M \supset \sigma(T)$ [**12**]. Also $T \in (\rho - G_1)$ for M denotes $T \in (\rho - G_1)$ for (M, M).

The main concern of this paper is to show that the natural definition for the numerical radius $w_\rho(\pi(T))$ in the Calkin algebra has properties analogous to those presented for $B(H)$ in [**12**], that is, to give some characterizations of essentially convexoid operators by using the theory of unitary ρ-dilations and to show some applications of these characterizations.

2. Characterizations of essentially convexoid operators. First, we improve the result [**26**, Theorem 2] and we give several characterizations of essentially convexoid operators.

THEOREM 2.1. *For a given $T \in B(H)$ and a closed convex set X in the complex plane, the following* (α), (β) *and* (γ) *are equivalent.*

(α) *$X \supset W_e(T)$;*

(β) *for all $\rho \geqq 1$ and all closed bounded sets $Y \supset X$, $T \in e(\rho - G_1)$ for (X, Y); that is,*

$$w_\rho(\pi(T - \mu)^{-1}) \leqq 1/d(\mu, X) \quad \textit{for all complex } \mu \notin Y;$$

(γ) *there exist ρ such that $1 \leqq \rho < \infty$ and bounded closed set Y such that $Y \supset X$ and $T \in e(\rho - G_1)$ for (X, Y); that is,*

$$w_\rho(\pi(T - \mu)^{-1}) \leqq 1/d(\mu, X) \quad \textit{for all complex } \mu \notin Y.$$

In order to prove Theorem 2.1 we need the following lemma.

LEMMA 2 1 *Let* $\nu : B(H)/K(H) \to B(H_0)$ *be an isometric isomorphism. Then*

(a) $w_\rho(\nu\pi(T)) \leqq w_\rho(T)$ *for all* $0 < \rho \leqq \infty$

(b) *for* $0 < \rho \leqq \infty$, $w_\rho(\nu\pi(T)) \leqq w_\rho(\pi(T))$;

in particular

$$w_\rho(\nu\pi(T)) = w_\rho(\pi(T)) \quad \textit{for } 0 < \rho \leqq 2 \textit{ and } \rho = \infty.$$

Proof. (a) Recall that and ν is an isometric isomorphism, and $\|\nu\pi(T)\| = \|\pi(T)\|$. By the absolute homogeneity of $w_\rho(\nu\pi(T))$, we have only to show that $w_\rho(T) \leqq 1$ implies $w_\rho(\nu\pi(T)) \leqq 1$. We divide the case $0 < \rho \leqq \infty$ into the following cases.

(i) $0 < \rho < 2$ $(\rho \neq 1)$. $w_\rho(T) \leqq 1$ if and only if $T \in C_\rho$, so that T satisfies (I_ρ') in Theorem B and

$$\|\mu - \nu\pi(T)\| = \|\mu - \pi(T)\| \leqq \|\mu - T\|$$

holds by the isometricity of ν, hence $\nu\pi(T)$ also satisfies (I_ρ'), namely, $\nu\pi(T) \in C_\rho$, i.e., $w_\rho(\nu\pi(T)) \leqq 1$.

(ii) $2 \leqq \rho < \infty$. $w_\rho(T) \leqq 1$ implies that T satisfies (I_ρ) in Theorem A and $\sigma(T) \subset D$ (the closed unit disk). $\sigma(\nu\pi(T)) = \sigma_e(T)$ [**4**] and $\sigma_e(T) \subset \sigma(T)$, so that

$$\sigma(\nu\pi(T)) \subset \sigma(T) \subset D.$$

On the other hand, we have

$$\|(\mu - \nu\pi(T))^{-1}\| = \|(\mu - \pi(T))^{-1}\| \leqq \|(\mu - T)^{-1}\|,$$

whence $\nu\pi(T) \in C_\rho$ by Theorem A, that is, $w_\rho(\nu\pi(T)) \leqq 1$.

(iii) $\rho = 1$.

$$\|\nu\pi(T)\| = \|\pi(T)\| \leqq \|T\|.$$

(iv) $\rho = \infty$.

$$w_\infty(\nu\pi(T)) = r(\nu\pi(T)) = r_e(T) \leqq r(T) = w_\infty(T).$$

Hence we have finished the proof of (a).

We need the following to show the latter half of (b).

THEOREM C [**3**]. *For each* $T \in B(H)$, *there exists a* $K_0 \in K(H)$ *such that*

$$\|T + K_0 + \lambda\| = \|\pi(T + \lambda)\| \textit{ for all complex } \lambda.$$

(b) We have only to show that $w_\rho(\pi(T)) \leqq 1$ implies $w_\rho(\nu\pi(T)) \leqq 1$ since the absolute homogeneity of $w_\rho(\nu\pi(T))$. And

$$w_\rho(\pi(T)) = \inf_K w_\rho(T + K) \leqq 1$$

implies that, for an arbitrary $\epsilon > 0$, there exists a compact operator K_0 such that

$$w_\rho(T + K_0) \leqq 1 + \epsilon.$$

By (a),

$$w_\rho(\nu\pi(T + K_0)) \leqq 1 + \epsilon,$$

that is, $w_\rho(\nu\pi(T)) \leqq 1 + \epsilon$ and ϵ is arbitrary, so we have

$$w_\rho(\nu\pi(T)) \leqq 1.$$

For the latter half of (b) we have only to show the opposite inequality for $0 < \rho \leqq 2$ and $\rho = \infty$.

(i) $0 < \rho < 2$ ($\rho \neq 1$). Assume $w_\rho(\nu\pi(T)) \leqq 1$, then this implies $\nu\pi(T) \in C_\rho$ and by Theorem B, $\nu\pi(T)$ satisfies (I_ρ'). By Theorem C, there exists a compact operator K_0 such that

$$\|\mu - \nu\pi(T)\| = \|\mu - \pi(T)\| = \|\mu - (T + K_0)\|$$

for all μ, so that $T + K_0$ satisfies (I_ρ'), hence $T + K_0 \in C_\rho$ by Theorem B, and so $w_\rho(T + K_0) \leqq 1$. By the definition of $w_\rho(\pi(T))$,

$$w_\rho(\pi(T)) = \inf_K w_\rho(T + K) \leqq w_\rho(T + K_0) \leqq 1,$$

whence the proof follows by the absolute homogeneity of $w_\rho(\pi(T))$.

(ii) $\rho = 1$. $\|\pi(T)\| = \|\nu\pi(T)\|$ is already well known.

(iii) $\rho = 2$. $w_2(\pi(T)) = w_2(\nu\pi(T))$ follows by continuity from (i).

(iv) $\rho = \infty$. $w_\infty(\nu\pi(T)) = r(\nu\pi(T)) = r_e(T)$ since $\sigma(\nu\pi(T)) = \sigma_e(T)$ and

$$w_\infty(\pi(T)) = \inf_K w_\infty(T + K) = \inf_K r(T + K).$$

On the other hand $r_e(T) = \inf_K r(T + K)$ [**24**, Lemma 2.2], so that we have

$$w_\infty(\nu\pi(T)) = w_\infty(\pi(T)).$$

Consequently we have finished the proof of Lemma 2.1.

We need only the first half of (b) to prove Theorem 2.1, but we cite the latter half of (b) for the sake of completeness.

In order to prove Theorem 2.1, we cite the following result, which improves the results of [**19**] and [**21**].

THEOREM D [**12**]. *If X is a closed convex subset of the complex plane, then $X \supset \overline{W(T)}$ if and only if there exist ρ such that $1 \leqq \rho < \infty$ and a closed bounded set Y such that $T \in (\rho - G_1)$ for (X, Y); that is,*

$$w_\rho((T - \mu)^{-1}) \leqq 1/d(\mu, X) \text{ for all complex } \mu \notin Y.$$

Proof of Theorem 2.1. $X \supset W_e(T)$ implies

$$\|\pi(T - \mu)^{-1}\| \leqq 1/d(\mu, X)$$

for all complex $\mu \notin X$ [**26**, Theorem 2], so that (α) implies (β) since $w_\rho(\pi(T))$ is a decreasing function of ρ. (β) $\Rightarrow$ (γ) is obvious. Finally assume (γ), that is, there exist ρ ($1 \leqq \rho < \infty$) and a closed bounded set Y such that $Y \supset X$ and $T \in e(\rho - G_1)$ for (X, Y). Then, by (b) of Lemma 2.1, we have that there exist ρ ($1 \leqq \rho < \infty$) and closed bounded Y such that $Y \supset X$ and $\nu\pi(T) \in (\rho - G_1)$ for (X, Y), whence we have $X \supset \bar{W}(\nu\pi(T))$ by Theorem D and the proof of (α) follows by the relation $W_e(T) = \bar{W}(\nu\pi(T))$ [**23**], so the proof is complete.

Remark 2.1. It is important to stress that $\rho \neq \infty$ in (γ), since (γ) for $\rho = \infty$ only ensures that $X \supset \sigma_e(T)$.

Next we show Theorem 2.2 and Theorem 2.3, which are both characterizations of essentially convexoid operators.

THEOREM 2.2. *Let $T \in B(H)$. T is essentially convexoid if and only if there exist ρ such that $1 \leqq \rho < \infty$ and a closed bounded set Y containing* co $\sigma_e(T)$ *such that $T \in e(\rho - G_1)$ for* (co $\sigma_e(T)$, Y)*; that is,*

(2.1) $\quad w_\rho(\pi(T - \mu)^{-1}) \leqq 1/d(\mu, \text{co } \sigma_e(T))$ *for all complex $\mu \notin Y$.*

THEOREM 2.3. *Let $T \in B(H)$. The following conditions are equivalent.*

(a) *T is essentially convexoid.*

(b) *There exists a compact operator K_0 such that the following* (2.2) *holds:*

(2.2) $\quad \bar{W}(T + K_0) = \text{co } \sigma(T + K_0) = \text{co } \sigma_e(T).$

In other words, $T + K_0$ is convexoid and co $\sigma(T + K_0) \subset$ co $\sigma(T + K)$ *holds for all compact operators K.*

(c) *There exist ρ such that $1 \leqq \rho < \infty$, a compact operator K_0 and a closed bounded set Y containing* co $\sigma(T + K_0)$ *such that the following* (2.3) *holds:*

(2.3) $\quad w_\rho((T + K_0 - \mu)^{-1}) \leqq 1/d(\mu, \text{co } \sigma_e(T))$ *for all complex $\mu \notin Y$.*

(d) *$T - \mu$ is essentially spectraloid for all complex μ* [**23**].

(e) *$T - \mu$ is essentially spectraloid for all complex μ whose absolute value is sufficiently large.*

(f) *The following $e(\Sigma - \theta)$ holds:*

$$e(\Sigma - \theta) \quad \text{Re} \textstyle\sum_e (e^{i\theta}T) = \textstyle\sum_e (\text{Re } e^{i\theta}T) \textit{ for all } 0 \leqq \theta \leqq 2\pi$$

where $\sum_e (S) = \text{co } \sigma_e(S)$ *and $e(\Sigma - \theta)$ is equivalent to*

$$\text{co Re } \sigma_e(e^{i\theta}T) = \text{co } \sigma_e (\text{Re } e^{i\theta}T) \quad \textit{for all } 0 \leqq \theta \leqq 2\pi.$$

Proof of Theorem 2.2. Take $X = \operatorname{co} \sigma_e(T)$ in Theorem 2.1. Then we have $\operatorname{co} \sigma_e(T) \supset W_e(T)$ and the reverse inclusion relation always holds, so the proof is complete.

Before we prove Theorem 2.3, we cite the following results.

THEOREM E [**7**]. *For $T \in B(H)$, there is $K_0 \in K(H)$ such that $\bar{W}(T + K_0) = W_e(T)$.*

THEOREM F [**7**]. *If T is essentially convexoid, then there exists $K_0 \in K(H)$ such that $T + K_0$ is convexoid.*

In [**7**] Theorem E is shown by using Theorem C and [**26**, Theorem 4].

Proof of Theorem 2.3. (a) $\Leftrightarrow$ (b). Let T be essentially convexoid. Then, by Theorem F, there exists a compact operator K_0 such that $T + K_0$ is convexoid and (2.2) holds and this proof is contained in the one of Theorem F and we cite it for the sake of completeness. By Theorem E and the hypothesis

$$\bar{W}(T + K_0) = W_e(T) = \operatorname{co} \sigma_e(T) \subset \operatorname{co} \sigma(T + K_0) \subset \bar{W}(T + K_0)$$

so that $T + K_0$ is convexoid and $\operatorname{co} \sigma(T + K_0) = \operatorname{co} \sigma_e(T)$, hence we have (b). Conversely, assume that (2.2) holds. Since

$$\operatorname{co} \sigma_e(T) \subset W_e(T) \quad \text{and} \quad W_e(T) = \cap_K \bar{W}(T + K)$$

always hold, we have

$$\bar{W}(T + K_0) = \operatorname{co} \sigma(T + K_0) = \operatorname{co} \sigma_e(T) \subset \bar{W}_e(T) \subset \bar{W}(T + K_0)$$

whence $W_e(T) = \operatorname{co} \sigma_e(T)$, and so T is essentially convexoid.

(b) $\Leftrightarrow$ (c). Assume (b). As $T + K_0$ is convexoid, there exist ρ such that $1 \leqq \rho < \infty$ and a closed bounded set Y such that

$$T + K_0 \in (\rho - G_1) \quad \text{for } (\operatorname{co} \sigma(T + K_0), Y)$$

by Theorem D and (c) follows by the assumption $\operatorname{co} \sigma(T + K_0) = \operatorname{co} \sigma_e(T)$. Conversely assume (c). Then we have $\operatorname{co} \sigma_e(T) \supset \bar{W}(T + K_0)$ by Theorem D. On the other hand

$$\operatorname{co} \sigma_e(T) \subset \operatorname{co} \sigma(T + K_0) \subset \bar{W}(T + K_0)$$

is always valid, and hence (2.2) holds.

(a) $\Leftrightarrow$ (d). This was shown in [**23**].

(a) $\Leftrightarrow$ (e). It is shown in [**11**] [**12**] that T is convexoid if and only if T is spectraloid for all u whose absolute values are sufficiently large, so the proof is an immediate consequence of the relations $W_e(T) = \bar{W}(\nu\pi(T))$ [**23**] and $\sigma_e(T) = \sigma(\nu\pi(T))$ [**4**].

(a) $\Leftrightarrow$ (f). As a self adjoint operator is always essentially convexoid, $e(\Sigma - \theta)$ implies

$$\begin{aligned}\operatorname{Re}\{e^{i\theta}\textstyle\sum_e(T)\} = \textstyle\sum_e(\operatorname{Re} e^{i\theta}T) &= W_e(\operatorname{Re} e^{i\theta}T)\\ &= \operatorname{Re} W_e(e^{i\theta}T) = \operatorname{Re}\{e^{i\theta}W_e(T)\}\end{aligned}$$

for any $0 \leqq \theta \leqq 2\pi$, this relation yields $W_e(T) = \Sigma_e(T)$ and the reverse relation is obvious. Hence the proof is complete.

Remark 2.2. Since $\operatorname{co}\omega(T) = \operatorname{co}\sigma_e(T)$ [**5**, Corollary of Theorem 2.4] and (b) of Theorem 2.3, we note that if there exists a compact operator K_0 such that $T + K_0$ is convexoid and $\omega(T) = \sigma(T + K_0)$, then T is essentially convexoid. In fact it is shown in [**25**, Theorem 4] that there exists a compact operator K_0 such that $\sigma(T + K_0) = \omega(T)$.

Remark 2.3. Given $T \in B(H)$, does there exists a $K_0 \in K(H)$ such that for any complex polynomial $p(z)$, $\|p(T + K_0)\| = \|p(\pi(T))\|$? This is an unsolved problem and recently much interest has been centered about the above one. In the special case $p(z) = z + \mu$ for all complex μ, the affirmative solution has been recently announced in Theorem C [**3**]. However if we put $p(z) = (z - \mu)^{-1}$ instead of the polynomial $p(z)$, then does

$$\|(T + K_0 - \mu)^{-1}\| = \|\pi(T - \mu)^{-1}\|$$

hold? This is also an unsolved problem. Compare (2.1) of Theorem 2.2 with (2.3) of Theorem 2.3. Then $w_\rho(\pi(T - \mu)^{-1})$ and $w_\rho((T + K_0 - \mu)^{-1})$ are closely related to the above unsolved problem even if $\rho = 1$.

Remark 2.4. We choose $Y = \operatorname{co}\sigma(T + K_0)$ in (2.3) of (c). Then (2.3) is equivalent to the existence of ρ such that $1 \leqq \rho < \infty$ and

$$T + K_0 \in (\rho - G_1) \quad \text{for } (\operatorname{co}\sigma_e(T), \operatorname{co}\sigma(T + K_0)).$$

Remark 2.5. It is shown in [**23**] that T is essentially convexoid if and only if $T \in e(1 - G_1)$ for $\operatorname{co}\sigma_e(T)$ and also T is essentially convexoid if and only if $T \in e(2 - G_1)$ for $\operatorname{co}\sigma_e(T)$. Theorem 2.2 improves these results by using the theory of unitary ρ-dilations.

Remark 2.6. The equivalence relation between (a) and (b) in Theorem 2.3 easily implies that T is essentially spectraloid if and only if there exists a compact operator K_0 such that

$$w(T + K_0) = r(T + K_0) = r_e(T)$$

holds, namely $T + K_0$ is spectraloid and $r(T + K_0) \leqq r(T + K)$ holds for all compact operators K.

3. Operators implying $\operatorname{Re}\sigma_e(T) = \sigma_e(\operatorname{Re} T)$. In this section we consider operators implying the equation of the title closely related to characterizations of essentially convexoid operators and we generalize a result of [**22**].

Definition 3.1. An operator T is said to satisfy the conditions $E - e(\rho - G_1)$ for (M, N) (in symbols, $T \in E - e(\rho - G_1)$ for (M, N)), if T satisfies the equality in (1.3):

$$(3.1) \quad w_\rho(\pi(T - \mu)^{-1}) = 1/d(\mu, M) \quad \text{for all complex } \mu \notin N$$

where M and N are two closed bounded sets such that $N \supset M \supset \sigma_e(T)$. $T \in E - e(\rho - G_1)$ for M denotes $T \in E - e(\rho - G_1)$ for (M, M).

Similarly $T \in E - (\rho - G_1)$ for (M, N) if T satisfies

$$w_\rho((T - \mu)^{-1}) = 1/d(\mu, M) \quad \text{for all } \mu \notin N,$$

where M and N are two closed bounded sets such that $N \supset M \supset \sigma(T)$ and $T \in E - (\rho - G_1)$ for M denotes $T \in E - (\rho - G_1)$ for (M, M) [**12**].

Remark 3.1. Since $r_e(T) \leqq w_\rho(\pi(T))$ holds for any ρ [**22**] and

$$1/d(\mu, \sigma_e(T)) = r_e((T - \mu)^{-1})$$

is valid for all $\mu \notin \sigma_e(T)$, it follows that $T \in e(\rho - G_1)$ for $(\sigma_e(T), N)$ is equivalent to $T \in E - e(\rho - G_1)$ for $(\sigma_e(T), N)$. That is, $(T - \mu)^{-1}$ is essentially ρ-oid for all complex $\mu \notin N$.

Definition 3.2. An operator T is called *of class* $e(M_\rho)$ $(\rho \geqq 1)$ if $(T - \mu)^{-1}$ is essentially ρ-oid for all $\mu \notin \sigma_e(T)$. That is, $T \in e(M_\rho)$ $(\rho \geqq 1)$ coincides with $T \in E - e(\rho - G_1)$ for $\sigma_e(T)$ by Remark 3.1 $(\rho \geqq 1)$.

We remark that $T \in e(G_1)$ for M means $T \in e(1 - G_1)$ for (M, M) and $T \in e(G_1)$ also means $T \in e(1 - G_1)$ for $\sigma_e(T)$. That is, $T \in E - e(1 - G_1)$ for $\sigma_e(T)$.

Definition 3.3. The class $e(R)$ is defined by the following: $T \in e(R)$ if

$$(3.2) \quad \|\pi(T - \mu)^{-1}\| = 1/d(\mu, W_e(T)) \quad \text{for all complex } \mu \notin W_e(T).$$

That is, $T \in e(R)$ if and only if $T \in E - e(1 - G_1)$ for $W_e(T)$.

We remark that the class R [**17**] is similarly defined by $T \in R$ if T satisfies

$$\|(T - \mu)^{-1}\| = 1/d(\mu, W(T)) \quad \text{for all } \mu \notin \overline{W(T)}.$$

The essentially hen-spectrum $\tilde{\sigma}_e(T)$ is defined by

$$\tilde{\sigma}_e(T) = [[\sigma_e(T)^c]_\infty]^c$$

where M^c is the complement of M and $[M]_\infty$ is the unbounded component of M. The hen-spectrum $\tilde{\sigma}(T)$ is defined by

$$\tilde{\sigma}(T) = [[\sigma(T)^c]_\infty]^c$$

in [**6**] and $\tilde{\sigma}(T)$ is a compact set and $\sigma(T) \subset \tilde{\sigma}(T) \subset \text{co}\,\sigma(T)$ holds [**6**]. $\tilde{\sigma}_e(T)$ is also a compact set such that $\sigma_e(T) \subset \tilde{\sigma}_e(T) \subset \text{co}\,\sigma_e(T)$ since $\sigma_e(T) = \sigma(\nu\pi(T))$ holds [**4**].

Definition 3.4. The class $e(H_1)$ is defined by the following. $T \in e(H_1)$ if

$$(3.3) \quad \|\pi(T-\mu)^{-1}\| \leqq 1/d(\mu, \tilde{\sigma}_e(T)) \quad \text{for all complex } \mu \notin \tilde{\sigma}_e(T).$$

That is, $T \in e(H_1)$ if and only if $T \in e(G_1)$ for $\tilde{\sigma}_e(T)$.

In [**6**], a class (H_1) is similarly defined as follows: $T \in (H_1)$ if T satisfies

$$\|(T-\mu)^{-1}\| \leqq 1/d(\mu, \tilde{\sigma}(T)) \quad \text{for all complex } \mu \notin \tilde{\sigma}(T).$$

It is shown in [**6**] that (H_1) contains both (G_1) and R. In [**12**] it is also shown that $T \in (H_1)$ if and only if $T \in (G_1)$ for $(\sigma(T), \tilde{\sigma}(T))$. On the other hand $\sigma_e(T) = \sigma(\nu\pi(T))$ [**4**], $\|\pi(T)\| = \|\nu\pi(T)\|$ and $W_e(T) = \bar{W}(\nu\pi(T))$ [**23**] hold, so that $T \in e(H_1)$ if and only if $T \in e(G_1)$ for $(\sigma_e(T), \tilde{\sigma}_e(T))$ and $e(H_1)$ contains both $e(G_1)$ and $e(R)$. $T \in R$ if and only if $\partial W(T) \subset \sigma(T)$ [**17**] and $T \in R$ if and only if $\overline{W(T)} = \tilde{\sigma}(T)$ [**6**], so that we remark that $T \in e(R)$ if and only if $\partial W_e(T) \subset \sigma_e(T)$ and $T \in e(R)$ if and only if $W_e(T) = \tilde{\sigma}_e(T)$ by the same reason.

THEOREM 3.1. *If there exists* $\rho \geqq 1$ *such that* $T \in e(\rho - G_1)$ *for* $(\sigma_e(T), \tilde{\sigma}_e(T))$ *and* $\text{Re}\,\sigma_e(T)$ *is connected, then*

$$e(*) \quad \text{Re}\,\sigma_e(T) = \sigma_e(\text{Re}\,T).$$

In order to prove Theorem 3.1, we cite the following result [**12**], which is an extension of the results of [**1**] and [**20**].

THEOREM G [**12**]. *If there exists* $\rho \geqq 1$ *such that* $T \in (\rho - G_1)$ *for* $(\sigma(T), \tilde{\sigma}(T))$ *and* $\text{Re}\,\sigma(T)$ *is connected, then*

$$(*) \quad \text{Re}\,\sigma(T) = \sigma(\text{Re}\,T).$$

Proof of Theorem 3.1. By (b) of Lemma 2.1,

$$w_\rho(\nu\pi(T)) \leqq w_\rho(\pi(T))$$

for all $\rho > 0$, and so the hypothesis implies that there exists $\rho \geqq 1$ such that

$$\nu\pi(T) \in (\rho - G_1) \quad \text{for } (\sigma(\nu\pi(T)), \tilde{\sigma}(\nu\pi(T)))$$

and $\text{Re}\,\sigma(\nu\pi(T))$ is connected since $\sigma_e(T) = \sigma(\nu\pi(T))$ [**4**], whence

$\nu\pi(T)$ satisfies (*) by Theorem G; that is,

$$\operatorname{Re}\sigma(\nu\pi(T)) = \sigma(\operatorname{Re}\nu\pi(T)).$$

It is easily seen that $\operatorname{Re}\nu\pi(T) = \nu\pi(\operatorname{Re} T)$, so that

$$\operatorname{Re}\sigma_e(T) = \operatorname{Re}\sigma(\nu\pi(T)) = \sigma(\nu\pi(\operatorname{Re} T)) = \sigma_e(\operatorname{Re} T)$$

and the proof is complete.

COROLLARY 3.1. *If* $T \in e(M_\rho)$ *and* $\operatorname{Re}\sigma_e(T)$ *is connected, then* $e(*)$ *holds.*

Proof. As $e(M_\rho)$ means $e(\rho - G_1)$ for $\sigma_e(T)$, the Corollary easily follows from Theorem 3.1.

Put $\rho = 1$ in Corollary 3.1. Then we have

COROLLARY 3.2. *If* $T \in e(H_1)$ *and* $\operatorname{Re}\sigma_e(T)$ *is connected, then* $e(*)$ *holds.*

As $e(H_1)$ contains $e(G_1)$ we have

COROLLARY 3.3. *If* $T \in e(G_1)$ *and* $\operatorname{Re}\sigma_e(T)$ *is connected, then* $e(*)$ *holds.*

COROLLARY 3.4. *If* $T \in e(R)$, *then* $e(*)$ *holds.*

Proof. $T \in e(R)$ if and only if $\partial W_e(T) \subset \sigma_e(T)$ and $\sigma_e(T) \subset W_e(T)$ holds, so that $T \in e(R)$ implies $\operatorname{Re}\sigma_e(T) = \operatorname{Re} W_e(T)$ is connected since $W_e(T)$ is convex. As $e(H_1) \supset e(R)$ holds in general, the Corollary follows by Corollary 3.2.

LEMMA 3.1. *The class* $e(R)$ *properly contains the class* R.

Proof. If $T \in R$, then the relations $W_e(T) = \overline{W(T)}$ and $\partial W(T) \subset \sigma_e(T)$ are shown in the proof of [**22**, Theorem 3]; that is, $\partial W_e(T) \subset \sigma_e(T)$, whence $T \in e(R)$. Take a compact operator $K \notin R$, then $\pi(K) = 0 \in e(R)$, so the proof is complete.

However, the result corresponding to Lemma 3.1 is not true for the essentially G_1 operators nor the essentially convexoid operators. We remark that (G_1) is not a subset of $e(G_1)$ and vice versa and (C) is not a subset of $e(C)$ and vice versa [**18**] where (C) denotes the set of all convexoid operators.

As an immediate consequence of Lemma 3.1 and Corollary 3.4, we have

COROLLARY 3.5 [**22**]. *If* $T \in R$, *then* $e(*)$ *holds.*

Remark 3.2. If $T \in e(R)$, then $\operatorname{Re}\omega(T) = \omega(\operatorname{Re} T)$. In fact, $T \in e(R)$ implies

$$\partial W_e(T) \subset \sigma_e(T) \subset \omega(T) \subset W_e(T),$$

so that $\operatorname{Re}\sigma_e(T) = \operatorname{Re}\omega(T) = \operatorname{Re} W_e(T)$ since $W_e(T)$ is convex. The conclusion follows by $\sigma_e(\operatorname{Re} T) = \omega(\operatorname{Re} T)$ [**2**, (7)] and Corollary 3.4. Hence the assertion is equivalent to Corollary 3.4. Moreover if $T \in R$, then $\operatorname{Re}\omega(T) = \omega(\operatorname{Re} T)$ [**22**] follows from Lemma 3.1 and the above remark.

With respect to [**20**, Corollaries 4 and 5], we remark that if $S, T \in e(R)$ such that $\operatorname{Re}\sigma_e(S) = \operatorname{Re}\sigma_e(T)$, then $\omega(\operatorname{Re} S) = \omega(\operatorname{Re} T)$ by Corollary 3.4 and [**2**, (7)]. By [**2**, Section 8], there exists a unitary operator U such that $U^*(\operatorname{Re} S)U - \operatorname{Re} T$ is compact; that is, $\operatorname{Re} S$ is essentially equivalent to $\operatorname{Re} T$.

The remainder of this section is devoted to showing a construction of operators satisfying $e(*)$.

THEOREM 3.2. *If A is any operator and B satisfies $e(*)$ such that*

(3.4) $\quad \operatorname{Re} W_e(A) \subset \operatorname{Re}\sigma_e(B)$

then $T = A \oplus B$ also satisfies $e(*)$.

Proof. Since the essential numerical range contains its essential spectrum, (3.4) implies

(3.5) $\quad \operatorname{Re}\sigma_e(A) \subset \operatorname{Re} W_e(A) \subset \operatorname{Re}\sigma_e(B)$

(3.6) $\quad \sigma_e(\operatorname{Re} A) \subset W_e(\operatorname{Re} A) = \operatorname{Re} W_e(A) \subset \operatorname{Re}\sigma_e(B),$

so that (3.5), (3.6) and the hypothesis yield

$$\begin{aligned}\sigma_e(\operatorname{Re} T) = \sigma_e(\operatorname{Re} A) \cup \sigma_e(\operatorname{Re} B) &= \sigma_e(\operatorname{Re} A) \cup \operatorname{Re}\sigma_e(B)\\ &= \operatorname{Re}\sigma_e(B)\end{aligned}$$

and

$$\operatorname{Re}\sigma_e(T) = \operatorname{Re}\sigma_e(A) \cup \operatorname{Re}\sigma_e(B) = \operatorname{Re}\sigma_e(B),$$

hence T satisfies $e(*)$.

Remark 3.3. The typical examples satisfying $e(*)$ are essentially normal. In fact, let T be an essentially normal operator, that is, let $\pi(T)$ be a normal element of the Calkin algebra $B(H)/K(H)$. Then we have

$$\sigma(f(\pi(T))) = f(\sigma(\pi(T)))$$

for every polynomial $p(z, z^*)$ [**29**] and if we put $f(z, z^*) = \frac{1}{2}(z + z^*) = \operatorname{Re} z$, then we have

$$\sigma(\pi(\operatorname{Re} T)) = \sigma(\operatorname{Re}\pi(T)) = \operatorname{Re}(\sigma(\pi(T))),$$

whence T satisfies $e(*)$.

4. Characterizations of operators belonging to $e(R)$. If $T \in e(R)$, then $T \in e(H_1)$ and $e(H_1) \subset e(C)$ by Theorem 2.2, so that $e(R)$ is a subclass of $e(C)$. We characterize $e(R)$ as an immediate consequence of characterizations of essentially convexoid operators.

THEOREM 4.1. *Let* $T \in B(H)$. *The following conditions are equivalent.*

(a) $T \in e(R)$, *i.e.*,

$$\|\pi(T-\mu)^{-1}\| = 1/d(\mu, W_e(T)) \quad \textit{for all } \mu \notin W_e(T).$$

(b_1) *for all* $1 \leqq \rho \leqq \infty$, $T \in E - e(\rho - G_1)$ *for* $W_e(T)$, *i.e.*,

$$w_\rho(\pi(T-\mu)^{-1}) = 1/d(\mu, W_e(T)) \quad \textit{for all } \mu \notin W_e(T).$$

(b_2) *there exists* ρ *such that* $1 \leqq \rho \leqq \infty$ *and* $T \in E - e(\rho - G_1)$ *for* $W_e(T)$, *i.e.*,

$$W_\rho(\pi(T-\mu)^{-1}) = 1/d(\mu, W_e(T)) \quad \textit{for all } \mu \notin W_e(T).$$

(c_1) *for all* $1 \leqq \rho \leqq \infty$, $T \in E - e(\rho - G_1)$ *for* $(\mathrm{co}\, \sigma_e(T), W_e(T))$, *i.e.*,

$$w_\rho(\pi(T-\mu)^{-1}) = 1/d(\mu, \mathrm{co}\, \sigma_e(T)) \quad \textit{for all } \mu \notin W_e(T).$$

(c_2) *there exists* ρ *such that* $1 \leqq \rho \leqq \infty$ *and* $T \in E - e(\rho - G_1)$ *for* $(\mathrm{co}\, \sigma_e(T), W_e(T))$., *i.e.*,

$$w_\rho(\pi(T-\mu)^{-1}) = 1/d(\mu, \mathrm{co}\, \sigma_e(T)) \quad \textit{for all } \mu \notin W_e(T).$$

(d_1) *for all* $1 \leqq \rho \leqq \infty$, $T \in E - e(\rho - G_1)$ *for* $\mathrm{co}\, \sigma_e(T)$, *i.e.*,

$$w_\rho(\pi(T-\mu)^{-1}) = 1/d(\mu, \mathrm{co}\, \sigma_e(T)) \quad \textit{for all } \mu \notin \mathrm{co}\, \sigma_e(T).$$

(d_2) *there exists* ρ *such that* $1 \leqq \rho \leqq \infty$ *and* $T \in E - e(\rho - G_1)$ *for* $\mathrm{co}\, \sigma_e(T)$, *i.e.*,

$$w_\rho(\pi(T-\mu)^{-1}) = 1/d(\mu, \mathrm{co}\, \sigma_e(T)) \quad \textit{for all } \mu \notin \mathrm{co}\, \sigma_e(T).$$

Proof. (a) $\Rightarrow$ (c_1). We see that if $T \in e(R)$, then $T \in e(C)$, that is, $W_e(T) = \mathrm{co}\, \sigma_e(T)$. On the other hand, if $T \in e(R)$, then $T \in e(H_1)$, that is, $\pi(T-\mu)^{-1}$ is normaloid for all $\mu \notin \tilde{\sigma}_e(T)$ and we have

$$\begin{aligned} 1/d(\mu, \mathrm{co}\, \sigma_e(T)) &= 1/d(\mu, W_e(T)) = \|\pi(T-\mu)^{-1}\| \\ &= w_\rho(\pi(T-\mu)^{-1}) = r(\pi(T-\mu)^{-1}) \end{aligned}$$

for all $1 \leqq \rho \leqq \infty$ and for all $\mu \notin W_e(T)$. Hence we have (c_1).

(c_1) $\Rightarrow$ (c_2) is obvious.

(c_2) $\Rightarrow$ (a). First of all, (4.1) below holds in general:

$$w_\rho(\pi(T-\mu)^{-1}) \leqq \|\pi(T-\mu)^{-1}\| \leqq 1/d(\mu, W_e(T)) \tag{4.1}$$

for all $\mu \notin W_e(T)$ and for any $1 \leqq \rho \leqq \infty$. Assume ($c_2$). Then T is essentially convexoid by Theorem 2.2, i.e., $W_e(T) = \mathrm{co}\, \sigma_e(T)$. This relation, (4.1) and the hypothesis (c_2) yield (a), that is,

$$\|\pi(T-\mu)^{-1}\| = 1/d(\mu, W_e(T)) \quad \text{for all } \mu \notin W_e(T).$$

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The proof of the equivalence among (a), (b_1) and (b_2) is similar to the one for (a), (c_1) and (c_2) and the proof of the equivalence among (a), (d_1) and (d_2) is also similar to the one for (a), (c_1) and (c_2), and we omit these.

THEOREM 4.2. *Let $T \in B(H)$. The following conditions are equivalent.*

(a) $T \in e(R)$.

(b) *$\partial\tilde{\sigma}_e(T)$ is a convex curve and $e(\sigma - \theta)$ holds:*

$$e(\sigma - \theta) \quad \sigma_e(\mathrm{Re}\, e^{i\theta}T) = \mathrm{Re}\, \sigma_e(e^{i\theta}T) \ (\textit{in symbols, } T \in e(\sigma - \theta))$$

for all $0 \leqq \theta \leqq 2\pi$, where ∂M denotes the boundary of M.

(c) *there exists a compact operator K_0 such that $T + K_0$ belongs to R.*

Proof. (a) $\Leftrightarrow$ (b). If $T \in e(R)$, then $\partial\tilde{\sigma}_e(T)$ is a convex curve since $W_e(T)$ is convex and $T \in e(R)$ if and only if $W_e(T) = \tilde{\sigma}_e(T)$. Moreover $T \in e(R)$ implies that $e^{i\theta}T$ also belongs to $e(R)$, so that $e(\sigma - \theta)$ holds by Corollary 3.4. Conversely, if (b) holds, then

$$\mathrm{co}\, \sigma_e(\mathrm{Re}\, e^{i\theta}T) = \mathrm{co}\, \mathrm{Re}\, \sigma_e(e^{i\theta}T) \text{ for all } 0 \leqq \theta \leqq 2\pi,$$

that is, $e(\Sigma - \theta)$ holds, and so $T \in e(C)$ by (f) of Theorem 2.3. In addition $\partial\tilde{\sigma}_e(T)$ is a convex curve, that is, $\tilde{\sigma}_e(T) = \mathrm{co}\, \sigma_e(T)$, and so

$$W_e(T) = \mathrm{co}\, \sigma_e(T) = \tilde{\sigma}_e(T),$$

whence $T \in e(R)$.

(a) $\Leftrightarrow$ (c). Assume (a). Since $T \in e(R)$ if and only if $\partial W_e(T) \subset \sigma_e(T)$ and there exists a compact operator K_0 such that $W_e(T) = \bar{W}(T + K_0)$ by Theorem E, we have

$$\partial W_e(T) = \partial W(T + K_0) \subset \sigma_e(T) \subset \sigma(T + K_0),$$

whence $T + K_0 \in R$ [**17**]. Conversely, assume (c), that is, there exists a compact operator K_0 such that $T + K_0 \in R$. As $R \subset e(R)$ by Lemma 3.1, $T + K_0 \in e(R)$, that is, $T \in e(R)$, so the proof is complete.

Remark 4.1. Compare (b) of Theorem 2.3 with (c) of Theorem 4.2; then it turns out to be the following fact that the corresponding relation to one between (a) and (c) of Theorem 4.2 is not true for essentially convexoid operators.

Remark 4.2. As we have seen, $T \in e(R)$ is characterized in terms of the essential numerical range and the essential spectrum of T as follows: $T \in e(R)$ if and only if $\partial W_e(T) \subset \sigma_e(T)$ and $T \in e(R)$ if and only if $W_e(T) = \tilde{\sigma}_e(T)$ and these characterizations show that $e(R)$ is the subclass of the essentially convexoid operators $e(C)$. That is to say, these characterizations may be considered as a "geometrical characterization" of $e(R)$.

On the other hand, T is an essentially convexoid operator if and only if T satisfies the inequality (2.1) of Theorem 2.2, and (c_2) and (d_2) of Theorem 4.1 are both the special cases of "the equality" in the inequality (2.1) of Theorem 2.2, so that (c_2) and (d_2) also indicate that $e(R)$ is the subclass of $e(C)$ in formula. That is to say, (c_2) and (d_2) may be also considered as a "characterization in formula" of $e(R)$.

5. ρ-essentially convexoid operators. Generalized essential numerical ranges $W_\rho(\pi(T))$ ($\rho \geqq 1$) are defined in [**22**] as follows:

$$W_\rho(\pi(T)) = \cap_\mu \{\lambda : |\lambda - \mu| \leqq w_\rho(\pi(T - \mu))\}.$$

$W_\rho(\pi(T))$ is a compact convex set containing co $\sigma_e(T)$, $W_\infty(\pi(T)) =$ co $\sigma_e(T)$, $W_\beta(\pi(T)) \subset W_\alpha(\pi(T))$ when $1 \leqq \alpha < \beta$, and in particular $W_\rho(\pi(T)) = W_e(T)$ for $1 \leqq \rho \leqq 2$ and $W_\rho(\pi(T)) = \cap_K W_\rho(T + K)$ [**22**].

Definition 5.1. T is called an *operator of class ρ-essentially convexoid operators* (in symbols, $T \in e(\rho - C)$) if $W_\rho(\pi(T)) =$ co $\sigma_e(T)$ ($\rho \geqq 1$). In particular $e(2 - C)$ and $e(1 - C)$ coincide with $e(C)$ together.

The function $w_\rho^0(\pi(T))$ is defined as follows:

$$w_\rho^0(\pi(T)) = \sup \{|\lambda| : \lambda \in W_\rho(\pi(T))\}$$

for $1 \leqq \rho \leqq \infty$. $w_\rho^0(\pi(T))$ satisfies the following properties:

$$r_e(T) \leqq w_\rho^0(\pi(T)) \leqq w_\rho(\pi(T)),$$
$$w_\infty^0(\pi(T)) = r_e(T) \quad \text{and}$$
$$w_\rho^0(\pi(T)) = w_e(T) \quad \text{for } 1 \leqq \rho \leqq 2$$

where $w_e(T)$ denotes the essential numerical radius of T defined by

$$w_e(T) = \sup \{|\lambda| : \lambda \in W_e(T)\}.$$

Moreover $w_\rho(\pi(T))$ and $w_\rho^0(\pi(T))$ together satisfy the absolute homogeneous property.

We show Theorem 5.1 and Theorem 5.2, which are extensions of (d) and (e) of Theorem 2.3.

THEOREM 5.1. *Let* $T \in B(H)$.

(a) *T is ρ-essentially convexoid if and only if*

$$w_\rho^0(\pi(T - \mu)) = r_e(T - \mu) \text{ for all } \mu,$$

where $\rho \geqq 1$.

(b) *T is ρ-essentially convexoid if and only if*

$$w_\rho^0(\pi(T - \mu)) = r_e(T - \mu)$$

for all μ whose absolute values are sufficiently large, where $\rho \geqq 1$.

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Proof. Since $W_\rho(\pi(T))$ is a compact convex set [**22**], then

$$(5.1)\quad W_\rho(\pi(T)) = \bigcap_\mu \{\lambda : |\lambda - \mu| \leqq w\rho^0(\pi(T - \mu))\}$$
$$= \bigcap\{\lambda : |\lambda - \mu| \leqq w_\rho^0(\pi(T - \mu)) \text{ for all } \mu \text{ whose absolute values are sufficiently large}\}$$

because the convex compact set X is the intersection of all the circles containing X and it is also the intersection of all the circles containing X with sufficiently large radii. Similarly we have

$$(5.2)\quad \text{co } \sigma_e(T) = \bigcap_\mu \{\lambda : |\lambda - \mu| \leqq r_e(T - \mu)\}$$
$$= \bigcap \{\lambda : |\lambda - \mu| \leqq r_e(T - \mu) \text{ for all } \mu \text{ whose absolute values are sufficiently large}\}.$$

We have only to show the necessity of (a) and the sufficiency of (b). The proof of the sufficiency of (b) easily follows by (5.1) and (5.2). Conversely, let $T \in e(\rho - C)$, that is, $W_\rho(\pi(T)) = \text{co } \sigma_e(T)$, then

$$W_\rho(\pi(T - \mu)) = \text{co } \sigma_e(T - \mu) \quad \text{for all } \mu,$$

hence

$$w_\rho^0(\pi(T - \mu)) = r_e(T - \mu) \quad \text{for all } \mu,$$

so the proof of the necessity of (a) is complete.

As $W_e(T) = W_\rho(\pi(T))$ $(1 \leqq \rho \leqq 2)$, we have the following result by a method similar to the one of Theorem 5.1.

THEOREM 5.2. *Let* $T \in B(H)$.

(a) $W_e(T) = W_\rho(\pi(T))$ $(1 \leqq \rho \leqq \infty)$ *if and only if*

$$w_e(T - \mu) = w_\rho^0(\pi(T - \mu)) \text{ for all } \mu.$$

(b) $W_e(T) = W_\rho(\pi(T))$ $(1 \leqq \rho \leqq \infty)$ *if and only if*

$$w_e(T - \mu) = w_\rho^0(\pi(T - \mu))$$

for all μ *whose absolute values are sufficiently large.*

We remark that both Theorem 5.1 and Theorem 5.2 imply (d) and (e) of Theorem 2.3 when we put $\rho = 2$ and $\rho = \infty$ respectively.

COROLLARY 5.1. *If* $T - \mu$ *is essentially* ρ*-oid for all* μ *whose absolute values are sufficiently large, then* T *is* ρ*-essentially convexoid, where* $\rho \geqq 1$.

Proof. The Corollary follows by Theorem 5.1 and the relation

$$r_e(T) \leqq w_\rho^0(\pi(T)) \leqq w_\rho(\pi(T)).$$

Remark 5.1. Let $T \in B(H)$. If

$$w_\rho^0(\pi(T - \mu)^{-1}) \leqq 1/d(\mu, \text{co } \sigma_e(T)) \quad \text{for all } \mu \notin \text{co } \sigma_e(T),$$

then T is a ρ-essentially convexoid operator.

We remark that "for all μ" in Remark 5.1 can be replaced by "for all μ whose absolute values are sufficiently large". Remark 5.1 and this remark follow by a method similar to that of [**16**, (1) of Theorem 4] and we omit it.

Remark 5.2. Compare (2.1) of Theorem 2.2 with Remark 5.1. Then this difference is naturally agreeable because $w_\rho^0(\pi(T)) \leqq w_\rho(\pi(T))$ and an essentially convexoid operator is always ρ-essentially convexoid.

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