RECURSIVE EQUIVALENCE TYPES OF VECTOR SPACES

ALAN G. HAMILTON

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We consider subspaces of a vector space U_F, which is countably infinite dimensional over a recursively enumerable field F with recursive operations, where the operations in U_F are also recursive, and where, of course, F and U_F are sets of natural numbers. It is the object of this paper to investigate recursive equivalence types of such vector spaces and the ways in which their properties are analogous to and depend on properties of recursive equivalence types of sets.

1. Introduction

The reader is referred to Dekker [2] for details of the construction of U_F. For the sake of convenience we shall write addition and scalar multiplication of vectors in U_F in the usual way, namely $v_1 + v_2$ and xv_1 (for $x \in F$), although of course these operations are not the same as addition and multiplication of natural numbers. There will be no confusion. Let N denote the set of natural numbers. If V and W are subspaces of U_F and V is a subspace of W, we write $V \subseteq W$. It will be useful to pick out one recursively enumerable basis of U_F. We shall call it the standard basis and denote it $\{p_i: i < \omega\}$.

The following is a brief exposition of the results from Dekker [2] and Hamilton [4] that we shall need. The reader is assumed to be familiar with recursive equivalence types, as in, say, Dekker and Myhill [3].

DEFINITION 1.1. If $V \subseteq U_F$ and A is a basis of V, then A is an *α-basis* of V if A is contained in a r.e. linearly independent subset of U_F.

Let D be the set of all subspaces of U_F which have an α-basis.

LEMMA 1.2. (i) *An r.e. basis is an α-basis.*

(ii) *If A is an α-basis of V then there is a uniform effective procedure for obtaining, given $v \in V$, the (finite) linear combination of vectors of A which is equal to v.*

If A is a subset of U_F, then the subspace generated by A will be denoted by $V(A)$. One of the principal results of [4] is:

THEOREM 1.3. *If $V \subseteq U_F$ and A and B are two α-bases of V, then A and B are recursively equivalent.*

This enables us to make the following definition.

DEFINITION 1.4. If $V \in D$, then the *α-dimension* of V (written $\dim_\alpha V$) is $RET(A)$ where A is any α-basis of V.

DEFINITION 1.5. If V and W are subspaces of U_F (not necessarily members of D), then V is *α-isomorphic* to W (written $V \simeq W$) if there is a one-one partial recursive function p such that

(i) δp and ρp are subspaces of U_F. (δp and ρp are the domain and range of p),

(ii) $V \subseteq \delta p$,

(iii) $p(V) = W$,

(iv) p is a classical isomorphism between δp and ρp.

It is easy to show that $\simeq$ is an equivalence relation. We call the equivalence classes *V-R.E.T.s.*

THEOREM 1.6. *If $V, W \in D$, then $V \simeq W$ if and only if* $\dim_\alpha V = \dim_\alpha W$. *Let $\Omega_V = \{V\text{-}RET(V) \colon V \in D\}$.*

DEFINITION 1.7. If $T \in \Omega_V$, then the *α-dimension* of T (written $\dim_\alpha T$) is $\dim_\alpha V$, where V is any element of T.

Let $\Lambda_V = \{T \in \Omega_V \colon \dim_\alpha T \in \Lambda\}$.

From Theorem 1.6 we immediately deduce:

COROLLARY 1.8. *If $T_1, T_2 \in \Omega_V$, then $T_1 = T_2$ if and only if* $\dim_\alpha T_1 = \dim_\alpha T_2$.

Finally, it is not difficult to show that for any R.E.T. X there is an element T of Ω_V such that $\dim_\alpha T = X$.

2. Operations

We define addition of subspaces of U_F analogously to the separable sum of subsets of N, in order that the sum may be well-defined on space-types.

Define functions $q_1, q_2 \colon U_F \to U_F$ as follows: If $v = (v_0, \cdots, v_k)$ relative to the standard r.e. basis $\{p_i\}$, let

$$q_1(v) = (v_0, 0, v_1, 0, \cdots, v_k),$$

and

$$q_2(v) = (0, v_0, 0, v_1, \cdots, 0, v_k).$$

Now if V and W are subspaces of U_F then $q_1(V) \cap q_2(W) = \{0\}$, and the sets $q_1(V) - \{0\}$ and $q_2(W) - \{0\}$ are recursively separable.

DEFINITION 2.1. (i) If V and W are subspaces of U_F, let

$$V + W = \{v + w : v \in q_1(V) \,\&\, w \in q_2(W)\}.$$

(ii) If $V \subseteq U_F$, let $V.1 = V$, and for $n > 1$, let $V.n = V.(n-1) + V$.

Note that this addition is neither commutative nor associative, but we are principally interested in the addition induced on V-R.E.T.s, which is both.

LEMMA 2.2. *If $V_1 \simeq V_2$ and $W_1 \simeq W_2$, then $V_1 + W_1 \simeq V_2 + W_2$.*

PROOF. Suppose that $p: V_1 \simeq V_2$ and $q: W_1 \simeq W_2$. Then let

$$p' = q_1 \circ p \circ q_1^{-1} : q_1(V_1) \simeq q_1(V_2),$$

and

$$q' = q_2 \circ q \circ q_2^{-1} : q_2(W_1) \simeq q_2(W_2).$$

Given $u \in V_1 + W_1$, i.e. $u = v + w$ for some $v \in V_1$ and $w \in W_1$, this expression is unique and we can effectively obtain v and w. Let

$$r(u) = p'(v) + q'(w)$$

for such u.

It is not difficult to verify that r is an α-isomorphism between $V_1 + W_1$ and $V_2 + W_2$.

COROLLARY 2.3. + and *$.n$ for each $n \in N$ induce well defined operations on V-R.E.T.s.*

DEFINITION 2.4. If T_1 and T_2 are V-R.E.T.s, then $T_1 + T_2$ is defined to be $V\text{-}RET(V_1 + V_2)$, where $V_1 \in T_1$ and $V_2 \in T_2$. As before, also, if T is a *V-R.E.T*, define $T.1 = T$ and, for $n > 1$, $T.n = T.(n-1) + T$.

THEOREM 2.5. *Addition of V-R.E.T.s is commutative and associative.*

THEOREM 2.6. *If $V_1, V_2 \in D$ and $V = V_1 + V_2$, then $V \in D$ and* $\dim_\alpha V = \dim_\alpha V_1 + \dim_\alpha V_2$.

PROOF. Suppose that A_1 and A_2 are α-bases of V_1 and V_2 respectively. Then $q_1(A_1)$ and $q_2(A_2)$ are recursively separable, and $q_1(A_1) \cup q_2(A_2)$ is an α-basis of V. Therefore $V \in D$. Now

$$\begin{aligned} \dim_\alpha V &= RET(q_1(A_1) \cup q_2(A_2)) \\ &= RET(q_1(A_1)) + RET(q_2(A_2)) \\ &= \dim_\alpha V_1 + \dim_\alpha V_2 . \end{aligned}$$

COROLLARY 2.7. *If $T_i \in \Omega_V$ and $n_i \in N$ for $1 \leqq i \leqq k$, and $T = \sum_{i=1}^k T_i . n_i$, then $T \in \Omega_V$ and* $\dim_\alpha T = \sum_{i=1}^k (\dim_\alpha T_i) . n_i$.

Now we consider the order relation $\leqq$ on spaces and space types. Note that our use of the symbol $\leqq$ here is different from Dekker's use in [2].

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DEFINITION 2.8. If $T_1, T_2 \in \Omega_V$, define $T_1 \leqq T_2$ to mean $(\exists U \in \Omega_V)(T_1 + U = T_2)$.

THEOREM 2.9. *If* $T_1, T_2 \in \Omega_V$, *then* $T_1 \leqq T_2$ *if and only if* $\dim_\alpha T_1 \leqq \dim_\alpha T_2$.

PROOF. Suppose that $T_1 \leqq T_2$. Then $T_1 + U = T_2$ for some $U \in \Omega_V$. Thus $\dim_\alpha T_1 + \dim_\alpha U = \dim_\alpha T_2$, by Corollary 2.7, and therefore $\dim_\alpha T_1 \leqq \dim_\alpha T_2$.

Now suppose that $\dim_\alpha T_1 \leqq \dim_\alpha T_2$. Then there is $X \in \Omega$ such that $\dim_\alpha T_1 + X = \dim_\alpha T_2$. Now X is equal to $\dim_\alpha U$ for some $U \in \Omega_V$, by the remark after Corollary 1.8, and so $\dim_\alpha T_1 + \dim_\alpha U = \dim_\alpha T_2$. By Corollaries 2.7 and 1.8, then, $T_1 + U = T_2$, i.e. $T_1 \leqq T_2$.

This theorem gives the reason why we had the restriction to Ω_V in Definition 2.8. If we had said $T_1 \leqq T_2$ if there exists U such that $T_1 + U = T_2$ for any T_1 and T_2, then there would have been no guarantee that if T_1 and T_2 were members of Ω_V then the U given by the definition would also be. We have not been able to discount the possibility that $\dim_\alpha T_1$ and $\dim_\alpha (T_1 + U)$ be defined, but that $\dim_\alpha U$ be not defined.

DEFINITION 2.10. If $V_1, V_2 \in D$, then define $V_1 \leqq V_2$ to mean there exists $V \in D$ such that $V_1 \hat{+} V = V_2$, and $V_1 - \{0\}$ is recursively separable from $V - \{0\}$. (Here $\hat{+}$ is the standard algebraic linear sum operation.)

THEOREM 2.11. *If* $V_1, V_2 \in D$, *then* $V_1 \leqq V_2$ *implies*

(i) $\dim_\alpha V_1 \leqq \dim_\alpha V_2$, *and*

(ii) $V\text{-}RET(V_1) \leqq V\text{-}RET(V_2)$.

Finally in this section we prove:

THEOREM 2.12. *If* $T_1, T_2 \in \Omega_V$, $T_1 \leqq T_2$ *and* $T_2 \leqq T_1$, *then* $T_1 = T_2$.

PROOF. By Theorem 2.9, we have $\dim_\alpha T_1 \leqq \dim_\alpha T_2$ and $\dim_\alpha T_2 \leqq \dim_\alpha T_1$. We can apply the Myhill-Cantor-Bernstein theorem (for R.E.T.s X and Y, if $X \leqq Y$ and $Y \leqq X$, then $X = Y$), to get $\dim_\alpha T_1 = \dim_\alpha T_2$. It follows that $T_1 = T_2$, by Corollary 1.8.

We shall see later how other theorems of this nature hold in Ω_V as a consequence of their holding in Ω.

3. Isolic spaces

Dekker [2] has given many of the properties of isolic spaces. Here I want mainly to point out analogies with the properties of isolated sets and isols. First note that in general an isolic space (one with an isolated α-basis) is not itself an isolated set, for if the field F is infinite and if $v \in V - \{0\}$, then the set $\{xv : x \in F - \{0\}\}$ is an infinite r.e. subset of V.

Let $D_\Lambda = \{V \in D : \dim_\alpha V \in \Lambda\}$.

THEOREM 3.1. *The following conditions are equivalent* (*assuming that* $V \in D$).

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(i) $V \in D_\Lambda$,
(ii) *V has no infinite dimensional r.e. subspace,*
(iii) *$W \subseteq V$ and $W \in D$ imply $W \in D_\Lambda$,*
(iv) *$W \subseteq V$, $W \in D$, $W' \subseteq W$ and $W' \simeq W$ together imply $W' = W$,*
(v) *$W \subseteq V$ and $a \in U_F - \{0\}$ imply $W \not\simeq W + V(\{a\})$, and*
(vi) *$W \subseteq V$ and $W + W_1 \simeq W + W_2$,* where $W, W_1, W_2 \in D$, *imply $W_1 = W_2$.*

PROOF. (i) $\Rightarrow$ (ii) was proved by Dekker in [2].

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) is Lemma 3.2 below.

(iv) $\Rightarrow$ (i): Suppose that (iv) holds and that (i) does not. Then V has an α-basis A which is not isolated. There is a partial recursive function q such that $q: A \simeq A_1$, where $A_1 \subseteq A$ properly. Obviously A_1 is an α-basis for $V(A_1)$, and it can be shown easily that for such a q, the restriction of q to A extends to a partial recursive isomorphism $q': V(A) \simeq V(A_1)$. Now $V(A_1) \subseteq V$ properly, contradicting (iv). Thus (i) $\cdots$ (iv) are equivalent.

(iii) $\Rightarrow$ (v): By (iii) and Theorem 2.6, $\dim_\alpha(W + V(\{a\}))$ is an isol. By Lemma 3.2, $W + V(\{a\}) \not\simeq W$, since $W + V(\{a\})$ has a proper subspace which is α-isomorphic to W.

(v) $\Rightarrow$ (i): By (v), $V \not\simeq V + V(\{a\})$ for any $a \in U_F - \{0\}$. But if $\dim_\alpha V$ exists, then so does $\dim_\alpha(V + V(\{a\}))$, and

$$\dim_\alpha(V + V(\{a\})) = \dim_\alpha V + 1 .$$

So we have $\dim_\alpha V \neq \dim_\alpha V + 1$. Hence $\dim_\alpha V$ is an isol.

(vi) $\Rightarrow$ (v) is trivial.

(iii) $\Rightarrow$ (vi): Suppose that $W \subseteq V$ and $W + W_1 \simeq W + W_2$, where W, W_1, $W_2 \in D$. Then we have

$$\dim_\alpha W + \dim_\alpha W_1 = \dim_\alpha W + \dim_\alpha W_2 .$$

By (iii), however, $\dim_\alpha W$ is an isol, so we can deduce $\dim_\alpha W_1 = \dim_\alpha W_2$. In turn this implies that $W_1 \simeq W_2$.

The next lemma completes the proof of the theorem.

LEMMA 3.2. *If $V \in D_\Lambda$, $W \subseteq V$* and *$p: V \simeq W$, then $W = V$.*

PROOF. Let $A = \{a_i : i < \omega\}$ be an α-basis of V and let $p: V \simeq W$, where $W \subseteq V$. Consider $a_i \in A$. Construct an r.e. subset $\mathrm{Cl}(a_i)$ of A thus:

$\mathrm{Cl}(a_i) = \{a_j \in A$: there is a chain $a_{i_0}, \cdots, a_{i_n}$ of elements of A such that $a_{i_0} = a$, $a_{i_n} = a_j$, and in the expression for $p(a_{i_l})$ in terms of the basis A, $a_{i_{l+1}}$ appears with a nonzero coefficient, for $0 \leqq l < n\}$.

A is isolated, so for each i, $\mathrm{Cl}(a_i)$ is finite. It is easily seen also that $p(V(\mathrm{Cl}(a_i))) = V(\mathrm{Cl}(a_i))$ for each i, since $V(\mathrm{Cl}(a_i))$ is finite dimensional and p is one-one. Now $a_i \in V(\mathrm{Cl}(a_i))$, so $a_i \in \rho p$ for each i, and further, A is contained in the image of V under p. It follows that $p(V) = V$, and so $W = V$.

LEMMA 3.3. *If* $V_1, V_2 \in D_\Lambda$, *then* $V_1 + V_2 \in D_\Lambda$.

LEMMA 3.4. (i) *If* $V_1 \in D$, $V_2 \in D_\Lambda$ *and* $V_1 \leqq V_2$, *then* $V_1 \in D_\Lambda$.
(ii) *If* $T_1 \in \Omega_\Lambda$, $T_2 \in \Lambda_V$ *and* $T_1 \leqq T_2$, *then* $T_1 \in \Lambda_V$.

4. Relations

Here we deal with V-R.E.T.s and relations among them. Our aim is the following:

THEOREM 4.1. *Suppose that* ϕ *is a (finite) formula involving* $+$, $\leqq$, *varibales, logical connectives and first order quantifiers. Then* ϕ *holds universally in* Ω_V *(respectively in* Λ_V*) if and only if* ϕ *holds universally in* Ω *(respectively in* Λ*).*

The remainder of the paper is devoted to proving this, but first some preliminaries.

NOTATION. If $R \subseteq \mathbf{X}^k\Omega$, i.e. if R is a relation among R.E.T.s, let

$$R_V = \{(T_1, \cdots, T_k) \in \mathbf{X}^k\Omega_V : (\dim_\alpha T_1, \cdots, \dim_\alpha T_k) \in R\}.$$

If

$$R = \{(X_1, \cdots, X_k) \in \mathbf{X}^k\Omega : \phi(X_1, \cdots, X_k)\},$$

where ϕ is a formula as described in Theorem 4.1, let

$$R' = \{(T_1, \cdots, T_k) \in \mathbf{X}^k\Omega_V : \phi(T_1, \cdots, T_k)\}.$$

THEOREM 4.2. *Suppose that* $R \subseteq \mathbf{X}^k\Omega$. *Then*
(i) $R = \mathbf{X}^k\Omega$ *if and only if* $R_V = \mathbf{X}^k\Omega_V$, *and*
(ii) $R = \mathbf{X}^k\Lambda$ *if and only if* $R_V = \mathbf{X}^k\Lambda_V$.

PROOF. Suppose that $R = \mathbf{X}^k\Omega$. If $(T_1, \cdots, T_k) \in \mathbf{X}^k\Omega_V$ then $(\dim_\alpha T_1, \cdots, \dim_\alpha T_k) \in \mathbf{X}^k\Omega = R$. Hence $(T_1, \cdots, T_k) \in R_V$.

Now suppose that $R_V = \mathbf{X}^k\Omega_V$. If $(X_1, \cdots, X_k) \in \mathbf{X}^k\Omega$ then there exist $(T_1, \cdots, T_k) \in \mathbf{X}^k\Omega_V$ such that for $1 \leqq i \leqq k$, $\dim_\alpha T_i = X_i$. By the definition of R_V, then, $(\dim_\alpha T_1, \cdots, \dim_\alpha T_k) \in R$, i.e. $(X_1, \cdots, X_k) \in R$. Hence $R = \mathbf{X}^k\Omega$.

Part (ii) is proved in the same way.

THEOREM 4.3. *If* $S \subseteq \mathbf{X}^k\Omega_V$, *then* $S = R_V$ *for some* $R \subseteq \mathbf{X}^k\Omega$.

PROOF. Let $S_D = \{(X_1, \cdots, X_k) \in \mathbf{X}^k\Omega$: there exists $(T_1, \cdots, T_k) \in \mathbf{X}^k\Omega_V$ such that for $1 \leqq i \leqq k$, $\dim_\alpha T_i = X_i$ and $(T_1, \cdots, T_k) \in S\}$.

It is easy to verify that $(S_D)_V = S$.

LEMMA 4.4. *Suppose that*

$$R = \{(X_1, \cdots, X_{m+n}) \in \mathbf{X}^{m+n}\Omega : X_1 + \cdots + X_m \leqq X_{m+1} + \cdots + X_{m+n}\},$$

and that R' *is as described above.*

Then $R' = R_V$.

PROOF. Using Corollary 2.7 and Theorem 2.9,

$$\begin{aligned}
R_V &= \{(T_1, \cdots, T_{m+n}) : (\dim_\alpha T_1, \cdots, \dim_\alpha T_{m+n}) \in R\} \\
&= \{(T_1, \cdots, T_{m+n}) : \dim_\alpha T_1 + \cdots + \dim_\alpha T_m \leqq \dim_\alpha T_{m+1} + \cdots + \dim T_{m+n}\} \\
&= \{(T_1, \cdots, T_{m+n}) : \dim_\alpha(T_1 + \cdots + T_m) \leqq \dim_\alpha(T_{m+1} + \cdots + T_{m+n})\} \\
&= \{(T_1, \cdots, T_{m+n}) : T_1 + \cdots + T_m \leqq T_{m+1} + \cdots + T_{m+n}\} \\
&= R'.
\end{aligned}$$

This is the base step of our inductive proof of Theorem 4.1, and now follow the induction steps.

LEMMA 4.5. *Suppose that* ϕ_1 *and* ϕ_2 *are formulas as described in Theorem* 4.1, *and that*

$$R_1 = \{X \in \mathbf{X}^k\Omega : \phi_1(X)\},$$
$$R_2 = \{X \in \mathbf{X}^k\Omega : \phi_2(X)\}.$$

Then if $R_1' = (R_1)_V$ *and* $R_2' = (R_2)_V$, *we have*
(i) $(\mathbf{X}^k\Omega - R_1)' = (\mathbf{X}^k\Omega - R_1)_V$,
(ii) $(R_1 \cup R_2)' = (R_1 \cup R_2)_V$, *and*
(iii) $(R_1 \cap R_2)' = (R_1 \cap R_2)_V$.

PROOF. It is easily shown that

$$\begin{aligned}
(\mathbf{X}^k\Omega - R_1)_V &= \mathbf{X}^k\Omega_V - (R_1)_V \\
&= \mathbf{X}^k\Omega_V - R_1' \text{ by hypothesis,} \\
&= (\mathbf{X}^k\Omega - R_1)'.
\end{aligned}$$

The proofs of (ii) and (iii) are similar.

LEMMA 4.6 (i) *Let* ϕ_1, R_1 *be as above, and let* ϕ *be the formula* $(\forall x_1)\phi_1(x_1)$, *where* x_1 *is a variable which does not occur bound in* ϕ_1. *Now let*

$$R = \{(X_2, \cdots, X_k) \in \mathbf{X}^{k-1}\Omega : (\forall X_1 \in \Omega)\, \phi_1(X_1, \cdots, X_k)\}.$$

If $R_1' = (R_1)_V$, *then* $R' = R_V$.
(ii) *As* (i), *but with an existential quantifier.*

PROOF.

$$\begin{aligned}
R_V &= \{(T_2, \cdots, T_k) : (\dim_\alpha T_2, \cdots, \dim_\alpha T_k) \in R\} \\
&= \{(T_2, \cdots, T_k) : (\forall X_1 \in \Omega)\phi_1(X_1, \dim_\alpha T_2, \cdots, \dim_\alpha T_k)\} \\
&= \{(T_2, \cdots, T_k) : (\forall T_1 \in \Omega_V)\phi_1(\dim_\alpha T_1, \cdots, \dim_\alpha T_k)\},
\end{aligned}$$

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by the final remark of Section 1,

$$= \{(T_2, \cdots, T_k): (\forall T_1 \in \Omega_V)((\dim_\alpha T_1, \cdots, \dim_\alpha T_k) \in R_1)\}$$

$$= \{(T_2, \cdots, T_k): (\forall T_1 \in \Omega_V)((T_1, \cdots, T_k) \in (R_1)_V)\}$$

$$= \{(T_2, \cdots, T_k): (\forall T_1 \in \Omega_V)((T_1, \cdots, T_k) \in R_1')\},$$

by our hypothesis,

$$= \{(T_2, \cdots, T_k): (\forall T_1 \in \Omega_V)\phi_1(T_1, \cdots, T_k)\}$$

$$= R'.$$

Part (ii) follows from (i) and the previous lemma.

We combine the last three lemmas now into:

LEMMA 4.7. *If ϕ is as described in Theorem* 4.1, *and* $R = \{X \in \mathbf{X}^k\Omega : \phi(X)\}$, *then* $R' = R_V$.

Now we invoke Theorem 4.2 to obtain

(i) $R = \mathbf{X}^k\Omega$ if and only if $R' = \mathbf{X}^k\Omega_V$, and

(ii) $R = \mathbf{X}^k\Lambda$ if and only if $R' = \mathbf{X}^k\Lambda_V$.

Theorem 4.1 is therefore proved.

NOTES: (i) *If* ϕ is a closed formula, then to apply the above we may just introduce a redundant free variable.

(ii) Because of Theorem 2.12, it was not necessary to include "$=$" as one of the symbols from which the admissible formulas ϕ were composed. However, we could just as well have included "$=$" in the statement of Theorem 4.1. Theorem 2.12 would then have been a corollary.

Finally, Theorem 4.1, along with results of Nerode [7] and Manaster and Nerode [6], enable us to state the following.

THEOREM 4.8. (i) *Universal sentences about the structure* $(\Omega_V, +, \leqq)$ *are decidable.*

(ii) *The first order theory of* $(\Omega_V, +, \leqq)$ *is not decidable.*

PROOF. Theorem 4.1 says that the first order theories of $(\Omega_V, +, \leqq)$ and $(\Omega, +, \leqq)$ are the same. In [7] it is proved that universal sentences about $(\Omega, +, \leqq)$ are decidable, so (i) follows. In [6] it is shown that the first order theory of $(\Omega, +)$ is not decidable, so (ii) follows.

Conclusion

The reason why vector spaces inherit properties of the kind discussed from their α-bases is that vector spaces are "freely generated" from sets with no structure. Linear independence is really just an absence of structure. This suggests that results similar to the above may be obtained for free algebraic systems more general than vector spaces. That is indeed the case, and some research in this direction is described in [5].

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Department of Mathematics
University of Stirling
Scotland