FORMAL CONTRACTION OF THE N-SIMPLEX

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1. If K is a finite geometric (i.e. admitting a rectilinear triangulation) n-complex and σ^n is an n-simplex of K which is not a face of any $n+1$ simplex of K, and if σ^{n-1} is an $n-1$ face of σ^n which is not a face of any other n-simplex in K, then the complex $K - \sigma^n - \sigma^{n-1}$ (the complex whose simplexes are those of K except for σ^n and σ^{n-1}) is called an elementary contraction of K of order n. The correspondence $K \to K - \sigma^n - \sigma^{n-1}$ will also be called an elementary contraction, there being no possibility of confusion. A sequence of elementary contractions is called a formal contraction. Formal contractions were first studied by Whitehead [3]. Many questions concerning this kind of contraction remain outstanding.

It is easy to show that a formal contraction may be arranged so that all its elementary contractions of order q precede all those of order p for $p < q$. A tree may be formally contracted to a given one of its vertices. If L is a formal contraction of K and if $|K|$ (the point set occupied by K) is n-connected, then $|L|$ is n-connected. Hence if a complex contracts formally to a vertex it contracts formally to a given one of its vertices. The cone over a complex contracts formally to its vertex, and any subdivision of the cone over a complex may be further subdivided so as to contract formally to the vertex. Despite all this, it is not known whether a subdivided n-simplex $(n > 2)$ can be contracted formally to a vertex.

2. THEOREM 1. If K is a subdivided n-simplex, there exists a subcomplex K' of the $n-1$ skeleton of K such that K contracts formally to K'.

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Proof. Each (n-1)-simplex of K lies on at most two n-simplexes of K, and in fact lies on exactly two unless it is on the boundary of K. We enumerate the n-simplexes of K; $\sigma_1^n, \sigma_2^n, \ldots, \sigma_q^n$ and we let $b_1, b_2, \ldots, b_q$ be their respective barycenters. If σ_i^n and σ_j^n intersect in an (n-1)-simplex, we denote their intersection by σ_{ij}^{n-1} and the barycenter of σ_{ij}^{n-1} by b_{ij}.

For each pair of n-simplexes intersecting in an (n-1)-simplex we construct a path from b_i to b_j consisting of the straight line segments $b_i b_{ij}$ and $b_{ij} b_j$. The union of all such paths is a linear graph containing all the barycenters of n-simplexes, and all the barycenters of (n-1)-simplexes which are not on the boundary of K. Moreover, each b_{ij} is an endpoint of exactly two segments in the graph. We denote this graph by G, and pick, in G, a tree T which contains all the b_i's and b_{ij}'s. This can always be done [2].

In forming T certain segments have been removed from G and, as b_{ij} is on exactly two segments of G, at most one of these could have been removed. The removal of the segment $b_i b_{ij}$ leaves b_{ij} an endpoint of the tree T, and the tree $T_1 = T - b_{ij} b_j - b_{ij}$ is an elementary contraction of T which contains all the b_i's. Continuing in this fashion we get a tree T' which contains all the b_i's and has only b_i's for endpoints. The idea of the proof is to contract T' to a particular barycenter and then, by reversing the order of contraction of T', contract K to K'.

If b_i is an endpoint of T', the removal of $b_i b_{ij}$ leaves b_{ij} an endpoint, so that the segment $b_{ij} b_j$ may be removed by elementary contraction. Such a sequence of two elementary contractions (i.e., removing the two segments containing a particular b_{ij}) will be called a modified contraction.

T' contracts to an arbitrary vertex, which we will assume to be a b_i. Hence, there is a sequence $T'_1, T'_2, \ldots, T'_{2(q-1)}$ of elementary contractions which contract T' to b_k.

Each endpoint of T' is a b_i, so that T'_1 must be $T' - b_i b_{ij} - b_i$ for some i and j. Since b_{ij} and $b_{ij}b_j$ play no further role in the contraction until they are removed, we can arrange the sequence so that $T'_2 = T'_1 - b_{ij}b_j - b_{ij}$. Hence the sequence $T'_1, T'_2, \ldots, T'_{2(q-1)}$ may be chosen so that T'_k, T'_{k+1}, for k odd, is a modified contraction. We now consider the sequence of modified contractions $T'_2, T'_4, \ldots, T'_{2(q-1)}$ in which $T'_{2(q-1)} = b_q$, which we will now assume to be the barycenter of an n-simplex having an (n-1)-face on the boundary of K. Note that each T'_{2j} has only b_i's for endpoints.

To further simplify matters we renumber the n-simplexes of K so that $T'_{2i} = T'_{2i-2} - b_i b_{ij} - b_i - b_{ij}b_j - b_{ij}$, where T'_0 will denote T'. In particular the first barycenter removed is b_1, the second b_2, etc. It is, of course, during this renumbering process that we justify the subscript $2(q-1)$ on the last modified contraction.

Now σ_q^n, the n-simplex having b_q as barycenter, has an (n-1)-face σ_q^{n-1} on the boundary of K, and the complex $K_1 = K - \sigma_q^n - \sigma_q^{n-1}$ is an elementary contraction of order n. The path from b_{q-1} to b_q in T' contains $b_{q-1,q}$, the barycenter of $\sigma_{q-1}^n \cap \sigma_q^n$. We let $\sigma_{q-1}^{n-1} = \sigma_{q-1}^n \cap \sigma_q^n$. Since σ_q^n is not in K_1, σ_{q-1}^n is the only simplex of K_1 containing σ_{q-1}^{n-1}. Hence $K_2 = K_1 - \sigma_{q-1}^n - \sigma_{q-1}^{n-1}$ is an elementary contraction of K_1 and a formal contraction of K.

$|K_2|$ contains b_i for $i = 1, 2, \ldots, q-2$. If $K_{j-1} = K_{j-2} - \sigma^n_{q-j+2} - \sigma^{n-1}_{q-j+2}$ has been defined and $|K_{j-1}|$ contains b_i for $i = 1, 2, \ldots, q-j+1$, and K_{j-1} is a formal contraction of K, we consider the modified contraction $T'_{2(q-j+1)}$. This removes b_{q-j+1} and a vertex $b_{q-j+1,k}$, where $k > q-j+1$. Since σ^n_k is not in K_{j-1}, σ^n_{q-j+1} is the only simplex in K_{j-1} which contains $\sigma^{n-1}_{q-j+1,k}$. Hence $K_j = K_{j-1} - \sigma^n_{q-j+1} - \sigma^{n-1}_{q-j+1,k}$ is an elementary contraction of K_{j-1} and a formal contraction of K. This process removes each n-simplex of K. The resulting complex is K'.

3. The preceding proof depends upon the fact that each (n-1)- simplex in K is on at most two n-simplexes in K. Since a similar statement is not necessarily the case in K', we cannot finish the contraction of the simplex except in the case of the 2-simplex, where K' is a tree. We can, however, say something more about the character of the complex K'.

THEOREM 2. If σ^{n-1} is an (n-1)-simplex of K' (the complex found in theorem 1), then σ^{n-1} has at most one (n-2)-face which is not on some other (n-1)-simplex in K'.

Proof. If σ^{n-1} is on the boundary of K, it has at most one (n-2)-face in common with a given (n-1)-simplex on the boundary of K. Since only one (n-1)-simplex on the boundary of K was removed in forming K', at most one (n-2)-face of any such (n-1)-simplex could have been freed in forming K'.

If σ^{n-1} is not on the boundary of K, it is on exactly two n-simplexes of K. We call them σ^n_i and σ^n_j; $\sigma^{n-1} = \sigma^n_i \cap \sigma^n_j$. Suppose σ^{n-2}_1 and σ^{n-2}_2 are (n-2)-faces of σ^{n-1}, each of which is on no other (n-1)-simplex in K'. If σ^{n-2}_1 were on Bd(K), it would be on σ^{n-1} and an

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(n-1)-simplex on Bd(K) ∩ K'. Hence we may assume that neither σ_1^{n-2} nor σ_2^{n-2} is on Bd(K).

There is, in addition to σ^{n-1}, exactly one (n-1)-face of σ_i^n containing σ_1^{n-2}; we call it σ_{11}^{n-1}. There is, in addition to σ_i^n, exactly one n-simplex of K containing σ_{11}^{n-1}; we call it σ_{11}^n. There is, in addition to σ_{11}^{n-1}, exactly one (n-1)-face of σ_{11}^n containing σ_1^{n-2}; we call it σ_{12}^{n-1}.

If σ_{1k}^{n-1} and σ_{1k}^n have been chosen so that $\sigma_1^{n-2} \subset \sigma_{1k}^{n-1} \subset \sigma_{1k}^n$, there is, in addition to σ_{1k}^{n-1}, exactly one (n-1)-face of σ_{1k}^n containing σ_1^{n-2}; we call it $\sigma_{1,k+1}^{n-1}$. There is, in addition to σ_{1k}^n, exactly one n-simplex of K containing $\sigma_{1,k+1}^{n-1}$; we call it $\sigma_{1,k+1}^n$. In this fashion we order all the n and (n-1)-simplexes containing σ_1^{n-2}.

Since σ^{n-1} is in K', and its face σ_1^{n-2} is in no other (n-1)-simplex in K', σ_{1k}^{n-1} is not in K' for k = 1, 2, ..., m, where m is number of (n-1)-simplexes in K, which are different from σ^{n-1} and contain σ_1^{n-2}. The contraction of K removed only those (n-1)-simplexes of K whose barycenters were in the tree T'. Hence the barycenter of σ_{1k}^{n-1} is in T' for k = 1, 2, ..., m. We will now denote this barycenter by $b(\sigma_{1k}^{n-1})$.

If $b(\sigma_{1k}^{n-1})$ is in T', the path made up of the segments

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$b(\sigma_{1,k-1}^{n})b(\sigma_{1k}^{n-1})$ and $b(\sigma_{1k}^{n-1})b(\sigma_{1k}^{n})$ is also in T'. Similarly the paths $b(\sigma_i^{n})b(\sigma_{11}^{n-1})$, $b(\sigma_{11}^{n-1})$, $b(\sigma_{11}^{n})$, $b(\sigma_{1,m-1}^{n})b(\sigma_{1m}^{n-1})$, and $b(\sigma_{1m}^{n-1})b(\sigma_j^{n})$ are in T'. The union of all these segments is the path in T' from $b(\sigma_i^{n})$ to $b(\sigma_j^{n})$.

In a similar fashion, we may define σ_{2k}^{n-1} and σ_{2k}^{n} so that $\sigma_2^{n-2} \subset \sigma_{2k}^{n-1} \subset \sigma_{2k}^{n}$ and $\sigma_{2,k+1}^{n-1} \subset 2k^{n}$ and get a path in T' from $b(\sigma_i^{n})$ to $b(\sigma_j^{n})$. Since T' is a tree these paths are identical. In particular $b(\sigma_{11}^{n-1}) = b(\sigma_{21}^{n-1})$ and $\sigma_{11}^{n-1} = \sigma_{21}^{n-1}$. Since $\sigma_{11}^{n-1} \cdot \sigma^{n-1} = \sigma_1^{n-2}$ and $\sigma_{21}^{n-1} \cdot \sigma^{n-1} = \sigma_2^{n-2}$, $\sigma_2^{n-2} = \sigma_1^{n-2}$. This completes the proof.

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