Proceedings of the Royal Society of Edinburgh, **153**, 1099–1117, *2023*
DOI:10.1017/prm.2022.33

On the spectrum of non-self-adjoint Dirac operators with quasi-periodic boundary conditions

Alexander Makin
Russian Technological University, Prospect Vernadskogo 78, Moscow 119454, Russia (alexmakin@yandex.ru)

(Received 24 August 2021; accepted 27 April 2022)

In this paper, we consider non-self-adjoint Dirac operators on a finite interval with complex-valued potentials and quasi-periodic boundary conditions. Necessary and sufficient conditions for a set of complex numbers to be the spectrum of the indicated problem are established.

Keywords: Dirac operator; inverse problem; quasi-periodic boundary conditions; spectrum

2020 *Mathematics subject classification:* Primary: 34L40
Secondary: 34A55

1. Introduction

The spectral theory of boundary value problems for first-order systems of ordinary differential equations of the form

$$\frac{1}{i}B\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = \lambda y, \quad y = \mathrm{col}(y_1, \ldots, y_n), \tag{1.1}$$

where B is a nonsingular diagonal $n \times n$ matrix,

$$B = \mathrm{diag}(b_1^{-1}I_{n_1}, \ldots, b_r^{-1}I_{n_r}) \in \mathbb{C}^{n\times n}, \quad n = n_1 + \cdots n_r,$$

with complex entries $b_j \neq b_k$, and $Q(x)$ is a potential matrix takes its origin in the paper by Birkhoff and Langer [2]. Afterwards their investigations were developed in many directions. In particular, one of the important classes of inverse spectral problems is the problem of recovering a system of differential equations from spectral data. The solution of such problems is considered in many papers (see [14, 17, 21, 32–38] and the references therein).

The main aim of the present article is to find necessary and sufficient conditions for solvability of inverse spectral problems for one-dimensional Dirac operators on a finite interval under possibly least restrictive assumptions on their potentials. We will consider canonical Dirac system

$$B\mathbf{y}' + V\mathbf{y} = \lambda\mathbf{y}, \tag{1.2}$$

where $\mathbf{y} = \mathrm{col}(y_1(x),\, y_2(x))$,

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

the complex-valued functions $p,\, q \in L_2(0,\, \pi)$, with two-point boundary conditions

$$U(\mathbf{y}) = C\mathbf{y}(0) + D\mathbf{y}(\pi) = 0, \tag{1.3}$$

where

$$C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad D = \begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix},$$

the coefficients a_{ij} are arbitrary complex numbers, and rows of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

are linearly independent.

Inverse self-adjoint problems (1.2), (1.3) have been studied in detail. In the cases of the Dirichlet and the Neumann boundary conditions reconstruction of a continuous potential from two spectra was carried out in [**7**], from one spectrum and the norming constants in [**6**], and from the spectral function in [**16**]. The analogous results for Dirac operator with summable potentials were established in [**1**]. The case of more general separated boundary conditions was considered in [**4**]. In the case of unseparated boundary conditions (including periodic, antiperiodic and quasi-periodic conditions) the indicated problem was solved in [**20**, **22**–**25**]. In non-self-adjoint case the problem of reconstructing the potential $V(x)$ from spectral data is much more complicated, since many methods successfully used to study self-adjoint operators are inapplicable. For example, the characterization of the spectra of the periodic (antiperiodic) problem for operator (1.2) with real coefficients is given in [**18**] in terms of special conformal mappings, which do not exist for complex-valued potentials. The property that the eigenvalues of corresponding Dirichlet problem and Neumann problem are interlaced, which is often used to prove the solvability of the basic equation, loses its meaning in the complex case. Non-self-adjoint inverse problems for system (1.2) with different types of boundary conditions with sufficiently smooth coefficients, which, however, could have singularities were investigated in [**3**, **9**, **27**, **31**].

Questions of uniqueness in inverse problems for operators of type (1.1) on a finite interval were studied in several papers. In particular, the uniqueness of the inverse problem for general Dirac-type systems ($B = B^*$) of order $2n$ was established in [**15**, **16**]. The most complete uniqueness result on general first-order systems (1.1) and (1.2) on a finite interval has been obtained recently in [**17**]. Also the solution to the inverse spectral problem (from the spectral matrix function) for Dirac-type operators on the axis and semiaxis was obtained in [**11**]. New inverse approach based on the A-function concept proposed by Gesztesy and Simon to Schrodinger operator has been recently extended in [**8**] to Dirac systems on the semiaxis.

In the present paper, we consider system (1.2), where complex-valued functions $p,\, q \in L_2(0,\, \pi)$ ($V \in L_2$) with quasi-periodic boundary conditions

$$\mathbf{y}(0) = \mathrm{e}^{it}\mathbf{y}(\pi), \tag{1.4}$$

where $t \in \mathbb{C},\, t \neq \pi k,\; k \in Z$. Section 2 contains some basic facts and definitions related to the studied problems. In § 3 by using a modified version of the Gelfand–Levitan–Marchenko method we prove solvability of the basic equation and establish necessary and sufficient conditions for an entire function to be the characteristic determinant of problems (1.2), (1.4). Furthermore, we obtain necessary and sufficient conditions for a set of complex numbers to be the spectrum of the mentioned problem.

2. Preliminaries

In what follows, we introduce the Euclidean norm $\|f\| = (|f_1|^2 + |f_2|^2)^{1/2}$ for vectors $f = \mathrm{col}(f_1,\, f_2) \in \mathbb{C}^2$, and set $\langle f,\, g\rangle = f_1 g_1 + f_2 g_2$. If W is 2×2 matrix, then we set $\|W\| = \sup_{\|f\|=1} \|Wf\|$ and denote by $L_{2,2}(a,\, b)$ and $L_{2,2}^{2,2}(a,\, b)$, respectively, the spaces of 2-coordinate vector functions $f(t) = \mathrm{col}(f_1(t),\, f_2(t))$ and 2×2 matrix functions $W(t)$ with finite norms

$$\|f\|_{L_{2,2}(a,b)} = \left(\int_a^b \|f(t)\|^2 \mathrm{d}t\right)^{1/2}, \quad \|W\|_{L_{2,2}^{2,2}(a,b)} = \left(\int_a^b \|W(t)\|^2 \mathrm{d}t\right)^{1/2}.$$

The operator $\mathbb{L}\mathbf{y} = B\mathbf{y}' + V\mathbf{y}$ is regarded as a linear operator in the space $L_{2,2}(0,\, \pi)$ with the domain $D(\mathbb{L}) = \{\mathbf{y} \in W_1^1[0,\, \pi] \times W_1^1[0,\, \pi] : \mathbb{L}\mathbf{y} \in L_{2,2}(0,\, \pi),\; U_j(\mathbf{y}) = 0$ $(j = 1,\, 2)\}$.

Denote by

$$E(x, \lambda) = \begin{pmatrix} c_1(x,\lambda) & -s_2(x,\lambda) \\ s_1(x,\lambda) & c_2(x,\lambda) \end{pmatrix} \tag{2.1}$$

the matrix of the fundamental solution system to equation (1.2) with boundary condition $E(0,\, \lambda) = I$, where I is the unit matrix, and by $E_0(x,\, \lambda)$ the fundamental solution system to the equation $B\mathbf{y}' = \lambda\mathbf{y}$ with boundary condition $E_0(0,\, \lambda) = I$. Obviously,

$$E_0(x,\lambda) = \begin{pmatrix} \cos\lambda x & -\sin\lambda x \\ \sin\lambda x & \cos\lambda x \end{pmatrix}.$$

Denote the second column of the matrix $E_0(x,\, \lambda)$ by

$$Y_0(x,\lambda) = \begin{pmatrix} -\sin\lambda x \\ \cos\lambda x \end{pmatrix}.$$

It is well known that the entries of the matrix $E(x,\, \lambda)$ are related by the identity

$$c_1(x,\lambda)c_2(x,\lambda) + s_1(x,\lambda)s_2(x,\lambda) = 1, \tag{2.2}$$

which is valid for any $x,\, \lambda$. The matrix $E(\pi,\, \lambda)$ is called the monodromy matrix of operator $\mathbb{L}\mathbf{y}$. For its entries we introduce the notation $c_j(\lambda) = c_j(\pi,\, \lambda)$,

$s_j(\lambda) = s_j(\pi, \lambda)$, $j = 1, 2$. We denote also the class of entire functions $f(z)$ of exponential type $\leqslant \sigma$ such that $\|f\|_{L_2(R)} < \infty$ by PW_σ. It is known [29] that the functions $c_j(\lambda)$, $s_j(\lambda)$ admit the representation

$$c_j(\lambda) = \cos \pi\lambda + g_j(\lambda), \quad s_j(\lambda) = \sin \pi\lambda + h_j(\lambda), \tag{2.3}$$

where g_j, $h_j \in PW_\pi$, $j = 1, 2$. For functions of type (2.3) the following statement is true:

LEMMA 2.1 [20]. *Entire functions $u(\lambda)$ and $v(\lambda)$ admit the representations*

$$u(\lambda) = \sin \pi\lambda + h(\lambda), \quad v(\lambda) = \cos \pi\lambda + g(\lambda), \tag{2.4}$$

where h, $g \in PW_\pi$, if and only if

$$u(\lambda) = -\pi(\lambda_0 - \lambda) \prod_{n=-\infty}^{\infty}{}' \frac{\lambda_n - \lambda}{n},$$

where $\lambda_n = n + \epsilon_n$, $\{\epsilon_n\} \in l_2$,

$$v(\lambda) = \prod_{n=-\infty}^{\infty} \frac{\lambda_n - \lambda}{n - 1/2},$$

where $\lambda_n = n - 1/2 + \kappa_n$, $\{\kappa_n\} \in l_2(\mathbb{Z})$. (The notation $\prod'$ means that $n = 0$ is missing in the product.)

Notice, that lemma 2.1 is a generalization of lemma 3.4.2 from [19]. In what follows, we will repeatedly use the following statement.

LEMMA 2.2 [30]. *If $f \in PW_\pi$, then for every sequence $\{\lambda_n\}$ $(n \in \mathbb{Z})$ with $\lambda_n - n = o(1)$ as $|n| \to \infty$ and every $R > 0$ the condition*

$$\sum_{n=-\infty}^{\infty} \max_{|t-\lambda_n| \leqslant R} |f(t)|^2 < \infty$$

if fulfilled. In particular,

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2 < \infty.$$

Denote by J_{jk} the determinant composed of the jth and kth columns of the matrix A. Denote also $J_0 = J_{12} + J_{34}$, $J_1 = J_{14} - J_{23}$, $J_2 = J_{13} + J_{24}$.

DEFINITION 2.3. *The boundary conditions (1.3) are called regular if*

$$J_1^2 + J_2^2 = (J_{14} + J_{32})^2 + (J_{13} + J_{24})^2 \neq 0, \tag{2.5}$$

and strongly regular if additionally

$$J_0^2 \neq J_1^2 - J_2^2. \tag{2.6}$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

DEFINITION 2.4. *The boundary conditions (1.3) are called regular but not strongly regular if (2.5) holds but (2.6) fails, i.e.*

$$J_0^2 = J_1^2 - J_2^2. \tag{2.7}$$

It is well known (see, for instance, [**5**]) that boundary conditions (1.4) are strongly regular, the characteristic determinant of problem (1.2), (1.4) can be reduced to the form

$$\Delta(\lambda) = -\cos t + \frac{c_1(\lambda) + c_2(\lambda)}{2}, \tag{2.8}$$

and the eigenvalues are specified by the asymptotic formulas

$$\lambda_n^{\pm} = 2n \pm \frac{t}{\pi} + \varepsilon_n^{\pm}, \tag{2.9}$$

where $\{\varepsilon_n^{\pm}\} \in l_2$, $n \in \mathbb{Z}$. Further $\Gamma(z, r)$ denotes a disc of radius r centred at the point z.

Next, we establish the necessary and sufficient conditions that an entire function must satisfy in order to be the characteristic determinant of some problem (1.2), (1.4). Then, we give an intrinsic description of sequences which are spectrum of operator (1.2), (1.4).

3. Main results

3.1. Characteristic determinant

THEOREM 3.1. *For a function $U(\lambda)$ to be the characteristic determinant of problem* (1.2), (1.4), *it is necessary and sufficient that it can be represented in the form*

$$U(\lambda) = -\cos t + \cos \pi\lambda + f(\lambda),$$

where $f \in PW_\pi$, and

$$\sum_{n=-\infty}^{\infty} |f(n)| < \infty. \tag{3.1}$$

Proof. Necessity. Assume that function U is the characteristic determinant, i.e. $U(\lambda) = \Delta(\lambda)$. Evidently, relations (2.3), (2.8) imply that $f \in PW_\pi$. To check inequality (3.1) we consider the monodromy matrix of problem (1.2), (1.4). Let the corresponding function $s_2(\lambda)$ have the roots λ_n, hence by [**30**, lemma 2.2],

$$\lambda_n = n + \delta_n, \tag{3.2}$$

where $\{\delta_n\} \in l_2$, $n \in \mathbb{Z}$. Relation (2.3) implies

$$c_j(\lambda_n) = \cos \pi\lambda_n + g_j(\lambda_n), \tag{3.3}$$

it follows from (2.3) and lemma 2.2 that

$$\sum_{n=-\infty}^{\infty} |g_j(\lambda_n)|^2 < \infty. \tag{3.4}$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

Denote

$$\chi(\lambda) = U(\lambda) + \cos t = \cos \pi\lambda + f(\lambda). \tag{3.5}$$

By virtue of (2.8),

$$c_1(\lambda_n) + c_2(\lambda_n) = 2\chi(\lambda_n).$$

It follows from (2.2) that $c_1(\lambda_n)c_2(\lambda_n) = 1$, consequently the numbers $c_1(\lambda_n)$, $c_2(\lambda_n)$ are the roots of the quadratic equation

$$w^2 - 2\chi(\lambda_n)w + 1 = 0. \tag{3.6}$$

Therefore we have

$$\begin{aligned} c_1(\lambda_n), c_2(\lambda_n) &= \chi(\lambda_n) \pm \sqrt{\chi^2(\lambda_n) - 1} \\ &= \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{(\cos \pi\lambda_n + f(\lambda_n))^2 - 1} \\ &= \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{\cos^2 \pi\lambda_n + 2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - 1} \\ &= \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - \sin^2 \pi\lambda_n}. \end{aligned} \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$(g_1(\lambda_n) - f(\lambda_n))^2 = 2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - \sin^2 \pi\lambda_n,$$

hence,

$$2\cos \pi\lambda_n f(\lambda_n) = g_1^2(\lambda_n) - 2g_1(\lambda_n)f(\lambda_n) + \sin^2 \pi\delta_n. \tag{3.8}$$

It follows from (3.2) that for all sufficiently large $|n|$ the inequality $|\cos \pi\lambda_n| > 1/2$ holds. This, together with (3.2), (3.4), and lemma 2.2 implies

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)| < \infty. \tag{3.9}$$

Since $f' \in PW_\pi$, then

$$|f(n)| \leqslant |f(\lambda_n)| + |f(n) - f(\lambda_n)| \leqslant |f(\lambda_n)| + |\delta_n||\tau_n| \leqslant |f(\lambda_n)| + (|\delta_n|^2 + |\tau_n|^2)/2,$$

where

$$\tau_n = \max_{\lambda \in \Gamma(n, |\delta_n|)} |f'(\lambda)|.$$

By lemma 2.2, $\{\tau_n\} \in l_2$. This and (3.9) imply (3.1).

Sufficiency. Let $f \in PW_\pi$ satisfy condition (3.1). It follows from the Paley–Wiener theorem and [**19**, lemma 1.3.1] that

$$\lim_{|\lambda| \to \infty} \mathrm{e}^{-\pi|\mathrm{Im}\,\lambda|} f(\lambda) = 0, \tag{3.10}$$

hence there exists a positive integer N_0 large enough that $|f(\lambda)| < 1/100$ if $\mathrm{Im}\,\lambda = 0$, $|\mathrm{Re}\,\lambda| \geqslant N_0$. Let λ_n $(n \in \mathbb{Z})$ be a strictly monotone increasing sequence of real

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

numbers such that for any $n \neq 0$ $\lambda_n = \lambda_{-n}$, $|\lambda_n - (N_0 + 1/2)| < 1/100$ if $0 \leqslant n \leqslant N_0$, and $\lambda_n = n$ if $n > N_0$. Denote

$$s(\lambda) = -\pi(\lambda_0 - \lambda) \prod_{n=-\infty}^{\infty}{}' \frac{\lambda_n - \lambda}{n}. \tag{3.11}$$

It follows from lemma 2.1 that

$$s(\lambda) = \sin \pi\lambda + h(\lambda), \tag{3.12}$$

where $h \in PW_\pi$, hence,

$$|s(\lambda)| \geqslant C_1 \mathrm{e}^{\pi|\operatorname{Im}\lambda|} \tag{3.13}$$

if $|\operatorname{Im}\lambda| \geqslant M$, where M is sufficiently large. It follows from (3.11) that

$$\dot{s}(\lambda_0) = \pi \prod_{n=-\infty}^{\infty}{}' \frac{\lambda_n - \lambda_0}{n} > 0.$$

One can readily see that the inequality $\dot{s}(\lambda_n)\dot{s}(\lambda_{n+1}) < 0$ holds for all $n \in \mathbb{Z}$. It follows from two last inequalities that

$$(-1)^n \dot{s}(\lambda_n) > 0. \tag{3.14}$$

Relation (3.12) and lemma 2.2 imply that

$$\dot{s}(\lambda_n) = \pi(-1)^n + \tau_n, \tag{3.15}$$

where $\{\tau_n\} \in l_2$, hence,

$$\frac{1}{\dot{s}(\lambda_n)} = \frac{(-1)^n}{\pi} + \sigma_n, \tag{3.16}$$

where $\{\sigma_n\} \in l_2$. Equation (3.6) has the roots

$$\begin{aligned} c_n^{\pm} &= \chi(\lambda_n) \pm \sqrt{\chi^2(\lambda_n) - 1} = \cos\pi\lambda_n + f(\lambda_n) \pm \sqrt{(\cos\pi\lambda_n + f(\lambda_n))^2 - 1} \\ &= \cos\pi\lambda_n + f(\lambda_n) \pm \sqrt{\cos^2\pi\lambda_n + 2\cos\pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - 1} \\ &= \cos\pi\lambda_n + f(\lambda_n) \pm \sqrt{2\cos\pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - \sin^2\pi\lambda_n}. \end{aligned} \tag{3.17}$$

It follows from (3.17) that if $0 < |n| \leqslant N_0$ the numbers c_n^+ are contained within the disc $\Gamma(i, 1/10)$, the numbers c_n^- are contained within the disc $\Gamma(-i, 1/10)$, and if $|n| > N_0$ the numbers $c_n^{\pm}$ are contained within the disc $\Gamma(1, 1/10)$ for even n, the numbers $c_n^{\pm}$ are contained within the disc $\Gamma(-1, 1/10)$ for odd n. Denote $c_n = c_n^+$ for even n and $c_n = c_n^-$ for odd n. Denote also

$$z_n = \frac{c_n}{\dot{s}(\lambda_n)}.$$

It follows from (3.14) that the numbers z_n lie strictly above the line $l : \operatorname{Im}\lambda = -\operatorname{Re}\lambda$.

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

Evidently,

$$\lambda_n = n + \rho_n, \tag{3.18}$$

where $\{\rho_n\} \in l_2$. It follows from (3.17) and (3.18) that

$$c_n = (-1)^n + \vartheta_n, \tag{3.19}$$

where $\{\vartheta_n\} \in l_2$. Let $\beta_n = c_n - \cos \pi\lambda_n$, then $\{\beta_n\} \in l_2$. Let us consider the function

$$g(\lambda) = s(\lambda) \sum_{n=-\infty}^{\infty} \frac{\beta_n}{\dot{s}(\lambda_n)(\lambda - \lambda_n)}.$$

By [**12**, p. 120] the function $g \in PW_\pi$ and $g(\lambda_n) = \beta_n$. Denote $c(\lambda) = \cos \pi\lambda + g(\lambda)$, then $c(\lambda_n) = c_n \neq 0$, hence the functions $s(\lambda)$ and $c(\lambda)$ have disjoint zero sets.

Denote

$$F(x,t) = \sum_{n=-\infty}^{\infty} \left(z_n Y_0(x,\lambda_n) Y_0^T(t,\lambda_n) - \frac{1}{\pi} Y_0(x,n) Y_0^T(t,n) \right).$$

It follows from [**29**] that

$$\|F(\cdot,x)\|_{L_{2,2}^{2,2}(0,\pi)} + \|F(x,\cdot)\|_{L_{2,2}^{2,2}(0,\pi)} < C_2,$$

where C_2 not depending on x.

Using the properties of the numbers z_n established above, we prove that for every $x \in [0, \pi]$ the homogeneous equation

$$\mathbf{f}^T(t) + \int_0^x \mathbf{f}^T(s) F(s,t) \mathrm{d}s = 0, \tag{3.20}$$

where $\mathbf{f}(t) = \mathrm{col}(f_1(t),\, f_2(t))$, $\mathbf{f} \in L_{2,2}(0,\, x)$, $\mathbf{f}(t) = 0$ if $x < t \leqslant \pi$ has the trivial solution only.

Multiplying equation (3.20) by $\overline{\mathbf{f}^T(t)}$ and integrating the resulting equation over segment $[0,\, x]$, we obtain

$$\|\mathbf{f}\|^2_{L_{2,2}(0,x)} + \int_0^x \left\langle \int_0^x \mathbf{f}^T(s) F(s,t) \mathrm{d}s, \mathbf{f}^T(t) \right\rangle \mathrm{d}t = 0. \tag{3.21}$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

Simple computations show

$$
\begin{aligned}
&\mathbf{f}^T(s)F(s,t)\\
&= (f_1(s), f_2(s)) \sum_{n=-\infty}^{\infty} \left(z_n \begin{pmatrix} \sin\lambda_n s \sin\lambda_n t & -\sin\lambda_n s\cos\lambda_n t \\ -\cos\lambda_n s \sin\lambda_n t & \cos\lambda_n s\cos\lambda_n t \end{pmatrix} \right.\\
&\quad \left. - \frac{1}{\pi}\begin{pmatrix} \sin ns \sin nt & -\sin ns\cos nt \\ -\cos ns \sin nt & \cos ns\cos nt \end{pmatrix}\right)\\
&= \sum_{n=-\infty}^{\infty} \Big\{ z_n[f_1(s)\sin\lambda_n s\sin\lambda_n t - f_2(s)\cos\lambda_n s\sin\lambda_n t,\\
&\quad - f_1(s)\sin\lambda_n s\cos\lambda_n t + f_2(s)\cos\lambda_n s\cos\lambda_n t]\\
&\quad - \frac{1}{\pi}[f_1(s)\sin ns\sin nt - f_2(s)\cos ns\sin nt,\\
&\quad - f_1(s)\sin ns\cos nt + f_2(s)\cos ns\cos nt] \Big\}\\
&= \sum_{n=-\infty}^{\infty} \Big\{ z_n[f_1(s)\sin\lambda_n s\sin\lambda_n t - f_2(s)\cos\lambda_n s\sin\lambda_n t]\\
&\quad - \frac{1}{\pi}[f_1(s)\sin ns\sin nt - f_2(s)\cos ns\sin nt],\\
&\quad z_n[-f_1(s)\sin\lambda_n s\cos\lambda_n t + f_2(s)\cos\lambda_n s\cos\lambda_n t]\\
&\quad - \frac{1}{\pi}[-f_1(s)\sin ns\cos nt + f_2(s)\cos ns\cos nt] \Big\},
\end{aligned} \tag{3.22}
$$

therefore, substituting the right-hand side of (3.22) into the second term in the left-hand side of (3.21), transforming the iterated integrals into products of integrals and using the reality of all numbers λ_n, we obtain

$$
\begin{aligned}
&\int_0^x \left\langle \int_0^x \mathbf{f}^T(s)F(s,t)\mathrm{d}s, \mathbf{f}^T(t)\right\rangle \mathrm{d}t\\
&= \sum_{n=-\infty}^{\infty} \int_0^x \left(\int_0^x \{z_n[f_1(s)\sin\lambda_n s\sin\lambda_n t - f_2(s)\cos\lambda_n s\sin\lambda_n t]\right.\\
&\quad \left. - \frac{1}{\pi}[f_1(s)\sin ns\sin nt - f_2(s)\cos ns\sin nt]\}\mathrm{d}s\right)\overline{f_1(t)}\mathrm{d}t\\
&\quad + \sum_{n=-\infty}^{\infty} \int_0^x \left(\int_0^x \{z_n[-f_1(s)\sin\lambda_n s\cos\lambda_n t + f_2(s)\cos\lambda_n s\cos\lambda_n t]\right.\\
&\quad \left. - \frac{1}{\pi}[-f_1(s)\sin ns\cos nt + f_2(s)\cos ns\cos nt]\}\mathrm{d}s\right)\overline{f_2(t)}\mathrm{d}t\\
&= \sum_{n=-\infty}^{\infty} \left(z_n \int_0^x [f_1(s)\sin\lambda_n s - f_2(s)\cos\lambda_n s]\mathrm{d}s \int_0^x \sin\lambda_n t\overline{f_1(t)}\mathrm{d}t \right.
\end{aligned}
$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

$$- \frac{1}{\pi}\int_0^x [f_1(s)\sin ns - f_2(s)\cos ns]\mathrm{d}s \int_0^x \sin nt\overline{f_1(t)}\mathrm{d}t\Big)$$

$$+ \sum_{n=-\infty}^{\infty}\Big(z_n\int_0^x [-f_1(s)\sin\lambda_n s + f_2(s)\cos\lambda_n s]\mathrm{d}s \int_0^x \cos\lambda_n t\overline{f_2(t)}\mathrm{d}t$$

$$- \frac{1}{\pi}\int_0^x [-f_1(s)\sin ns + f_2(s)\cos ns]\mathrm{d}s \int_0^x \cos nt\overline{f_2(t)}\mathrm{d}t\Big)$$

$$= \sum_{n=-\infty}^{\infty} z_n\Big(\int_0^x [f_1(s)\sin\lambda_n s - f_2(s)\cos\lambda_n s]\mathrm{d}s \int_0^x \sin\lambda_n t\overline{f_1(t)}\mathrm{d}t$$

$$+ \int_0^x [-f_1(s)\sin\lambda_n s + f_2(s)\cos\lambda_n s]\mathrm{d}s \int_0^x \cos\lambda_n t\overline{f_2(t)}\mathrm{d}t\Big)$$

$$- \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\Big(\int_0^x [f_1(s)\sin ns - f_2(s)\cos ns]\mathrm{d}s \int_0^x \sin nt\overline{f_1(t)}\mathrm{d}t$$

$$+ \int_0^x [-f_1(s)\sin ns + f_2(s)\cos ns]\mathrm{d}s \int_0^x \cos nt\overline{f_2(t)}\mathrm{d}t\Big)$$

$$= \sum_{n=-\infty}^{\infty} z_n\Big(\int_0^x [f_1(t)\sin\lambda_n t - f_2(t)\cos\lambda_n t]\mathrm{d}t \int_0^x \sin\lambda_n t\overline{f_1(t)}\mathrm{d}t$$

$$+ \int_0^x [-f_1(t)\sin\lambda_n t + f_2(t)\cos\lambda_n t]\mathrm{d}t \int_0^x \cos\lambda_n t\overline{f_2(t)}\mathrm{d}t\Big)$$

$$- \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\Big(\int_0^x [f_1(t)\sin nt - f_2(t)\cos nt]\mathrm{d}t \int_0^x \sin nt\overline{f_1(t)}\mathrm{d}t$$

$$+ \int_0^x [-f_1(t)\sin nt + f_2(t)\cos nt]\mathrm{d}t \int_0^x \cos nt\overline{f_2(t)}\mathrm{d}t\Big)$$

$$= \sum_{n=-\infty}^{\infty} z_n\int_0^x [f_1(t)\sin\lambda_n t - f_2(t)\cos\lambda_n t]\mathrm{d}t \int_0^x [\overline{f_1(t)}\sin\lambda_n t - \overline{f_2(t)}\cos\lambda_n t]\mathrm{d}t$$

$$- \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\int_0^x [f_1(t)\sin nt - f_2(t)\cos nt]\mathrm{d}t \int_0^x [\overline{f_1(t)}\sin nt - \overline{f_2(t)}\cos nt]\mathrm{d}t$$

$$= \sum_{n=-\infty}^{\infty} z_n\left|\int_0^x \langle\mathbf{f}(t), Y_0(t,\lambda_n)\rangle\mathrm{d}t\right|^2 - \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\left|\int_0^x \langle\mathbf{f}(t), Y_0(t,n)\rangle\mathrm{d}t\right|^2. \qquad (3.23)$$

It is well known that the function system $\{\frac{1}{\sqrt{\pi}}Y_0(t,\,n)\}$ $(n\in\mathbb{Z})$ is an orthonormal basis in $L_{2,2}(0,\,\pi)$, hence it follows from the Parseval equality that

$$\|\mathbf{f}\|^2_{L_{2,2}(0,x)} = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\left|\int_0^x \langle\mathbf{f}(t), Y_0(t,n)\rangle\mathrm{d}t\right|^2. \qquad (3.24)$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

It follows from (3.21),(3.23) and (3.24) that

$$\sum_{n=-\infty}^{\infty} z_n \left| \int_0^x \langle \mathbf{f}(t), Y_0(t, \lambda_n) \rangle \mathrm{d}t \right|^2 = 0.$$

Since all the numbers z_n are located strictly in the same half-plane relative to a line which passes through the origin, we see that

$$\int_0^x \langle \mathbf{f}(t), Y_0(t, \lambda_n) \rangle \mathrm{d}t = 0$$

for all $n \in \mathbb{Z}$. It follows from (3.12) that the function $s(\lambda)$ is a sin-type function [**13**], therefore [**1**, lemma 5.3], the system $Y_0(t,\, \lambda_n)$ is a Riesz basis of $L_{2,2}(0,\, \pi)$, hence the system $Y_0(t,\, \lambda_n)$ is complete in $L_{2,2}(0,\, \pi)$, it follows now that $\mathbf{f}(t) \equiv 0$.

By [**29**, theorem 5.1], the functions $c(\lambda)$ and $-s(\lambda)$ are the entries of the first line of the monodromy matrix

$$\tilde{E}(\pi, \lambda) = \begin{pmatrix} \tilde{c}_1(\pi, \lambda) & -\tilde{s}_2(\pi, \lambda) \\ \tilde{s}_1(\pi, \lambda) & \tilde{c}_2(\pi, \lambda) \end{pmatrix}$$

for problem (1.2), (1.4) with a potential $\tilde{V} \in L_2$, i.e.

$$c(\lambda) = \tilde{c}_1(\pi, \lambda), s(\lambda) = \tilde{s}_2(\pi, \lambda). \tag{3.25}$$

The corresponding characteristic determinant

$$\tilde{\Delta}(\lambda) = -\cos t + (\tilde{c}_1(\pi, \lambda) + \tilde{c}_2(\pi, \lambda))/2 = -\cos t + \cos \pi\lambda + \tilde{f}(\lambda),$$

where $\tilde{f} \in PW_\pi$. It follows from (2.2), (3.5), (3.6), (3.25) that

$$\begin{aligned} \tilde{\Delta}(\lambda_n) &= -\cos t + (\tilde{c}_1(\pi, \lambda_n) + \tilde{c}_2(\pi, \lambda_n))/2 \\ &= -\cos t + \left(\tilde{c}_1(\pi, \lambda_n) + \frac{1}{\tilde{c}_1(\pi, \lambda_n)} \right)/2 = -\cos t + \left(c(\lambda_n) + \frac{1}{c(\lambda_n)} \right)/2 \\ &= -\cos t + \chi(\lambda_n) = U(\lambda_n). \end{aligned}$$

This implies that the function

$$\Phi(\lambda) = \frac{U(\lambda) - \tilde{\Delta}(\lambda)}{s(\lambda)} = \frac{f(\lambda) - \tilde{f}(\lambda)}{s(\lambda)}$$

is an entire function in the whole complex plane. Since by the Paley–Wiener theorem

$$|f(\lambda) - \tilde{f}(\lambda)| < C_3 \mathrm{e}^{\pi|\mathrm{Im}\,\lambda|}, \tag{3.26}$$

then by (3.13) $|\Phi(\lambda)| \leqslant C_4$ if $|\mathrm{Im}\,\lambda| \geqslant M$. We denote by Ω the set

$$\Gamma(N_0 + 1/2, 1/10) \bigcup \Gamma(-N_0 - 1/2, 1/10)) \bigcup \Gamma_{|n|>N_0}(n, 1/10).$$

Since the function $s(\lambda)$ is a sin-type function [**13**], then $|s(\lambda)| > C_5 > 0$ if $\lambda \notin \Omega$. From this inequality, (3.26) and the maximum principle we obtain that $|\Phi(\lambda)| < C_6$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

in the strip $|\operatorname{Im}\lambda| \leqslant M$, hence the function $\Phi(\lambda)$ is bounded in the whole complex plane and, by virtue of Liouville theorem, it is a constant. Let $|\operatorname{Im}\lambda| = M$, then it follows from (3.10) that $\lim_{|\lambda|\to\infty}(f(\lambda) - \tilde{f}(\lambda)) = 0$, consequently $\Phi(\lambda) \equiv 0$, therefore $U(\lambda) \equiv \tilde{\Delta}(\lambda)$. □

3.2. Spectrum

THEOREM 3.2. *For a set* Λ *to be the spectrum of some Dirac operator* (1.2), (1.4) *with a complex-valued potential* $V \in L_2(0, \pi)$ *it is necessary and sufficient that it consists of two sequences of eigenvalues* $\lambda_n^{\pm}$ *satisfying condition* (2.9) *and the inequality*

$$\sum_{k=-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \left(\frac{\varepsilon_n^+}{2n + t/\pi - k} + \frac{\varepsilon_n^-}{2n - t/\pi - k} \right) \right| < \infty. \tag{3.27}$$

Proof. Sufficiency. Let two sequences $\lambda_n^{\pm}$ satisfy conditions (2.9) and (3.27). Evidently, there exists a constant M such that

$$\sup |\varepsilon_n^{\pm}| < M, \qquad \sum_{n=-\infty}^{\infty} |\varepsilon_n^{\pm}|^2 < M. \tag{3.28}$$

It is well known that

$$\sin \pi\lambda = \pi\lambda \prod_{n=-\infty}^{\infty}{}' \frac{n-\lambda}{n} = \pi\lambda \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{\lambda}{n}\right),$$

therefore the function $\Delta_0(\lambda) = \cos \pi\lambda - \cos t$ has the representation

$$\Delta_0(\lambda) = -2 \sin \frac{\pi\lambda + t}{2} \sin \frac{\pi\lambda - t}{2} = -\frac{\pi^2(\lambda^2 - (t/\pi)^2)}{2} \prod_{n=-\infty}^{\infty}{}' \frac{(2n + t/\pi - \lambda)(2n - t/\pi - \lambda)}{4n^2}.$$

Denote

$$\Delta(\lambda) = -\frac{\pi^2}{2}(\lambda_0^+ - \lambda)(\lambda_0^- - \lambda) \prod_{n=-\infty}^{\infty}{}' \frac{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}{4n^2}.$$

Evidently,

$$|\Delta_0(\lambda)| < c_1 \mathrm{e}^{\pi|\operatorname{Im}\lambda|}. \tag{3.29}$$

Let $f(\lambda) = \Delta(\lambda) - \Delta_0(\lambda)$. Investigation of the properties of the function $f(\lambda)$ is based on the following propositions.

PROPOSITION 3.3. *The function* $f(\lambda)$ *is an entire function of exponential type not exceeding* π.

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

Denote Γ the union of the discs $\Gamma(2n \pm t/\pi,\, 1/4)$ $(n \in Z)$. If $\lambda \notin \Gamma$, then

$$f(\lambda) = -\Delta_0(\lambda)\left(1 - \frac{\Delta(\lambda)}{\Delta_0}\right) = -\Delta_0(\lambda)(1 - \phi(\lambda)), \tag{3.30}$$

where

$$\phi(\lambda) = \frac{(\lambda_0^+ - \lambda)(\lambda_0^- - \lambda)}{(\lambda^2 - (t/\pi)^2)} \prod_{n=-\infty}^{\infty}{}' \frac{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}{(2n + t/\pi - \lambda)(2n - t/\pi - \lambda)}$$
$$= \prod_{n=-\infty}^{\infty} \left(1 + \frac{\varepsilon_n^+}{2n + t/\pi - \lambda}\right)\left(1 + \frac{\varepsilon_n^-}{2n - t/\pi - \lambda}\right).$$

Let us estimate the function $\phi(\lambda)$. Denote $\alpha_n^{\pm}(\lambda) = \frac{\varepsilon_n^{\pm}}{2n \pm t/\pi - \lambda}$. It follows from (3.28) that

$$\sum_{n=-\infty}^{\infty} (|\alpha_n^+(\lambda)| + |\alpha_n^-(\lambda)|) \leqslant \sum_{n=-\infty}^{\infty} (|\varepsilon_n^+|^2 + |\varepsilon_n^-|^2 + |2n + t/\pi - \lambda|^{-2}$$
$$+ |2n - t/\pi - \lambda|^{-2})/2 < c_3. \tag{3.31}$$

It is easy to see that for all $|n| > n_0$, where n_0 is a sufficiently large number, we have

$$|\alpha_n^{\pm}(\lambda)| < 1/4 \tag{3.32}$$

for all $\lambda \notin \Gamma$. If $|n| \leqslant n_0$, then inequality (3.32) holds for all sufficiently large $|\lambda|$, hence inequality (3.32) is valid for all $|\lambda| \geqslant C_0$. It follows from (3.31), (3.32) and elementary inequality

$$|\ln(1 + z)| \leqslant 2|z|, \tag{3.33}$$

which is valid if $|z| \leqslant 1/4$ that

$$\sum_{n=-\infty}^{\infty} (|\ln(1 + \alpha_n^+(\lambda)| + |\ln(1 + \alpha_n^-(\lambda))| \leqslant c_4.$$

Here and throughout the following, we choose the branch of $\ln(1 + z)$ that is zero for $z = 0$. In view of [**10**, p. 433], we rewrite the last relation in the form

$$|\phi(\lambda)| \leqslant \prod_{n=-\infty}^{\infty} |1 + \alpha_n^+(\lambda)|1 + \alpha_n^-(\lambda)| \leqslant \mathrm{e}^{c_4}. \tag{3.34}$$

It follows from (3.29), (3.30), (3.34) that

$$|f(\lambda)| < c_5 \mathrm{e}^{\pi|\mathrm{Im}\,\lambda|} \tag{3.35}$$

outside the domain $\Gamma' = \Gamma \cup \{|\lambda| < C_0\}$.

Denote $x_0^{\pm} = |\mathrm{Re}\, t/\pi| \pm 1/3$, $T^+ = \cup_n [2n + |\mathrm{Re}\, t/\pi| - 1/4,\, 2n + |\mathrm{Re}\, t/\pi| + 1/4]$, $T^- = \cup_n [2n - |\mathrm{Re}\, t/\pi| - 1/4,\, 2n - |\mathrm{Re}\, t/\pi| + 1/4]$. It easy to see that the points

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

$x_0^\pm \notin T^+$ and at least one of these point does not belong T^- since $x_0^+ - x_0^- = 2/3 > 1/2$. Denote this point by x_0 then all points $x_0 + 2k$, $k \in Z$ lie outside the set $T^+ \cup T^-$.

In particular, inequality (3.35) is valid if λ belongs lines $\operatorname{Im}\lambda = \pm\hat{C}_0$, where $\hat{C}_0 = C_0 + |t|$, and vertical segments with vertexes $(x_0 + 2k, -\hat{C}_0)$, $(x_0 + 2k, \hat{C}_0)$, $|2k-1| > C_0$, $k \in \mathbb{Z}$. By the maximum principle, inequality (3.35) holds in whole complex plane, hence the function $f(\lambda)$ is an entire function of exponential type not exceeding π.

Proposition 3.4. *The function f belongs to PW_π.*

Denote

$$W(\lambda) = \ln\phi(\lambda) = \sum_{n=-\infty}^{\infty} (\ln(1+\alpha_n^+(\lambda)) + \ln(1+\alpha_n^-(\lambda)),$$

then

$$f(\lambda) = -\Delta_0(\lambda)\left(1 - \mathrm{e}^{W(\lambda)}\right). \tag{3.36}$$

Let us estimate the function $W(\lambda)$ if $\lambda \notin \Gamma'$. It follows from (3.28), (3.32), (3.33) that

$$\begin{aligned}
|W(\lambda)| &\leqslant \sum_{n=-\infty}^{\infty} (|\ln(1+\alpha_n^+(\lambda)| + |\ln(1+\alpha_n^-(\lambda))| \\
&\leqslant \frac{2M}{|\lambda|} + \sum_{n=-\infty}^{\infty}\left(\frac{|\varepsilon_n^+|^2 + |\varepsilon_n^-|^2}{10M} + \frac{10M}{|2n-\lambda|^2}\right) \\
&\leqslant \frac{2M}{|\lambda|} + \frac{1}{10} + 20M\sum_{n=0}^{\infty}\frac{1}{n^2 + |\operatorname{Im}\lambda|^2} \\
&\leqslant \frac{2M}{|\lambda|} + \frac{1}{10} + 20M\left(\frac{2}{|\operatorname{Im}\lambda|^2} + \int_1^\infty \frac{\mathrm{d}x}{x^2 + |\operatorname{Im}\lambda|^2}\right) \\
&\leqslant \frac{2M}{|\operatorname{Im}\lambda|} + \frac{1}{10} + 20M\left(\frac{2}{|\operatorname{Im}\lambda|^2} + \frac{\pi}{2|\operatorname{Im}\lambda|}\right).
\end{aligned}$$

The last inequality implies that

$$|W(\lambda)| < 1/4 \tag{3.37}$$

if $|\operatorname{Im}\lambda| \geqslant M_1 = 10(\pi + 2 + 22M) + \hat{C}_0$. Then from the trivial inequality

$$\frac{|z|}{2} \leqslant |1 - \mathrm{e}^z| \leqslant 2|z|, \tag{3.38}$$

which holds for $|z| \leqslant 1/4$, we obtain the inequality $|1 - \mathrm{e}^{W(\lambda)}| \leqslant 2|W(\lambda)|$, which, together with (3.29) and (3.36) implies that

$$|f(\lambda)| \leqslant c_6|W(\lambda)| \tag{3.39}$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

for $\lambda \in l$, where l is the line $\operatorname{Im}\lambda = M_1$. Let us prove that

$$\int_l |W(\lambda)|^2 \mathrm{d}\lambda < \infty. \tag{3.40}$$

From the elementary inequality $|\ln(1+z) - z| \leqslant |z|^2$ true for $|z| \leqslant 1/2$, we obtain

$$\ln(1+z) - z = r(z),$$

where $|r(z)| \leqslant |z|^2$, hence,

$$W(\lambda) = S_1(\lambda) + S_2(\lambda), \tag{3.41}$$

where

$$S_1(\lambda) = \sum_{n=-\infty}^{\infty} (\alpha_n^+(\lambda) + \alpha_n^-(\lambda)),$$
$$|S_2(\lambda)| \leqslant \sum_{n=-\infty}^{\infty} (|\alpha_n^+(\lambda)|^2 + |\alpha_n^-(\lambda)|^2).$$

Evidently,

$$|W(\lambda)| \leqslant |S_1(\lambda)| + |S_2(\lambda)|. \tag{3.42}$$

Set

$$I_m = \int_l |S_m(\lambda)|^2 \mathrm{d}\lambda$$

($m = 1, 2$). First consider the integral I_1. It follows from [**28**, p. 221] that

$$\begin{aligned} I_1 &= \int_l \left| \sum_{n=-\infty}^{\infty} \left(\frac{\varepsilon_n^+}{2n + t/\pi - \lambda} + \frac{\varepsilon_n^-}{2n - t/\pi - \lambda} \right) \right|^2 \mathrm{d}\lambda \\ &\leqslant 2 \left(\int_l \left| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_n^+}{2n + t/\pi - \lambda} \right|^2 \mathrm{d}\lambda + \int_l \left| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_n^+}{2n - t/\pi - \lambda} \right|^2 \mathrm{d}\lambda \right) \\ &= 2 \left(\int_{l^+} \left| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_n^+}{2n - \lambda} \right|^2 \mathrm{d}\lambda + \int_{l^-} \left| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_n^-}{2n - \lambda} \right|^2 \mathrm{d}\lambda \right) < \infty, \end{aligned} \tag{3.43}$$

where $l^\pm$ are the lines $\operatorname{Im}\lambda = M_1 \mp t/\pi$ correspondingly.

It is readily seen that

$$|S_2(\lambda)| \leqslant \sum_{n=-\infty}^{\infty} \frac{|\varepsilon_n^+|^2}{|2n + t/\pi - \lambda|^2} + \sum_{n=-\infty}^{\infty} \frac{|\varepsilon_n^-|^2}{|2n - t/\pi - \lambda|^2} \leqslant c_7,$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

hence,

$$I_2 \leqslant c_7 \int_l \left(\sum_{n=-\infty}^{\infty} \frac{|\varepsilon_n^+|^2}{|2n + t/\pi - \lambda|^2} + \sum_{n=-\infty}^{\infty} \frac{|\varepsilon_n^-|^2}{|2n - t/\pi - \lambda|^2} \right) \mathrm{d}\lambda$$
$$\leqslant c_8 \sum_{n=-\infty}^{\infty} (|\varepsilon_n^+|^2 + |\varepsilon_n^-|^2) \int_{\tilde{l}} \frac{\mathrm{d}\lambda}{|2n - \lambda|^2} < c_9 \sum_{n=-\infty}^{\infty} (|\varepsilon_n^+|^2 + |\varepsilon_n^-|^2) < c_{10}, \tag{3.44}$$

where $\tilde{l} = l^+ \cup l^-$. Relations (3.42)–(3.44) imply (3.40). It follows from (3.39), (3.40) and [**26**, p. 115] that

$$\int_R |f(\lambda)|^2 \mathrm{d}\lambda < \infty. \tag{3.45}$$

Proposition 3.5. *The function $f(\lambda)$ satisfies condition (3.1).*

Let $k \in \mathbb{Z}$. Obviously,

$$0 < c_{11} < |\Delta_0(k)| < c_{12}. \tag{3.46}$$

Denote

$$\epsilon_n = \max(|\varepsilon_n^+|, |\varepsilon_n^-|).$$

There exists a number $n_0 > 0$ such that

$$\sum_{|n|>n_0} \epsilon_n^2 < 1/1000,$$

and for any $|n| > n_0$ the inequality $\epsilon_n^{2/3} < 1/1000$ holds. Let $\lambda \notin \Gamma'$. Supplementary suppose that

$$|\lambda| > M_2 = 1000(2n_0 + 1)n_0 M.$$

Then, using the well-known inequality $ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$ $(a, b > 0,\ p, q > 1,\ 1/p + 1/q = 1)$, we obtain

$$\sum_{n=-\infty}^{\infty} (|\alpha_n^+(\lambda)| + |\alpha_n^-(\lambda)|) \leqslant \sum_{|n|\leqslant n_0} \left(\frac{\epsilon_n}{|2n + t/\pi - \lambda|} + \frac{\epsilon_n}{|2n - t/\pi - \lambda|} \right)$$
$$+ \sum_{|n|>n_0} \left(\frac{\epsilon_n}{|2n + t/\pi - \lambda|} + \frac{\epsilon_n}{|2n - t/\pi - \lambda|} \right)$$
$$\leqslant 2M \sum_{|n|\leqslant n_0} \frac{1}{|2n - \lambda|} + 2 \sum_{|n|>n_0} \left(\epsilon_n^2 + \frac{\epsilon_n^{2/3}}{|2n - \lambda|^{4/3}} \right)$$
$$\leqslant \frac{1}{50} + \frac{1}{500} \sum_{n=1}^{\infty} \frac{1}{n^{4/3}} < \frac{1}{10}, \tag{3.47}$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

hence inequality (3.37) is valid for all λ belonging to the considered domain. Arguing as above, we see that

$$|f(\lambda)| \leqslant c_{13}\left(\left|\sum_{n=-\infty}^{\infty}(\alpha_n^+(\lambda)+\alpha_n^-(\lambda))\right| + \sum_{n=-\infty}^{\infty}(|\alpha_n^+(\lambda)|^2+|\alpha_n^-(\lambda)|^2)\right).$$

The last inequality implies that for all $|k|>k_0$, where $k_0=\max(C_0,\, M_2)$,

$$|f(k)| \leqslant c_{14}\left(\left|\sum_{n=-\infty}^{\infty}\left(\frac{\varepsilon_n^+}{2n+t/\pi-k}+\frac{\varepsilon_n^-}{2n-t/\pi-k}\right)\right| + \sum_{n=-\infty}^{\infty}\left(\frac{|\varepsilon_n^+|^2}{|2n+t/\pi-k|^2}+\frac{|\varepsilon_n^-|^2}{|2n-t/\pi-k|^2}\right)\right). \tag{3.48}$$

Clearly,

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left(\frac{|\varepsilon_n^+|^2}{|2n+t/\pi-k|^2}+\frac{|\varepsilon_n^-|^2}{|2n-t/\pi-k|^2}\right)\\ &\quad=\sum_{n=-\infty}^{\infty}|\varepsilon_n^+|^2\sum_{k=-\infty}^{\infty}\frac{1}{|2n+t/\pi-k|^2}+\sum_{n=-\infty}^{\infty}|\varepsilon_n^-|^2\sum_{k=-\infty}^{\infty}\frac{1}{|2n-t/\pi-k|^2}\\ &\quad< c_{15}\sum_{n=-\infty}^{\infty}(|\varepsilon_n^+|^2+|\varepsilon_n^-|^2)<c_{16}.\end{aligned} \tag{3.49}$$

It follows from (3.27), (3.46), (3.48), (3.49) that (3.1) holds.

Necessity. If a set $\{\Lambda\}$ is the spectrum of a Dirac operator (1.2), (1.4), then relation (2.9) takes place [**5**]. Let us prove that condition (3.27) holds. Since $f(\lambda)=\Delta(\lambda)-\Delta_0(\lambda)$, then by theorem 3.1 relation (3.1) is valid.

Let $\lambda=k$, $k\in\mathbb{Z}$, $|k|>k_0$, hence inequality (3.47) holds. It follows from (3.36), (3.38) and (3.46) that

$$|W(k)|\leqslant|f(k)|.$$

This, together with (3.41) implies

$$|S_1(k)|\leqslant|f(k)|+\sum_{n=-\infty}^{\infty}(|\alpha_n^+(k)|^2+|\alpha_n^-(k)|^2). \tag{3.50}$$

Using (3.49), we find that

$$\sum_{n=-\infty}^{\infty}(|\alpha_n^+(k)|^2+|\alpha_n^-(k)|^2)<c_{17}. \tag{3.51}$$

It follows from (3.50), (3.51) and (3.1) that

$$\sum_{|k|>k_0}|S_1(k)|<c_{18}.$$

https://doi.org/10.1017/prm.2022.33 Published online by Cambridge University Press

It is easy to see that

$$\sum_{|k|\leqslant k_0} |S_1(k)| < k_0 c_{19}.$$

The last two inequalities imply (3.27).

Acknowledgement

The author expresses his deep gratitude to the referee.

References

1 A. Albeverio, R. Hryniv and Ya. Mykytyuk. Inverse spectral problems for Dirac operators with summable potentials. *Russ. J. Math. Phys.* **12** (2005), 406–423.

2 G. D. Birkhoff and R. E. Langer. The boundary problems and developments associated with a system of ordinary differential equations of the first order. *Proc. Am. Acad. Arts Sci.* **58** (1923), 49–128.

3 N. Bondarenko and S. Buterin. An inverse spectral problem for integro-differential Dirac operators with general convolution kernels. *Appl. Anal.* **99** (2020), 700–716.

4 Vasil B. Daskalov and Evgeni Kh. Khristov. Explicit formulae for the inverse problem for the regular Dirac operator. *Inverse Probl.* **16** (2000), 247–258.

5 P. Djakov and B. Mityagin. Unconditional convergence of spectral decompositions of 1D Dirac operators with regular boundary conditions. *Indiana Univ. Math. J.* **61** (2012), 359–398.

6 T. T. Dzabiev. The inverse problem for the Dirac equation with a singularity. *Acad. Nauk. Azerbaidzan. SSR. Dokl.* **22** (1966), 8–12 (in Russian).

7 M. G. Gasymov and T. T. Dzabiev. Solution of the inverse problem by two spectra for the Dirac equation on a finite interval. *Acad. Nauk. Azerbaidzan. SSR. Dokl.* **22** (1966), 3–7 (in Russian).

8 F. Gesztesy and A. Sakhnovich. The inverse approach to Dirac-type systems based on the A-function concept. *J. Funct. Anal.* **279** (2020), 108609.

9 O. Gorbunov and V. Yurko. Inverse problem for Dirac system with singularities in interior points. *Anal Math. Phys.* **6** (2016), 1–29.

10 M. A. Lavrentiev and B. V. Shabat. *Methods of Theory of Complex Variable* (Nauka, Moscow, 1973) (in Russian).

11 M. Lesch and M. Malamud, The inverse spectral problem for first order systems on the half line, Differential operators and related topics. Proceedings of the Mark Krein international conference on operator theory and applications, Odessa, Ukraine, August 18–22, 1997. Volume I (Basel, Birkhauser, 2000). Oper. Theory, Adv. Appl. 117, 199–238.

12 B. Ya. Levin, Lectures on Entire Functions, Am. Math. Soc. Transl. Math. Monographs Vol. 150 (Am. Math. Providence, RI, 1996).

13 B. Ya. Levin and I. V. Ostrovskii. On small perturbations of the set of zeros of functions of sine type. *Math. USSR-Izv.* **14** (1980), 79–101.

14 B. M. Levitan and I. S. Sargsyan. *Sturm–Liouville and Dirac operators* (Kluwer Academic Publishers, Dordrecht, 1991).

15 M. M. Malamud. On Borg-type theorems for first-order systems on a finite interval. *Funktsional. Anal. i Prilozhen.* **33** (1999), 75–80 (in Russian); Engl. transl.: Funct. Anal. Appl. **33** (1999), no. 1, 64–68.

16 M. M. Malamud. Uniqueness questions in inverse problems for systems of differential equations on a finite interval. *Trans. Moscow Math. Soc.* **60** (1999), 173–224.

17 M. M. Malamud. Unique determination of a system by a part of the Monodromy matrix. *Func. Anal. Appl.* **49** (2015), 264–278.

18 S. G. Mamedov. The inverse boundary problem on a finite interval for the Dirac's systems of equations. *Azerbaidzan Gos. Univ. Uchen. Zap. Ser. Fiz-Mat. Nauk.* **57** (1975), 61–67 (in Russian).

19 V. A. Marchenko. *Sturm–Liouville operators and their applications* (Birkhauser Verlag, Basel, 1986).

20 T. V. Misyura. Characterization of spectra of periodic and anti-periodic problems generated by Dirac's operators. II. *Theoriya functfii, funct. analiz i ikh prilozhen.* **31** (1979), 102–109.

21 Y. V. Mykytyuk and D. V. Puyda. Inverse spectral problems for Dirac operators on a finite interval. *J. Math. Anal. Appl.* **386** (2012), 177–194.

22 I. M. Nabiev. Solution of the class of inverse problems for the Dirac operators. *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. Mech.* **21** (2001), 146–157.

23 I. M. Nabiev. Characteristic of spectral data of Dirac operators. *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Math. Mech.* **24** (2004), 161–166.

24 I. M. Nabiev. Solution of the quasiperiodic problem for the Dirac system. *Math. Notes* **89** (2011), 845–852.

25 I. M. Nabiev. The inverse periodic problem for the Dirac operator. *Proc. IMM NAS Azerbaijan* **XIX** (2003), 177–180.

26 S. M. Nikolskii. *Approximation of Functions of Several Variables and Embedding Theorems* (Nauka, Moscow, 1977) (in Russian).

27 W. Ning. An inverse spectral problem for a nonsymmetric differential operator: Reconstruction of eigenvalue problem. *J. Math. Anal. Appl.* **327** (2007), 1396–1419.

28 J.-J. Sansug and V. Tkachenko. Characterization of the periodic and antiperiodic spectra of nonselfadjoint Hill's operators. *Oper. Theory Adv. Appl.* **98** (1997), 216–224.

29 V. Tkachenko. Non-self-adjoint periodic Dirac operators. *Oper. Theory: Adv. Appl.* **123** (2001), 485–512.

30 V. Tkachenko. Non-self-adjoint periodic Dirac operators with finite-band spectra. *Int. Equ. Oper. Theory* **36** (2000), 325–348.

31 C.-Fu Yang and V. Yurko. Recovering Dirac operator with nonlocal boundary conditions. *J. Math. Anal. Appl.* **440** (2016), 156–166.

32 V. A. Yurko, Method of spectral mappings in the inverse problem theory, Inverse and Ill-posed Problems Series (VSP, Utrecht, 2002).

33 V. A. Yurko. *Introduction to the theory of inverse spectral problems* (Fizmatlit, Moscow, 2007) (in Russian).

34 V. A. Yurko. *Inverse spectral problems for differential operators and their applications* (Gordon and Breach Science Publishers, Amsterdam, 2000) (in Russian).

35 V. A. Yurko. An inverse spectral problem for singular non-selfadjoint differential systems. *Sbornik: Math.* **195** (2004), 1823–1854.

36 V. A. Yurko. Inverse spectral problems for differential systems on a finite interval. *Results Math.* **48** (2005), 371–386.

37 V. A. Yurko. An inverse problem for differential systems on a finite interval in the case of multiple roots of the characteristic polynomial. *Differ. Eqns.* **41** (2005), 818–823.

38 V. A. Yurko. An inverse problem for differential systems with multiplied roots of the characteristic polynomial. *J. Inv. Ill-Posed Probl.* **13** (2005), 503–512.