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Sobolev trace-type inequalities via time-space fractional heat equations

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Abstract. This article aims to establish fractional Sobolev trace inequalities, logarithmic Sobolev trace inequalities, and Hardy trace inequalities associated with time-space fractional heat equations. The key steps involve establishing dedicated estimates for the fractional heat kernel, regularity estimates for the solution of the time-space fractional equations, and characterizing the norm of $\dot{W}_p^{\nu/2}(\mathbb{R}^n)$ in terms of the solution $u(x,t)$. Additionally, fractional logarithmic Gagliardo–Nirenberg inequalities are proven, leading to L^p–logarithmic Sobolev inequalities for $\dot{W}_p^{\nu/2}(\mathbb{R}^n)$. As a byproduct, Sobolev affine trace-type inequalities for $\dot{H}^{-\nu/2}(\mathbb{R}^n)$ and local Sobolev-type trace inequalities for $Q_{\nu/2}(\mathbb{R}^n)$ are established.

1 Introduction

Analytic inequalities, including Sobolev inequalities, logarithmic Sobolev inequalities, Hardy inequalities, and their fractional counterparts, play crucial roles in harmonic analysis, mathematical physics, and partial differential equations (PDEs). Interested readers can explore the works of Beckner and Pearson [4], Cotsiolis and Tavoularis [9], Talenti [36], Xiao and Zhai [42], and the references therein for further insights into Sobolev-type inequalities.

Trace inequalities of Sobolev type, logarithmic Sobolev type, and Hardy type, particularly in the context of operators and equations, have also been extensively studied. Xiao established sharp fractional Sobolev trace inequalities linked to the Poisson equation in [39]. Einav and Loss [15] proved Sobolev trace inequalities involving the projector τ_k. More recently, Li, Hu, and Zhai [27] contributed to the field by establishing fractional Sobolev, logarithmic Sobolev, and Hardy trace inequalities associated with fractional harmonic extensions.

In this article, our objective is to establish Sobolev, logarithmic Sobolev, and Hardy trace-type inequalities associated with the solution of the following time-space fractional equations:

$$\begin{cases} \partial_t^\beta u(x,t) + (-\Delta_x)^{\alpha/2} u(x,t) = 0, & (x,t) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

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where $\alpha > n$ and $\beta \in (0,1]$. Here, the Caputo fractional derivative, denoted by ∂_t^β, is defined as

$$\partial_t^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_r u(r,x) \frac{dr}{(t-r)^\beta}, \quad \beta \in (0,1).$$

Additionally, the fractional Laplace operator $(-\Delta_x)^{\alpha/2}$ in $\mathbb{R}^n$ is defined on the Schwartz class through the Fourier transform:

$$[\widehat{(-\Delta_x)^{\alpha/2} f}](\xi) = |\xi|^\alpha \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx = |\xi|^\alpha \widehat{f}(\xi).$$

When $\beta = 1$, the equations (1.1) become fractional diffusion equations:

$$\begin{cases} \partial_t u(x,t) + (-\Delta_x)^{\alpha/2} u(x,t) = 0, & (x,t) \in \mathbb{R}_+^{n+1}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

The Carleson embedding associated with (1.2) has been extensively studied in various works, including Adams and Hedberg [1], Chang and Xiao [8], Liu, Wu, Xiao, and Yuan [29], Xiao [38], Xiao and Zhai [42], and Zhai [46]. These studies have contributed to the understanding of the Carleson embedding properties related to the fractional diffusion equations. When $\beta = 1$ and $\alpha = 2$, the equation (1.1) corresponds to the classical heat equation, a fundamental equation with widespread applications in various fields, including mathematics, physics, fluid dynamics, and engineering.

If $\beta = 1$ and $\alpha \in (0,2)$, the equations (1.1) transform into the spatial fractional heat equation. This equation has found applications in the study of fluid dynamics, contributing to the understanding of heat transfer processes.

When $\beta \in (0,1)$ and $\alpha = 2$, the equations (1.1) become the so-called "time fractional" heat equations:

$$\begin{cases} \partial_t^\beta u(x,t) + (-\Delta_x) u(x,t) = 0, & (x,t) \in \mathbb{R}_+^{n+1}, \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

The equations (1.3) exhibit sub-diffusive behavior and are associated with anomalous diffusion or diffusion in non-homogeneous media with random fractal structures.

The introduction of the time-fractional derivative ∂_t^β by Caputo in [7] marked a significant development for investigating the analytic expression of a linear dissipative mechanism. In mathematical physics and engineering, Caputo fractional derivatives and their generalizations have become instrumental in addressing unconventional physical phenomena, capturing the attention of numerous researchers. For further exploration of generalizations of Caputo derivatives, readers can refer to works by Bernardis, Martín-Reyes, Stinga, and Torrea [5], Gorenflo, Luchko, and Yamamoto [18], Kilbas, Srivastava, and Trujillo [25], and Li and Liu [26].

Fractional derivatives offer distinct advantages compared to integer-order derivatives. They capture the history-dependent development of a system function more accurately due to global correlation. The fractional derivative model also addresses the limitations of classical differential model theory, providing better agreement with experimental results. Additionally, in describing complex physical and mechanical problems, fractional-order models often offer clarity and conciseness compared to nonlinear models. Leveraging these advantages, time-fractional calculus finds widespread application in various scientific branches, including statistical mechanics, theoretical physics, theoretical neuroscience, the theory of complex chemical reactions, fluid dynamics, hydrology, and mathematical finance. For an extensive list of references, readers can consult Khoshnevisan [24].

In Section 2.2, we utilize the subordinative formula to estimate the higher-order derivatives of the integral kernels associated with the fractional heat semigroups $e^{-t(-\Delta_x)^{\alpha/2}}$ for $t > 0$, denoted by $K_{\alpha,t}(\cdot)$ (refer to Lemmas 2.2 and 2.8). The time-space fractional heat kernel, denoted by $G_t^{\alpha,\beta}(\cdot)$, is introduced as the fundamental solution to equations (1.1). Through the representation (2.1), we establish that for $m \in \mathbb{Z}_+$, the following estimates hold:

$$\left|t^m \frac{\partial^m G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)}{\partial x_i^m}\right| + \left|t^m \frac{\partial^m G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)}{\partial t^m}\right| \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}}, \quad \alpha > n+m \;\&\; \beta \in (0,1].$$

These estimates are detailed in Propositions 2.10 and 2.11.

Let $C_0^\infty(\mathbb{R}^n)$ represents the space of infinitely differentiable functions on $\mathbb{R}^n$ with compact support. For $\nu \in (0,1)$ and $p \in (1, n/\nu)$, the homogeneous Sobolev space $\dot{W}^\nu_p(\mathbb{R}^n)$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\dot{W}^\nu_p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p\nu}}dxdy\right)^{1/p}.$$

Specially, when $p = 2$, $\dot{W}^\nu_2(\mathbb{R}^n)$ is also denoted by $\dot{H}^\nu(\mathbb{R}^n)$. Moreover, $\dot{W}^{-\nu}_{p'}(\mathbb{R}^n)$ is the dual of $\dot{W}^\nu_p(\mathbb{R}^n)$. In Section 3, considering f in the homogeneous Sobolev space $\dot{H}^{\nu/2}(\mathbb{R}^n)$, and utilizing the Fourier transform of $G_t^{\alpha,\beta}(\cdot)$, we establish equivalent characterizations of the norm of $\dot{H}^{\nu/2}(\mathbb{R}^n)$ as follows:

$$\int_{\mathbb{R}^{n+1}_+} \left|\widetilde{\nabla}^m_x G^{\alpha,\beta}_{t^{\alpha/\beta}} * f(x)\right|^2 t^{2m-1-\nu}dxdt \approx \int_{\mathbb{R}^n} |\xi|^\nu \left|\widehat{f}(\xi)\right|^2 d\xi.$$

Here, $\widetilde{\nabla}^m_x := (\partial^m_{x_1}, \partial^m_{x_2}, \ldots, \partial^m_{x_n})$ (refer to (3.4)). Building upon this result, we establish fractional Sobolev, logarithmic Sobolev, and Hardy trace-type inequalities. For any $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ and $u(x,t) = G_t^{\alpha,\beta} * f(x)$, the fractional Sobolev trace inequality is given by

$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}dx\right)^{(n-\nu)/n} \lesssim \int_{\mathbb{R}^n} \left|\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})\right|^2 t^{2m-1-\nu}dxdt. \tag{1.4}$$

This result is detailed in Theorem 3.1. Furthermore, when $\|f\|_{L^2(\mathbb{R}^n)} = 1$, the following fractional logarithmic Sobolev inequality and the fractional Hardy inequality (or the Kato inequality) are established:

$$\begin{cases} \exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2)dx\right) \lesssim \int_{\mathbb{R}^n} \left|\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})\right|^2 t^{2m-1-\nu}dxdt; \\ \qquad \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^\nu} \lesssim \int_{\mathbb{R}^n} \left|\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})\right|^2 t^{2m-1-\nu}dxdt. \end{cases} \tag{1.5}$$

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These results are detailed in Theorem 3.1. Xiao [39] established inequalities akin to (1.4) and the first inequality of (1.5) for the Poisson extension. Li, Hu, and Zhai in [27] investigated corresponding inequalities related to the Caffarelli–Silvestre extensions. Inequalities similar to the second inequality of (1.5) have been examined in [4, 14, 21, 23, 34, 42, 43]. We will demonstrate that the right-hand side of (1.4) and (1.5) can be replaced by

$$\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^m u(x, t^{\alpha/\beta})}{\partial t^m} \right|^2 t^{2m-1-\nu} dxdt$$

(see Theorem 3.3). Via a change of variable, inequalities similar to (1.4) and (1.5) in Theorems 3.1 and 3.3 can be proven for $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0, \min\{2m, n\})$ and $u(x,t) = G^{\alpha,\beta}_t * f(x)$ (see Theorems 3.2 and 3.4).

Moreover, Theorems 3.1 and 3.3 imply the following Sobolev-type trace inequalities:

$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)} dx \right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left\{ \left| t^m \widetilde{\nabla}^m_x u(x, t^{\alpha/\beta}) \right|^2 + \left| t^m \frac{\partial^m u(x, t^{\alpha/\beta})}{\partial t^m} \right|^2 \right\} \frac{dxdt}{t^{1+\nu}}$$

for $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0, \min\{2m, n\})$ and $u(x,t) = G^{\alpha,\beta}_t * f(x)$. Via a change of variable, for $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0, \min\{2m, n\})$ and $u(x,t) = G^{\alpha,\beta}_t * f(x)$, the following equivalent version:

(1.6)
$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)} dx \right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left\{ \left| t^{\beta m/\alpha} \widetilde{\nabla}^m_x u(x, t) \right|^2 + \left| t^m \frac{\partial^m u(x, t)}{\partial t^m} \right|^2 \right\} \frac{dxdt}{t^{1+\beta\nu/\alpha}}$$

can be deduced from Theorems 3.2 and 3.4 immediately.

To generalize the Sobolev-type trace inequalities in Theorem 3.1 to $\dot{W}^{\nu/2}_p(\mathbb{R}^n)$, in Theorem 3.10, we characterize $\dot{W}^{\nu/2}_p(\mathbb{R}^n)$ as follows:

$$\left(\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}^m_x u(x, t^{\alpha/\beta})|^p t^{pm-p\nu/2-1} dxdt \right)^{1/p} \approx \|f\|_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)}$$

and

$$\left(\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^m u(x, t^{\alpha/\beta})}{\partial t^m} \right|^p t^{pm-p\nu/2-1} dxdt \right)^{1/p} \approx \|f\|_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)}$$

with $p > 1$, $\nu \in (0,2)$ and $u(x,t) = G^{\alpha,\beta}_t * f(x)$. Moreover, we establish fractional logarithmic Gagliardo–Nirenberg inequalities which imply the L^p-logarithmic Sobolev inequalities for $\dot{W}^{\nu/2}_p(\mathbb{R}^n)$.

A direct computation indicates that the inequality (1.6) is invariant under the transform $\phi(x) = \lambda x + x_0$ for $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, i.e.,

$$\left(\int_{\mathbb{R}^n} |(f \circ \phi)(x)|^{2n/(n-\nu)} dx \right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^m (u \circ \phi)(x, t^{\alpha/\beta})}{\partial t^m} \right|^2 t^{2m-1-\nu} dxdt.$$

However, both the Lebesgue space $L^{2n/(n-\nu)}(\mathbb{R}^n)$ with

$$\|f\|_{L^{2n/(n-\nu)}} := \left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\, dx\right)^{(n-\nu)/(2n)} < \infty$$

and the Sobolev space $\dot{H}^{\nu/2}(\mathbb{R}^n)$ with

$$\begin{aligned}\|f\|_{\dot{H}^{\nu/2}(\mathbb{R}^n)} &\approx \left(\int_{\mathbb{R}^{n+1}_+} \left|\widetilde{\nabla}_x^m u(x,t^{\alpha/\beta})\right|^2 t^{2m-1-\nu} dx dt\right)^{1/2}\\ &\approx \left(\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu} dx dt\right)^{1/2} < \infty\end{aligned}$$

are not invariant under the transform ϕ. In [39], using the characterization of Q-type space $Q_\kappa(\mathbb{R}^n)$, Xiao obtained a revised conformal invariant Sobolev-type trace inequality (see [39, Theorem 4.1]). In Theorem 5.4, following the idea of [39], we prove the local versions of (1.4) for $f \in Q_{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0, \min\{2, n\})$ and $u(x,t) = G_t^{\alpha,\beta} * f(x)$:

$$\begin{aligned}\sup_I \left(\frac{1}{|I|}\int_I |f(x) - f_I|^{2n/(n-\nu)}\, dx\right)^{(n-\nu)/(2n)}\\ \lesssim \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r \int_{|y-x_0|<r} \left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu} dx dt\right)^{1/2}\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}\sup_I \left(\frac{1}{|I|}\int_I |f(x) - f_I|^{2n/(n-\nu)}\, dx\right)^{(n-\nu)/(2n)}\\ \lesssim \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r \int_{|y-x_0|<r} \left|\widetilde{\nabla}_x^m u(x,t^{\alpha/\beta})\right|^2 t^{2m-1-\nu} dx dt\right)^{1/2}.\end{aligned} \tag{1.8}$$

Notations: *In this paper, $A \lesssim B$ means $A \le CB$ for a positive constant C. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Let $k \in \mathbb{N}$. Here, $\mathbb{N}$ denotes the set of natural numbers. The symbol $C^k(\mathbb{R}^n)$ denotes the class of all functions $f : \mathbb{R}^n \to \mathbb{R}$ with k continuous partial derivatives. Denote by $f * g$ the convolution of functions f and g, i.e.,*

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

2 Preliminaries

2.1 Basic lemmas

First, we investigate the integrability of the Fourier transform of the time-space fractional heat kernel $G_t^{\alpha,\beta}(\cdot)$.

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Definition 2.1 Let $\alpha > n/2$ and $\beta \in (0,1]$. We say $G_t^{\alpha,\beta}(\cdot)$ is the time-space fractional heat kernel if for $(x,t) \in \mathbb{R}^n \times (0,\infty)$,

$$G_t^{\alpha,\beta}(x) := \int_0^\infty K_{\alpha,(t/u)^\beta}(x) g_\beta(u)\,du, \tag{2.1}$$

where $g_\beta(\cdot)$ is the density function of D_1 and is infinitely differentiable on the entire real line with $g_\beta(u) = 0$ for $u \le 0$ (cf. [17]) and $K_{\alpha,t}(\cdot)$ denotes the fractional heat kernel defined as

$$\widehat{K_{\alpha,t}}(\xi) := e^{-t|\xi|^{\alpha/2}}.$$

It follows from [17, (2.5)] that, when $u \to \infty$, there holds

$$g_\beta(u) \approx \frac{\beta}{\Gamma(1-\beta)} u^{-\beta-1}.$$

Following [17, p. 8], the Fourier transform of the kernel $G_t^{\alpha,\beta}(\cdot)$ can be represented as

$$\widehat{G_t^{\alpha,\beta}}(\xi) = E_\beta(-|\xi|^\alpha t^\beta),$$

where

$$E_\beta(t) := \sum_{k=0}^\infty \frac{t^k}{\Gamma(1+\beta k)}. \tag{2.2}$$

Here, the symbol $\Gamma(\cdot)$ denotes the Gamma function and $E_\beta(\cdot)$ is Mittag-Leffler function.

Remark 2.2 In [17], the time-space fractional heat kernel $G_t^{\alpha,\beta}(\cdot)$ is defined by

$$G_t^{\alpha,\beta}(x) := \int_0^\infty p(x,s) f_{E_t}(s)\,ds,$$

where $\widehat{p(\cdot,s)}(\xi) = e^{-s|\xi|^\alpha} = \widehat{K_{\alpha,s}}(\xi)$ and

$$f_{E_t}(s) := t\beta^{-1} s^{-1-1/\beta} g_\beta(ts^{-1/\beta}).$$

By the change of variable: $u = ts^{-1/\beta}$, we have

$$\begin{aligned} G_t^{\alpha,\beta}(x) &= \int_0^\infty -p(x,s) g_\beta(ts^{-1/\beta})\,d(ts^{-1/\beta}) \\ &= \int_0^\infty p(x,(t/u)^\beta) g_\beta(u)\,du \\ &= \int_0^\infty K_{\alpha,(t/u)^\beta}(x) g_\beta(u)\,du. \end{aligned}$$

Thus, we use $(t/u)^\beta$ as the subscript in (2.1). Such a representation of $G_t^{\alpha,\beta}(\cdot)$ was also used by Foondun and Nane [17]. Precisely,

$$G_t^{\alpha,\beta}(x) = \int_0^\infty p(x,(t/u)^\beta) g_\beta(u)\,du$$

(see [17, p. 502, line 10]).

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For $\alpha = 2$ and $\beta = 1$, Foondun and Nane [17, p. 501, (2.12)] pointed out that when $n = 1$, the kernel $G_t^{2,1}(\cdot)$ becomes

$$G_t^{2,1}(x) = H_{1,1}^{1,0}\left[\frac{|x|^2}{t}\middle|\begin{matrix}(1,1)\\(1,2)\end{matrix}\right] = \frac{1}{(4\pi t)^{1/2}}e^{-|x|^2/(4t)}.$$

The cases $n \geq 2$ are similar. In fact, by (2.2), we have

$$E_1(u) = \sum_{k=0}^{\infty}\frac{u^k}{\Gamma(1+k)} = e^u.$$

Then for $n \geq 2$, we can obtain

$$\begin{aligned}G_t^{2,1}(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}E_1(-|\xi|^2 t)d\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-|\xi|^2 t}d\xi\\ &= \prod_{j=1}^{n}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_j\xi_j}E_1(-\xi_j^2 t)d\xi_j\\ &= \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/(4t)},\end{aligned}$$

which indicates that $G_t^{2,1}(\cdot)$ becomes the classical heat kernel.

Remark 2.3 Let $\mathcal{L}$ be the Laplace transform, i.e.,

$$\mathcal{L}(g)(s) := \int_0^{\infty} e^{-st}g(t)dt.$$

After applying the Fourier transform and the Laplace transform to $G_t^{\alpha,\beta}(\cdot)$, we have

$$\begin{aligned}\mathcal{L}\left(\widehat{G_t^{\alpha,\beta}}\right)(\xi,s) &:= (2\pi)^{-n}\int_0^{\infty}\int_{\mathbb{R}^n} e^{-st-ix\cdot\xi}\left(\int_0^{\infty}K_{\alpha,w}(x)f_{E_t}(w)dw\right)dxdt\\ &= \int_0^{\infty}e^{-w|\xi|^{\alpha}}\left(\int_0^{\infty}e^{-st}f_{E_t}(w)dt\right)dw.\end{aligned}$$

By [32, p. 3, (2.9) and (2.10)], the Laplace transform $t \to s$ of $f_{E_t}(w)$ is

$$\mathcal{L}(f_{E_t})(w,s) = s^{\beta-1}e^{-ws^{\beta}}.$$

Then

$$\mathcal{L}\left(\widehat{G_t^{\alpha,\beta}}\right)(\xi,s) = s^{\beta-1}\int_0^{\infty}e^{-w\left(s^{\beta}+|\xi|^{\alpha}\right)}dw = \frac{s^{\beta-1}}{s^{\beta}+|\xi|^{\alpha}}.$$

Thus, by the inverse Laplace transform, we can get

$$\widehat{G_t^{\alpha,\beta}}(\xi) = E_{\beta}(-|\xi|^{\alpha}t^{\beta}).$$

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Remark 2.4 Applying the Laplace transform to $\partial_t^\beta u(x,t)$, we have

$$\begin{aligned}\mathcal{L}\left(\partial_t^\beta u\right)(x,s) &= \int_0^\infty e^{-st}\partial_t^\beta u(x,t)dt\\ &= \frac{1}{\Gamma(1-\beta)}\int_0^\infty e^{-st}\left(\int_0^t \partial_r u(x,r)\frac{dr}{(t-r)^\beta}\right)dt\\ &= \frac{1}{\Gamma(1-\beta)}\int_0^\infty \partial_r u(x,r)\left(\int_r^\infty e^{-st}\frac{dt}{(t-r)^\beta}\right)dr\\ &= \frac{1}{\Gamma(1-\beta)}\int_0^\infty s^{\beta-1}e^{-sr}\partial_r u(x,r)\left(\int_0^\infty e^{-m}m^{-\beta}dm\right)dr\\ &= s^\beta\mathcal{L}(u)(x,s) - s^\beta u(x,0).\end{aligned}$$

After applying the Fourier transform and the Laplace transform to (1.1), we can get

$$s^\beta\mathcal{L}(\widehat{u})(\xi,s) - s^{\beta-1}\widehat{f}(\xi) + |\xi|^\alpha\mathcal{L}_s(\widehat{u})(\xi,s) = 0,$$

which indicates that

$$\mathcal{L}(\widehat{u})(\xi,s) = \widehat{f}(\xi)\frac{s^{\beta-1}}{s^\beta + |\xi|^\alpha}.$$

Applying the inverse Laplace transform, we obtain

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)E_\beta(-|\xi|^\alpha t^\beta).$$

We can use the inverse Fourier transform to deduce that

$$u(x,t) := G_t^{\alpha,\beta} * f(x) = \int_{\mathbb{R}^n} G_t^{\alpha,\beta}(x-y)f(y)dy. \tag{2.3}$$

Thus, the solution to equations (1.1) can be represented as (2.3).

Lemma 2.5 *Let $m \in \mathbb{Z}_+$, $\alpha > n+m$ and $\beta \in (0,1]$.*

(i) *It holds*

$$\begin{cases}\int_0^\infty |E_\beta(-t)|^2 t^\delta dt < \infty, & -1<\delta<1,\\ \int_0^\infty \left|\frac{d^m E_\beta(-t)}{dt^m}\right|^2 t^\delta dt < \infty, & -1<\delta<2m+1.\end{cases} \tag{2.4}$$

(ii) *If $\delta \in (-1, 2\alpha-1)$, there exists a constant $M(n,\alpha,\beta,\delta)$ such that*

$$\int_0^\infty \left|\widehat{G}^{\alpha,\beta}_{t^{\alpha/\beta}}(\xi)\right|^2 t^\delta dt = M(n,\alpha,\beta,\delta)|\xi|^{-\delta-1}.$$

(iii) *If $\delta \in (2m-1, 2m+2\alpha-1)$, $m \in \mathbb{Z}_+$ and $\alpha > m$, then*

$$\int_0^\infty \left|\frac{\partial^m \widehat{G}^{\alpha,\beta}_{t^{\alpha/\beta}}(\xi)}{\partial t^m}\right|^2 t^\delta dt \approx |\xi|^{2m-\delta-1}.$$

Proof It follows from [17, (2.7)] that for $t>0$,

$$\frac{1}{1+\Gamma(1-\beta)t} \le E_\beta(-t) \le \frac{1}{1+\Gamma(1+\beta)^{-1}t}.$$

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Assume that $-1 < \delta < 1$. Then

$$\int_0^\infty |E_\beta(-t)|^2 t^\delta dt = \int_0^1 |E_\beta(-t)|^2 t^\delta dt + \int_1^\infty |E_\beta(-t)|^2 t^\delta dt$$
$$\lesssim \int_0^1 t^\delta dt + \int_1^\infty \frac{t^\delta}{(1+\Gamma(1+\beta)^{-1}t)^2} dt < \infty.$$

Following Haubold, Mathai, and Saxena [22], as $t \to \infty$, for $M \in \mathbb{N}_+$, we have

$$E_\beta(-t) = -\sum_{l=1}^{M} \frac{1}{\Gamma(1-\beta l)} \frac{1}{(-t)^l} + O\left[\frac{1}{(-t)^{M+1}}\right]. \tag{2.5}$$

Then

$$\left|\frac{d^m E_\beta(-t)}{dt^m}\right|^2 \lesssim \sum_{l=1}^{M} \left|\frac{1}{\Gamma(1-\beta l)} \frac{1}{(t)^{2l+2m}}\right| + O\left[\frac{1}{(-t)^{2M+2m}}\right].$$

There exists a constant A_β for t in a neighborhood of 0 such that

$$\begin{aligned}\left|\frac{d^m E_\beta(-t)}{dt^m}\right| &= \left|\sum_{k=m}^{\infty} \frac{(-1)^k k(k-1)\ldots(k-m)t^{k-m}}{\Gamma(1+\beta k)}\right| \\ &= \left|\sum_{k=0}^{\infty} \frac{(-1)^{k+m}(k+m)(k+m-1)\ldots(k+1)t^k}{\Gamma(1+\beta(k+m))}\right| \\ &= \left|\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(k+m-1)\ldots(k+1)t^k}{\beta\Gamma(\beta+\beta k)}\right| \\ &\lesssim |A_\beta| + \left|\sum_{k=[\beta]+1}^{\infty} \frac{(k+m-1)\ldots(k+1)}{\Gamma(\beta+\beta k)}\right| < \infty,\end{aligned} \tag{2.6}$$

which indicates that when $-1 < \delta < 2m+1$,

$$\begin{aligned}&\int_0^\infty \left|\frac{d^m E_\beta(-t)}{dt^m}\right|^2 t^\delta dt \\ &= \int_0^1 \left|\frac{d^m E_\beta(-t)}{dt^m}\right|^2 t^\delta dt + \int_1^\infty \left|\frac{d^m E_\beta(-t)}{dt^m}\right|^2 t^\delta dt \\ &\lesssim \int_0^1 t^\delta dt + \int_1^\infty \left\{\sum_{l=1}^{M} \left|\frac{1}{\Gamma(1-\beta l)} \frac{1}{(-t)^{2l+2m}}\right| + O\left[\frac{1}{(-t)^{2M+2m}}\right]\right\} t^\delta dt < \infty.\end{aligned}$$

(ii) For $\delta \in (-1, 2\alpha - 1)$, by the change of variable $u = |\xi|^\alpha t^\alpha$, we can obtain

$$\begin{aligned}\int_0^\infty \left|\widehat{G}^{\alpha,\beta}_{t^{\alpha/\beta}}(\xi)\right|^2 t^\delta dt &= \int_0^\infty \left|E_\beta(-|\xi|^\alpha t^\alpha)\right|^2 t^\delta dt \\ &= \alpha^{-1}|\xi|^{-\delta-1} \int_0^\infty \left|E_\beta(-u)\right|^2 u^{(\delta+1-\alpha)/\alpha} du \\ &= M(n,\alpha,\beta,\delta)|\xi|^{-\delta-1}.\end{aligned}$$

(iii) Notice that

$$\int_0^\infty \left|\frac{\partial^m \widehat{G}^{\alpha,\beta}_{t^{\alpha/\beta}}(\xi)}{\partial t^m}\right|^2 t^\delta dt = \int_0^\infty \left|\frac{\partial^m E_\beta(-|\xi|^\alpha t^\alpha)}{\partial t^m}\right|^2 t^\delta dt$$
$$\approx |\xi|^{2m-\delta-1}\int_0^\infty \left|\sum_{i=1}^m u^{(\delta+1-\alpha+2i\alpha-2m)/(2\alpha)}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du.$$

For $\delta \in (2m-1, 2\alpha+2m-1)$ and $m \in \mathbb{Z}_+$, (2.4) implies

$$\int_0^\infty \left|\sum_{i=1}^m u^{(\delta+1-\alpha+2i\alpha-2m)/(2\alpha)}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du \lesssim \sum_{i=1}^m \int_0^\infty \left|\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 u^{(\delta+1-\alpha+2i\alpha-2m)/\alpha} du < \infty.$$

Then

$$\int_0^\infty \left|\frac{\partial^m \widehat{G}^{\alpha,\beta}_{t^{\alpha/\beta}}(\xi)}{\partial t^m}\right|^2 t^\delta dt \approx |\xi|^{2m-\delta-1}.$$

This proves (2.5). ∎

Denote by $\mathcal{M}$ the Hardy–Littlewood maximal operator, i.e.,

$$\mathcal{M}f(x) := \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

Lemma 2.6 *For $m \in \mathbb{Z}_+$, $\alpha > 1/2$ and $\beta \in (0,1]$, there exists a constant C depending only on n, α and β such that for $x, y \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$,*

$$|G_t^{\alpha,\beta} * f(x-y)| \le C\Big(1+|y|/t^{\beta/\alpha}\Big)^n \mathcal{M}f(x).$$

Proof Let $f_{t,\alpha,\beta}(x) := f(t^{\beta/\alpha}x)$. For any ball $B \subset \mathbb{R}^n$ centered at x_B with radius r_B, define $B_{t,\alpha,\beta}$ as $B_{t,\alpha,\beta} := B(t^{\beta/\alpha}x_B, t^{\beta/\alpha}r_B)$. If $x \in B$, then $t^{\beta/\alpha}x \in B_{t,\alpha,\beta}$. Hence, it is easy to see that

$$\mathcal{M}f_{t,\alpha,\beta}(x/t^{\beta/\alpha}) \le \sup_{x\in B_{t,\alpha,\beta}}\frac{1}{|B_{t,\alpha,\beta}|}\int_{B_{t,\alpha,\beta}}|f(y)|dy \le \mathcal{M}f(x). \tag{2.7}$$

We only need to prove

$$|G_1^{\alpha,\beta} * f(x-y)| \lesssim (1+|y|)^n \mathcal{M}f(x). \tag{2.8}$$

In fact, if (2.8) holds, (2.7) and the change of variable $z = t^{\alpha/\beta}u$ give

$$|G_t^{\alpha,\beta} * f(x-y)| \approx \left|\int_{\mathbb{R}^n}\frac{t^\beta}{(t^{\beta/\alpha}+|x-y-z|)^{n+\alpha}}f(z)dz\right|$$
$$\approx \left|\int_{\mathbb{R}^n} t^{-n\beta/\alpha}\frac{1}{(1+|\frac{x-y}{t^{\beta/\alpha}}-u|)^{n+\alpha}}f(t^{\beta/\alpha}u)t^{n\beta/\alpha}du\right|$$
$$\lesssim (1+|y|/t^{\beta/\alpha})^n \mathcal{M}f_{t,\alpha,\beta}(x/t^{\beta/\alpha})$$
$$\lesssim (1+|y|/t^{\beta/\alpha})^n \mathcal{M}f(x).$$

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Below, we prove (2.8). If $|x| < |y|$, it is obvious that $G_1^{\alpha,\beta}(x-y) < 1$. On the other hand, for $|x| \geq |y|$, it can be deduced from the triangle inequality that

$$G_1^{\alpha,\beta}(x-y) \lesssim \frac{1}{(1+(|x|-|y|))^{n+\alpha}}.$$

Set the decreasing radial majorant function of $G_1^{\alpha,\beta}(x-y)$ as

$$\psi_y^{\alpha,\beta}(x) := \begin{cases} \frac{1}{(1+(|x|-|y|))^{n+\alpha}}, & |x| \geq |y|, \\ 1, & |x| < |y|. \end{cases}$$

With a slight abuse of notation, let us write $\psi_y^{\alpha,\beta}(x) = \psi_y^{\alpha,\beta}(r)$ if $|x| = r$. We can get

$$\begin{aligned} |G_1^{\alpha,\beta} * f(x-y)| &\lesssim \int_{\mathbb{R}^n} \frac{1}{(1+|x-y-z|)^{n+\alpha}} |f(z)| dz \\ &\lesssim \int_{\mathbb{R}^n} \psi_y^{\alpha,\beta}(x-z)|f(z)|dz \\ &\approx \sum_{k=-\infty}^{\infty} \int_{2^k<|x-z|\leq 2^{k+1}} \psi_y^{\alpha,\beta}(x-z)|f(z)|dz \\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{n(k+1)} \psi_y^{\alpha,\beta}(2^k) \int_{|x-z|\leq 2^{k+1}} 2^{-n(k+1)}|f(z)|dz, \end{aligned}$$

which, together with the inequality:

$$\int_{|x-z|\leq 2^{k+1}} 2^{-n(k+1)}|f(z)|dz \leq \mathcal{M}f(x),$$

implies that

$$\begin{aligned} |G_1^{\alpha,\beta} * f(x-y)| &\lesssim \mathcal{M}f(x) \sum_{k=-\infty}^{\infty} \psi_y^{\alpha,\beta}(2^k) 2^{n(k+1)} \\ &\lesssim \mathcal{M}f(x) \sum_{k=-\infty}^{\infty} \psi_y^{\alpha,\beta}(2^k) \int_{2^{k-1}}^{2^k} r^{n-1} dr \\ &\lesssim \mathcal{M}f(x) \|\psi_y^{\alpha,\beta}\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

It follows from a direct computation that

$$\|\psi_y^{\alpha,\beta}\|_{L^1(\mathbb{R}^n)} \approx \int_{|x|<|y|} 1 dx + \int_{|x|\geq|y|} \frac{1}{(1+|x|-|y|)^{n+\alpha}} dx.$$

Then we use the change of variable: $|x| - |y| = r$ to deduce that

$$\begin{aligned} \|\psi_y^{\alpha,\beta}\|_{L^1(\mathbb{R}^n)} &\lesssim |y|^n + \int_0^\infty \frac{(r+|y|)^{n-1} dr}{(1+r)^{n+\alpha}} \\ &\lesssim |y|^n + \sum_{k=0}^{n-1} C_{n-1}^k |y|^{n-1-k} \int_0^\infty \frac{r^k}{(1+r)^{n+\alpha}} dr. \end{aligned}$$

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Here, C_{n-1}^k denotes the number of combinations of choosing k many objects from a group of $n-1$ many objects. Notice that $k \leq n-1$ and $n+\alpha-k \geq \alpha+1 > 1$. There exists $A > 0$ such that

$$\sup_{0\leq k\leq n-1} \left\{ \int_0^\infty \frac{r^k}{(1+r)^{n+\alpha}} dr \right\} \leq A,$$

which makes

$$\|\psi_y^{\alpha,\beta}\|_{L^1(\mathbb{R}^n)} \lesssim |y|^n + \sum_{k=-\infty}^{\infty} C_{n-1}^k |y|^{n-1-k} \lesssim (1+|y|)^n. \qquad \blacksquare$$

Lemma 2.7 *Let $m \in \mathbb{Z}_+$, $\alpha > 1/2$ and $\beta \in (0,1]$. There exists a constant C depending only on n, α and β such that for $x, y \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$,*

$$\left|G_{t^{\alpha/\beta}}^{\alpha,\beta} * f(x-y)\right| \leq C\Big(1+|y|/t\Big)^n \mathcal{M}f(x).$$

Proof Let $f_t(x) := f(tx)$. For any ball $B \subset \mathbb{R}^n$ centered at x_B with radius r_B, denote by B_t the ball $B(tx_B, tr_B)$. If $x \in B$, then $tx \in B_t$. Hence, it is easy to see that

$$\mathcal{M}f_t(x/t) \leq \sup_{x\in B_t} \frac{1}{|B_t|} \int_{B_{t,\alpha,\beta}} |f(y)|dy \leq \mathcal{M}f(x). \tag{2.9}$$

By (2.8) and (2.9), we can obtain

$$\begin{aligned}
|G_{t^{\alpha/\beta}}^{\alpha,\beta} * f(x-y)| &\approx \left| \int_{\mathbb{R}^n} \frac{t^\beta}{(t+|x-y-z|)^{n+\alpha}} f(z)dz \right| \\
&\approx \left| \int_{\mathbb{R}^n} t^{-n} \frac{1}{(1+|\frac{x-y}{t}-u|)^{n+\alpha}} f(tu)t^n du \right| \\
&\lesssim (1+|y|/t)^n \mathcal{M}f_t(x/t) \\
&\lesssim (1+|y|/t)^n \mathcal{M}f(x),
\end{aligned}$$

which indicates Lemma 2.7. $\blacksquare$

2.2 Regularity of time-space fractional heat kernels

For $m \in \mathbb{Z}_+$, define

$$\widetilde{\nabla}_x^m f(x) := \left(\frac{\partial^m f(x)}{\partial x_1^m}, \frac{\partial^m f(x)}{\partial x_2^m}, \dots, \frac{\partial^m f(x)}{\partial x_n^m} \right).$$

Lemma 2.8 *Let $\alpha > 1/2$, $m \in \mathbb{Z}_+$, $x \in \mathbb{R}^n$ and $t > 0$.*

$$\left| t^{m/\alpha} \frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m} \right| \lesssim \frac{t}{\big(t^{1/\alpha}+|x|\big)^{n+\alpha}}.$$

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Proof The subordinative formula [19, (5.31)] indicates that $K_{\alpha,t}(\cdot)$ can be expressed as

$$\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m} = \int_0^\infty \eta_t^\alpha(s)\frac{\partial^m K_s(x)}{\partial x_j^m}ds,$$

where $K_s(\cdot)$ denotes the integral kernel of the heat semigroup $\{e^{-s(-\Delta_x)}\}_{s>0}$. Here, the nonnegative continuous function $\eta_t^\alpha(\cdot)$ satisfies

$$\begin{cases} \eta_t^\alpha(s) = (1/t^{2/\alpha})\eta_1^\alpha(s/t^{2/\alpha}), \\ \eta_t^\alpha(s) \le t/s^{1+\alpha/2} \quad \forall\ s,\ t>0, \\ \int_0^\infty s^{-\gamma}\eta_1^\alpha(s)ds < \infty, \quad \gamma>0, \\ \eta_t^\alpha(s) \approx t/s^{1+\alpha/2} \quad \forall\ s \ge t^{2/\alpha} > 0. \end{cases} \tag{2.10}$$

When m is odd, we can get

$$\left|\frac{\partial^m K_s(x)}{\partial x_j^m}\right| = \left|\frac{\partial^m\left(s^{-n/2}e^{-|x|^2/s}\right)}{\partial x_j^m}\right| \approx \left|s^{-n/2}e^{-|x|^2/s}\sum_{i=0}^{(m-1)/2}\frac{x_j^{2i+1}}{s^{(m-1)/2+1+i}}\right|.$$

Then

$$\begin{aligned}\left|\frac{\partial^m K_s(x)}{\partial x_j^m}\right| &\approx s^{-n/2}e^{-|x|^2/s}\left|\sum_{i=0}^{(m-1)/2}\frac{x_j^{2i+1}}{s^{(m-1)/2+i+1}}\right| \\ &\lesssim s^{-(m+n)/2}e^{-|x|^2/s}\sum_{i=0}^{(m-1)/2}\left|\left(|x|^2/s\right)^{(2i+1)/2}\right| \\ &\lesssim s^{-(m+n)/2}e^{-c_m|x|^2/s},\end{aligned}$$

where c_m is a constant depending on m.

When m is even, we have

$$\frac{\partial^m K_s(x)}{\partial x_j^m} \approx s^{-n/2}e^{-|x|^2/s}\sum_{i=0}^{m/2}\frac{x_j^{2i}}{s^{m/2+i}}.$$

For this case, we can obtain

$$\left|\frac{\partial^m K_s(x)}{\partial x_j^m}\right| \lesssim s^{-(m+n)/2}e^{-c_m|x|^2/s}.$$

Then letting $m \in \mathbb{Z}_+$, we can get

$$\left|\frac{\partial^m K_s(x)}{\partial x_j^m}\right| \lesssim s^{-(n+m)/2}e^{-c_m|x|^2/s}. \tag{2.11}$$

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By $s = t^{2/\alpha}h$, (2.10) and (2.11) imply

$$\begin{aligned}\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| &= \left|\int_0^\infty \eta_t^\alpha(s)\frac{\partial^m K_s(x)}{\partial x_j^m}ds\right| \\ &\lesssim \int_0^\infty \left|\frac{t}{s^{1+\alpha}}\frac{\partial^m K_s(x)}{\partial x_j^m}\right| ds \\ &\lesssim \int_0^\infty \frac{t}{s^{1+\alpha}} s^{-(n+m)/2} e^{-c_m|x|^2/s}\Big|_{s=t^{2/\alpha}h} ds \\ &\lesssim \int_0^\infty \frac{t}{(t^{2/\alpha}h)^{1+\alpha/2+(n+m)/2}} e^{-c_m|x|^2/(t^{2/\alpha}h)} t^{2/\alpha} dh \\ &= \int_0^\infty \frac{1}{t^{(n+m)/2}h^{1+\alpha/2+(n+m)/2}} e^{-c_m|x|^2/(t^{2/\alpha}h)} dh.\end{aligned}$$

Let $y = t^{2/\alpha}h/|x|^2$. It holds

$$\begin{aligned}\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| &\lesssim \int_0^\infty \frac{e^{-c_m/y}|x|^2}{t^{(n+m)/\alpha}(y|x|^2/t^{2/\alpha})^{1+\alpha/2+(n+m)/2}t^{2/\alpha}}dy \\ &= \frac{t}{|x|^{\alpha+n+m}}\left(\int_0^1 \frac{1}{y^{1+\alpha/2+(n+m)/2}}e^{-c_m/y}dy + \int_1^\infty \frac{1}{y^{1+\alpha/2+(n+m)/2}}e^{-c_m/y}dy\right) \\ &\lesssim \frac{t}{|x|^{\alpha+n+m}}\left(\int_0^1 \frac{1}{y^{1+\alpha/2+(n+m)/2}}y^{1+\alpha/2+(n+m)/2}dy + \int_1^\infty \frac{1}{y^{1+\alpha/2+(n+m)/2}}dy\right) \\ &\lesssim \frac{t}{|x|^{\alpha+n+m}}.\end{aligned}$$

On the other hand, the fact $\eta_t^\alpha(s) = (1/t^{2/\alpha})\eta_1^\alpha(s/t^{2/\alpha})$ implies

$$\begin{aligned}\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| &\lesssim \int_0^\infty \frac{1}{t^{2/\alpha}}\eta_1^\alpha\left(\frac{s}{t^{2/\alpha}}\right)\frac{1}{s^{(n+m)/2}}ds \\ &\lesssim \frac{1}{t^{(n+m)/\alpha}}\int_0^\infty \frac{\eta_1^\alpha(h)}{h^{(n+m)/2}}dh.\end{aligned}$$

By (2.10), we get

$$\int_0^\infty \frac{\eta_1^\alpha(h)}{h^{(n+m)/2}}dh < \infty,$$

which gives

$$\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| \lesssim \frac{1}{t^{(n+m)/\alpha}}.$$

Thus, it indicates that

$$\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| \lesssim \min\left\{\frac{t}{|x|^{\alpha+n+m}}, \quad \frac{1}{t^{(n+m)/\alpha}}\right\}.$$

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Case 1: $0 \le t^{1/\alpha} \le |x|$. We have

$$\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| \lesssim \frac{t}{2|x|^{\alpha+n+m}} \lesssim \frac{t^{m/\alpha}}{|x|^m}\frac{t^{1-m/\alpha}}{|x|^{\alpha+n}+t^{(n+\alpha)/\alpha}} \lesssim \frac{t^{1-m/\alpha}}{\left(t^{1/\alpha}+|x|\right)^{n+\alpha}}.$$

Case 2: $|x| < t^{1/\alpha}$. It can be deduced that

$$\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| \lesssim \frac{1}{t^{(n+m)/\alpha}} \lesssim \frac{t^{1-m/\alpha}}{\left(t^{1/\alpha}+|x|\right)^{n+\alpha}},$$

which gives

$$\left|t^{m/\alpha}\frac{\partial^m K_{\alpha,t}(x)}{\partial x_j^m}\right| \lesssim \frac{t}{\left(t^{1/\alpha}+|x|\right)^{n+\alpha}}. \qquad \blacksquare$$

Lemma 2.9 *For $m \in \mathbb{Z}_+$, $\alpha > 1/2$, $\beta \in (0,1]$ and $t > 0$, there holds*

$$\left|t^m\frac{\partial^m K_{\alpha,t}(x)}{\partial t^m}\right| \lesssim \frac{t}{\left(t^{1/\alpha}+|x|\right)^{n+\alpha}}.$$

Proof By (2.10), we get

$$K_{\alpha,t}(x) = \int_0^\infty \frac{\eta_1^\alpha(s/t^{2/\alpha})K_s(x)}{t^{2/\alpha}}ds = \int_0^\infty \eta_1^\alpha(h)K_{t^{2/\alpha}h}(x)dh.$$

Then

$$\frac{\partial^m K_{\alpha,t}(x)}{\partial t^m} = \frac{\partial^m}{\partial t^m}\left(\int_0^\infty \eta_1^\alpha(h)K_{t^{2/\alpha}h}(x)dh\right) = \int_0^\infty \eta_1^\alpha(h)\frac{\partial^m K_{t^{2/\alpha}h}(x)}{\partial t^m}dh.$$

By (2.2) and the higher-order derivative formula of composite functions, we can obtain

$$\begin{aligned}\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial t^m}\right| &\lesssim \sum_{i=1}^m\left|\int_0^\infty \eta_1^\alpha(h)t^{2i/\alpha-m}h^i\left.\frac{\partial^i K_s(x)}{\partial s^i}\right|_{s=t^{2/\alpha}h}dh\right|\\ &\lesssim \int_0^\infty \frac{\eta_1^\alpha(h)}{t^m(t^{2/\alpha}h)^{n/2}e^{c|x|^2/(t^{2/\alpha}h)}}dh.\end{aligned}$$

Letting $s = t^{2/\alpha}h/|x|^2$ and $y = t^{2/\alpha}h/|x|^2$, we can get

$$\begin{aligned}\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial t^m}\right| &\lesssim t^{-m}\int_0^\infty \frac{t(|x|^2/t^{2/\alpha})}{t^{n/\alpha}(y|x|^2/t^{2/\alpha})^{n/2+1+\alpha/2}e^{c/y}}dy\\ &\lesssim t^{-m}\left(\int_0^1 \frac{t}{|x|^{\alpha+n}}dy + \int_1^\infty \frac{t}{|x|^{\alpha+n}y^{n/2+1+\alpha/2}}dy\right)\\ &\lesssim \frac{1}{t^{m-1}|x|^{\alpha+n}}.\end{aligned}$$

On the other hand, the fact $\int_0^\infty \eta_1^\alpha(h)h^{-n/2}dh < \infty$ implies

$$\left|\frac{\partial^m K_{\alpha,t}(x)}{\partial t^m}\right| \lesssim \int_0^\infty \frac{\eta_1^\alpha(h)}{t^m(t^{2/\alpha}h)^{n/2}e^{c|x|^2/(t^{2/\alpha}h)}}dh$$
$$\lesssim t^{-(m+n/\alpha)}\int_0^\infty \eta_1^\alpha(h)/h^{n/2}dh$$
$$\lesssim t^{-(m+n/\alpha)}.$$

Thus, we can get

$$\left|t^m\frac{\partial^m K_{\alpha,t}(x)}{\partial t^m}\right| \lesssim \min\left\{\frac{t}{(t^{1/\alpha})^{n+\alpha}}, \frac{t}{|x|^{\alpha+n}}\right\} \lesssim \frac{t}{(t^{1/\alpha}+|x|)^{n+\alpha}}. \qquad \blacksquare$$

Below, we estimate the regularity of the time-space fractional heat kernel $G_t^{\alpha,\beta}(\cdot)$.

Proposition 2.10 *Let $\alpha > n+m$, $m \in \mathbb{Z}_+$ and $\beta \in (0,1]$. Then*

$$\left|t^m\frac{\partial^m\left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial x_i^m}\right| \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}}.$$

Proof By Lemma 2.8, we know

$$\left|\frac{\partial^m\left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial x_i^m}\right| = \left|\int_0^\infty \frac{\partial^m K_{\alpha,(t^{\alpha/\beta}/u)^\beta}(x)}{\partial x_i^m}g_\beta(u)du\right|$$
$$\lesssim \int_0^\infty \frac{(t^{\alpha/\beta}/u)^{\beta(1-m/\alpha)}g_\beta(u)}{((t^{\alpha/\beta}/u)^{\beta/\alpha}+|x|)^{n+\alpha}}du$$
$$= I + II,$$

where

$$\begin{cases} I := \int_0^1 \frac{(t^{\alpha/\beta}/u)^{\beta(1-m/\alpha)}g_\beta(u)}{((t^{\alpha/\beta}/u)^{\beta/\alpha}+|x|)^{n+\alpha}}du,\\ II := \int_1^\infty \frac{(t^{\alpha/\beta}/u)^{\beta(1-m/\alpha)}g_\beta(u)}{((t^{\alpha/\beta}/u)^{\beta/\alpha}+|x|)^{n+\alpha}}du.\end{cases}$$

For the term I, by Definition 2.1, since $\lim_{u\to 0} g_\beta(u) = 0$, we can get

$$I \lesssim \int_0^1 \frac{t^{\alpha-m}u^{-\beta(1-m/\alpha)}}{((t^{\alpha/\beta}/u)^{\beta/\alpha}+|x|))^{n+\alpha}}du.$$

Since $\alpha > 0$,

$$I \lesssim \frac{t^{\alpha-m}}{|x|^{n+\alpha}}\int_0^1 u^{-\beta(1-m/\alpha)}du \lesssim \frac{t^{\alpha-m}}{|x|^{n+\alpha}}$$

and

$$I \lesssim t^{-n-m}\int_0^1 u^{\beta(m+n)/\alpha}du \lesssim t^{-n-m}.$$

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When $u \to \infty$ and $m < \alpha$, we know $g_\beta(u) \approx \frac{\beta}{\Gamma(1-\beta)} u^{-\beta-1}$. Then

$$II \lesssim \int_1^\infty \frac{(t^{\alpha/\beta}/u)^{\beta - m\beta/\alpha} u^{-\beta-1}}{((t^{\alpha/\beta}/u)^{\beta/\alpha} + |x|)^{n+\alpha}} du.$$

We can obtain

$$II \lesssim \frac{t^{\alpha-m}}{|x|^{n+\alpha}} \int_1^\infty u^{-\beta(2-m/\alpha)-1} du \lesssim \frac{t^{\alpha-m}}{|x|^{n+\alpha}} \quad \text{when } \alpha > m/2$$

and

$$II \lesssim t^{-n-m} \int_1^\infty u^{-\beta+\beta(n+m)/\alpha-1} du \lesssim t^{-n-m} \quad \text{when } \alpha > n+m.$$

Let $\alpha > n+m$. If $|x| < t$,

$$\left| t^m \frac{\partial^m \left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial x_i^m} \right| \lesssim \min\left\{ \frac{t^\alpha}{|x|^{n+\alpha}}, \frac{t^\alpha}{t^{n+\alpha}} \right\} \lesssim \frac{t^\alpha}{2t^{n+\alpha}} \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}}.$$

On the other hand, if $|x| \geq t$,

$$\left| t^m \frac{\partial^m \left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial x_i^m} \right| \lesssim \min\left\{ \frac{t^\alpha}{|x|^{n+\alpha}}, \frac{t^\alpha}{t^{n+\alpha}} \right\} \lesssim \frac{t^\alpha}{2|x|^{n+\alpha}} \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}}.$$

Then

$$\left| t^m \frac{\partial^m \left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial x_i^m} \right| \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}},$$

which proves Proposition 2.10. ∎

Similarly, we can obtain the following result.

Proposition 2.11 *For $m \in \mathbb{Z}_+$, $\alpha > n+m$, $\beta \in (0,1]$ and $t > 0$, there holds*

$$\left| t^m \frac{\partial^m \left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial t^m} \right| \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}}.$$

Proof At first, (2.1) implies

$$\left| \frac{\partial^m \left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial t^m} \right| = \left| \int_0^\infty \frac{\partial^m \left(K_{\alpha,(t^{\alpha/\beta}/u)^\beta}(x)\right)}{\partial t^m} g_\beta(u) du \right|$$
$$\lesssim \int_0^\infty \sum_{j=1}^m \left| \left(u^{-j\beta} t^{j\alpha-m} \frac{\partial^m K_{\alpha,s}(x)}{\partial s^m}\Big|_{s=(t^{\alpha/\beta}/u)^\beta} \right) g_\beta(u) \right| du.$$

Then it follows from Proposition 2.10 that

$$\left| \frac{\partial^j K_{\alpha,s}(x)}{\partial s^j} \right| \lesssim \frac{s^{1-j}}{(s^{1/\alpha}+|x|)^{n+\alpha}},$$

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which yields

$$\begin{aligned}\left|\frac{\partial^m\left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial t^m}\right| &\lesssim \int_0^\infty \sum_{j=1}^m \left(u^{-j\beta}t^{j\alpha-m}\frac{(t^{\alpha/\beta}/u)^{(1-j)\beta}}{((t^{\alpha/\beta}/u)^{\beta/\alpha}+|x|)^{n+\alpha}}\right)g_\beta(u)du\\ &\lesssim t^{\alpha-m}\int_0^\infty \frac{g_\beta(u)/u^\beta}{(t/u^{\beta/\alpha}+|x|)^{n+\alpha}}du\\ &\lesssim I+II,\end{aligned}$$

where

$$\begin{cases} I:=t^{\alpha-m}\int_0^1 \frac{g_\beta(u)/u^\beta}{(t/u^{\beta/\alpha}+|x|)^{n+\alpha}}du,\\ II:=t^{\alpha-m}\int_1^\infty \frac{g_\beta(u)/u^\beta}{(t/u^{\beta/\alpha}+|x|)^{n+\alpha}}du.\end{cases}$$

According to [17, (2.4) and (2.5)], we have

$$\begin{cases}\lim_{u\to\infty} g_\beta(u)=u^{-\beta-1},\\ \lim_{u\to 0} g_\beta(u)=0.\end{cases}$$

Noting that $\alpha>n$ and $\beta(n+\alpha)/\alpha<2\beta$, we can obtain

$$I\lesssim t^{-m-n}\int_0^1 u^{\beta n/\alpha}du\lesssim t^{-n-m}$$

and

$$II\lesssim t^{-m-n}\int_1^\infty u^{-2\beta-1+\beta(n+\alpha)/\alpha}du\lesssim t^{-m-n}.$$

On the other hand, since

$$\left|\frac{\partial^m\left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial t^m}\right|\lesssim \frac{t^{\alpha-m}}{|x|^{n+\alpha}}\int_1^\infty u^{-2\beta-1}du+\frac{t^{\alpha-m}}{|x|^{n+\alpha}}\int_0^1 u^{-\beta}du\lesssim\frac{t^{\alpha-m}}{|x|^{n+\alpha}},$$

we get

$$\left|\frac{\partial^m\left(G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)\right)}{\partial t^m}\right|\lesssim \min\left\{\frac{t^{\alpha-m}}{|x|^{n+\alpha}},\frac{t^{-m}}{t^n}\right\}\lesssim\frac{t^{\alpha-m}}{(t^{n+\alpha}+|x|^{n+\alpha})}\lesssim\frac{t^{\alpha-m}}{(t+|x|)^{n+\alpha}}.$$

The proof of Proposition 2.11 is completed. ∎

3 Fractional trace inequalities via the time-space fractional extension

3.1 Fractional trace inequalities involving $\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})$ and $\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}$

Theorem 3.1 *Let $m\in\mathbb{Z}_+$, $\alpha>n+m$, $\beta\in(0,1]$, $f\in\dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu\in(0,\min\{2m,n\})$. For $(x,t)\in\mathbb{R}^n\times(0,\infty)$, denote by $u(x,t)=G^{\alpha,\beta}_t * f(x)$ the time-space fractional extension of f.*

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(i) *There holds*

$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\,dx\right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu}\,dx\,dt. \tag{3.1}$$

(ii) *If* $\|f\|_{L^2(\mathbb{R}^n)} = 1$, *there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2)\,dx\right) \lesssim \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu}\,dx\,dt. \tag{3.2}$$

(iii) *There holds*

$$\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^\nu} \lesssim \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu}\,dx\,dt. \tag{3.3}$$

Proof In order to prove (i) of Theorem 3.1, we need to establish the following result: for $\nu \in (0, \min\{2m, n\})$,

$$\int_{\mathbb{R}^{n+1}_+} \left|\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})\right|^2 t^{2m-1-\nu}\,dx\,dt \approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2\,d\xi. \tag{3.4}$$

In fact, notice that $u(x,t) = G_t^{\alpha,\beta} * f(x)$. It holds

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu}\,dx\,dt &= \int_{\mathbb{R}^{n+1}_+} |\widehat{\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})}|^2 t^{2m-1-\nu}\,dx\,dt \\
&\approx \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2m} |\widehat{u}(\xi, t^{\alpha/\beta})|^2 t^{2m-1-\nu}\,d\xi\,dt \\
&\approx \int_{\mathbb{R}^n} \left(\int_0^\infty (E_\beta(-\omega))^2 \omega^{(2m-\nu-\alpha)/\alpha}\,d\omega\right) |\xi|^\nu |\widehat{f}(\xi)|^2\,d\xi.
\end{aligned}$$

In (2.4) of Lemma 2.5, take $\nu \in (0, \min\{2m, n\})$ and $m < \alpha$. We get

$$\int_0^\infty (E_\beta(-\omega))^2 \omega^{(2m-\nu-\alpha)/\alpha}\,d\omega < \infty.$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu}\,dx\,dt &\approx \int_{\mathbb{R}^n} \left(\int_0^\infty (E_\beta(-\omega))^2 \omega^{(2-\nu-\alpha)/\alpha}\,d\omega\right) |\xi|^\nu |\widehat{f}(\xi)|^2\,d\xi \\
&\approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2\,d\xi.
\end{aligned}$$

We know that

$$\begin{aligned}
\int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2\,d\xi &= \|\widehat{f}(\cdot)|\cdot|^{\nu/2}\|_{L^2(\mathbb{R}^n)} \\
&= \|\widehat{(-\Delta_x)^{\nu/4} f}\|_{L^2(\mathbb{R}^n)} \\
&= \|(-\Delta_x)^{\nu/4} f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

It follows from the well-known fractional Sobolev inequality:

$$\|f\|_{L^{2n/(n-\nu)}(\mathbb{R}^n)} \le B(n,\nu)\|(-\Delta_x)^{\nu/4} f\|_{L^2(\mathbb{R}^n)}$$

for $\nu \in (0, n)$ and some constant $B(n,\nu)$ that (3.1) holds.

Now, we want to prove (ii) of Theorem 3.1. Let $p = n(r-2)/\nu$, $2 < r < 2n/(n-\nu)$ and $\nu \in (0, \min\{n, 2m\})$. The Hölder inequality implies that

(3.5)

$$\|f\|^r_{L^r(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^p |f(x)|^{r-p} dx \le \|f\|^p_{L^{2n/(n-\nu)}(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |f(x)|^2 dx\right)^{1-p(n-\nu)/(2n)}.$$

If $\|f\|_{L^2(\mathbb{R}^n)} = 1$, it can be deduced from (3.5) that

$$\left(\int_{\mathbb{R}^n} |f(x)|^{r-2}|f(x)|^2 dx\right)^{1/(r-2)} = \left(\int_{\mathbb{R}^n} |f(x)|^r dx\right)^{1/(r-2)} \le \|f\|^{n/\nu}_{L^{2n/(n-\nu)}(\mathbb{R}^n)}.$$

The inequality (3.1) implies that for a positive constant $A(n, s, \nu)$,

$$\begin{aligned}&\left(\int_{\mathbb{R}^n} |f(x)|^{r-2}|f(x)|^2 dx\right)^{1/(r-2)} \\ &\quad\le \left(A(n,s,\nu) \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}^m_x u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu} dx dt\right)^{n/(2\nu)},\end{aligned}$$

which yields

$$\begin{aligned}&\exp\left(\frac{2\nu}{n(r-2)} \ln\left(\int_{\mathbb{R}^n} |f(x)|^{r-2}|f(x)|^2 dx\right)\right) \\ &\quad\le \left(A(n,s,\nu) \int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}^m_x u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu} dx dt\right)^{n/(2\nu)}.\end{aligned}$$

Since $\|f\|_{L^2(\mathbb{R}^n)} = 1$, $d\mu(x) := |f(x)|^2 dx$ can be treated as a probability measure on $\mathbb{R}^n$. Thus (3.2) can be obtained by letting $r \to 2$. In fact,

$$\begin{aligned}&\lim_{r\to 2} \exp\left(\frac{2\nu}{n(r-2)} \ln\left(\int_{\mathbb{R}^n} |f(x)|^{r-2}|f(x)|^2 dx\right)\right) \\ &= \exp\left(\frac{\nu \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2)}{n \int_{\mathbb{R}^n} |f(x)|^2 dx}\right) \\ &= \exp\left(\frac{\nu}{n} \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) dx\right),\end{aligned}$$

which implies (3.2).

At last, the inequality (3.3) follows from (3.4) and the fractional Hardy inequality.

$$\left\|\frac{f(\cdot)}{|\cdot|^{\nu/2}}\right\|_{L^2(\mathbb{R}^n)} \le H\|(-\Delta_x)^{\nu/4} f\|_{L^2(\mathbb{R}^n)},$$

which is a special case of [42, (3.1) in Theorem 3.1] ∎

As an immediate corollary of Theorem 3.1, we can obtain the following result.

Theorem 3.2 *Let $m \in \mathbb{Z}_+$, $\alpha > n + m$, $\beta \in (0, 1]$, $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ and $\nu \in (0, \min\{2m, n\})$. For $(x,t) \in \mathbb{R}^n \times (0, \infty)$, denote by $u(x,t) = G^{\alpha,\beta}_t * f(x)$ the time-space fractional extension of f. Then, the following statements are true.*

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(i) *There holds*

$$\left(\int_{\mathbb{R}^n}|f(x)|^{2n/(n-\nu)}\,dx\right)^{1-\nu/n}\lesssim\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t)|^2 t^{\beta(2m-\nu)/\alpha-1}dxdt.$$

(ii) *If* $\|f\|_{L^2(\mathbb{R}^n)}=1$, *there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx\right)\lesssim\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t)|^2 t^{\beta(2m-\nu)/\alpha-1}dxdt.$$

(iii) *There holds*

$$\int_{\mathbb{R}^n}|f(x)|^2\frac{dx}{|x|^\nu}\lesssim\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t)|^2 t^{\beta(2m-\nu)/\alpha-1}dxdt.$$

Proof By the change of variable: $t=\omega^{\alpha/\beta}$, we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t)|^2 t^{\beta(2m-\nu)/\alpha-1}dxdt&=\frac{\alpha}{\beta}\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,\omega^{\alpha/\beta})|^2\omega^{2m-1-\nu}dxd\omega\\&\approx\int_{\mathbb{R}^n}|\xi|^\nu|\widehat{f}(\xi)|^2d\xi.\end{aligned}$$

Similarly to the proof of Theorem 3.1, we can prove Theorem 3.2. ■

Theorem 3.3 *Let* $m\in\mathbb{Z}_+$, $\alpha>n+m$, $\beta\in(0,1]$, $f\in\dot{H}^{\nu/2}(\mathbb{R}^n)$ *with* $\nu\in(0,\min\{2\alpha,n\})$. *For* $(x,t)\in\mathbb{R}^n\times(0,\infty)$, *denote by* $u(x,t)=G^{\alpha,\beta}_t*f(x)$ *the time-space fractional extension of f.*

(i) *There holds*

$$\left(\int_{\mathbb{R}^n}|f(x)|^{2n/(n-\nu)}\,dx\right)^{1-\nu/n}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt. \tag{3.6}$$

(ii) *If* $\|f\|_{L^2(\mathbb{R}^n)}=1$, *there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx\right)\lesssim\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt. \tag{3.7}$$

(iii) *There holds*

$$\int_{\mathbb{R}^n}|f(x)|^2\frac{dx}{|x|^\nu}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt. \tag{3.8}$$

Proof In order to prove (3.6), we need to prove that for $\nu\in(0,\min\{2\alpha,n\})$, there exists a constant $a(n,\alpha,\beta,\nu)$ such that

$$\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt=a(n,\alpha,\beta,\nu)\int_{\mathbb{R}^n}|\xi|^\nu|\widehat{f}(\xi)|^2d\xi.$$

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In fact, noting that $u(x,t^{\alpha/\beta}) = G^{\alpha,\beta}_{t^{\alpha/\beta}} * f(x)$, we can apply (2.5) to deduce that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}\,dxdt\\ &= \int_0^\infty\int_{\mathbb{R}^n} t^{2m-1-\nu}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 d\xi dt\\ &= \int_{\mathbb{R}^n}\int_0^\infty t^{2m-1-\nu}\left|\frac{\partial^m E_\beta(-|\xi|^\alpha t^\alpha)}{\partial t^m}\right|^2 |\widehat{f}(\xi)|^2 d\xi dt.\end{aligned}$$

Denote

$$I := \int_{\mathbb{R}^n}\int_0^\infty t^{2m-1-\nu}\left|\frac{\partial^m E_\beta(-|\xi|^\alpha t^\alpha)}{\partial t^m}\right|^2 |\widehat{f}(\xi)|^2 d\xi dt.$$

Let $u = |\xi|^\alpha t^\alpha$. Then

$$\left|\frac{\partial^m E_\beta(-|\xi|^\alpha t^\alpha)}{\partial t^m}\right|^2 \approx \left|\sum_{i=1}^m |\xi|^{i\alpha} t^{i\alpha-m}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 = |\xi|^{2m}\left|\sum_{i=1}^m u^{i-m/\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2.$$

We get

$$\begin{aligned} I &\approx \int_{\mathbb{R}^n}\left(\int_0^\infty\left|\sum_{i=1}^m u^{i-m/\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 u^{(2m-\nu-\alpha)/\alpha}|\xi|^\nu du\right)|\widehat{f}(\xi)|^2 d\xi\\ &= \int_{\mathbb{R}^n}\left(\int_0^\infty\left|\sum_{i=1}^m u^{i-1/2-\nu/2\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du\right)|\xi|^\nu|\widehat{f}(\xi)|^2 d\xi.\end{aligned}$$

By Lemma 2.5, for $\nu \in (0,\min\{2\alpha,n\})$ and $\alpha > m$, we can obtain $-1 < 2i-1-\nu/\alpha < 2i+1$ and

$$\int_0^\infty u^{2i-1-\nu/\alpha}\left|\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du < \infty.$$

Then

$$\int_0^\infty\left|\sum_{i=1}^m u^{i-1/2-\nu/2\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du \lesssim \sum_{i=1}^m\int_0^\infty u^{2i-1-\nu/\alpha}\left|\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du < \infty,$$

which indicates

$$\int_0^\infty\int_{\mathbb{R}^n}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu} d\xi dt \approx \int_{\mathbb{R}^n}|\xi|^\nu|\widehat{f}(\xi)|^2 d\xi.$$

Theorem 3.3 can be proved in a way similar to that of Theorem 3.1. ■

Theorem 3.4 *Let $m \in \mathbb{Z}_+$, $\alpha > n+m$, $\beta \in (0,1]$, $f \in \dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0,\min\{2\alpha,n\})$. For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, denote by $u(x,t) = G^{\alpha,\beta}_t * f(x)$ the time-space fractional extension of f. Then the following statements are true.*

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(i) *There holds*

$$\left(\int_{\mathbb{R}^n} |f(x)|^{2n/(n-\nu)}\,dx\right)^{1-\nu/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2 t^{2m-1-\beta\nu/\alpha}\,dxdt.$$

(ii) *If* $\|f\|_{L^2(\mathbb{R}^n)} = 1$, *there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2)\,dx\right) \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2 t^{2m-1-\beta\nu/\alpha}\,dxdt.$$

(iii) *There holds*

$$\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^\nu} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2 t^{2m-1-\beta\nu/\alpha}\,dxdt.$$

Proof The proof of this theorem is similar to that of Theorem 3.3. Notice that

$$\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2 t^{2m-1-\beta\nu/\alpha}\,dxdt = \int_0^\infty \int_{\mathbb{R}^n} t^{2m-1-\nu\alpha/\beta} \left|\frac{\partial^m E_\beta(-|\xi|^\alpha t^\beta)}{\partial t^m}\right|^2 |\widehat{f}(\xi)|^2\,d\xi dt.$$

Letting $u = |\xi|^\alpha t^\beta$, we can get

$$\left|\frac{\partial^m E_\beta(-|\xi|^\alpha t^\beta)}{\partial t^m}\right|^2 \approx \left|\sum_{i=1}^m |\xi|^{i\alpha} t^{i\beta-m} \frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 \approx |\xi|^{2m\alpha/\beta} \left|\sum_{i=1}^m u^{i-m/\beta} \frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2 t^{2m-1-\beta\nu/\alpha}\,dxdt \\
&\approx \int_{\mathbb{R}^n}\left(\int_0^\infty \left|\sum_{i=1}^m u^{i-m/\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 u^{(2m-\nu-\alpha)/\alpha}|\xi|^\nu\,du\right)|\widehat{f}(\xi)|^2\,d\xi \\
&= \int_{\mathbb{R}^n}\left(\int_0^\infty \left|\sum_{i=1}^m u^{i-1/2-\nu/2\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du\right)|\xi|^\nu|\widehat{f}(\xi)|^2\,d\xi.
\end{aligned}$$

By Lemma 2.5, for $\nu \in (0, \min\{2\alpha, n\})$ and $\alpha > m$, we can obtain $-1 < 2i - 1 - \nu/\alpha < 2i + 1$ and

$$\int_0^\infty u^{2i-1-\nu/\alpha}\left|\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du < \infty.$$

Then

$$\left(\int_0^\infty \left|\sum_{i=1}^m u^{i-1/2-\nu/2\alpha}\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du\right) \lesssim \sum_{i=1}^m \int_0^\infty u^{2i-1-\nu/\alpha}\left|\frac{\partial^i E_\beta(-u)}{\partial u^i}\right|^2 du < \infty,$$

which indicates

$$\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m u(x,t)}{\partial t^m}\right|^2 t^{2m-1-\beta\nu/\alpha}\,dxdt \approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2\,d\xi.$$

The rest of the proof is similar to that of Theorem 3.3 and so is omitted. ■

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3.2 Fractional trace inequalities involving $(-\Delta_x)^{s/2}u(x,t^{\alpha/\beta})$

Theorem 3.5 *Let $m\in\mathbb{Z}_+$, $\alpha>n$, $\beta\in(0,1]$, $f\in\dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu\in(0,n)$ and $s\in(\nu/2,\alpha+\nu/2)$. For $(x,t)\in\mathbb{R}^n\times(0,\infty)$, denote by $u(x,t)=G_t^{\alpha,\beta}*f(x)$ the extension of f via the time-space fractional heat kernel. Then the following statements are true.*

(i) *There holds*

$$\left(\int_{\mathbb{R}^n}|f(x)|^{2n/(n-\nu)}\,dx\right)^{1-\nu/n}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t^{\alpha/\beta})\right|^2t^{2s-\nu-1}dxdt.$$

(ii) *If $\|f\|_{L^2(\mathbb{R}^n)}=1$, there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx\right)\lesssim\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t^{\alpha/\beta})\right|^2t^{2s-\nu-1}dxdt.$$

(iii) *There holds*

$$\int_{\mathbb{R}^n}|f(x)|^2\frac{dx}{|x|^\nu}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t^{\alpha/\beta})\right|^2t^{2s-\nu-1}dxdt.$$

Proof We only need to prove

$$\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t^{\alpha/\beta})\right|^2t^{2s-\nu-1}dxdt\approx\int_{\mathbb{R}^n}|\xi|^\nu|\widehat{f}(\xi)|^2d\xi. \tag{3.9}$$

By (2.4), we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t^{\alpha/\beta})\right|^2t^{2s-\nu-1}dxdt&\approx\int_0^\infty\int_{\mathbb{R}^n}|\xi|^{2s}|\widehat{u}(\xi,t^{\alpha/\beta})|^2t^{2s-\nu-1}d\xi dt\\&\approx\int_0^\infty\int_{\mathbb{R}^n}|\xi|^{2s}(E_\beta(-|\xi|^\alpha t^\alpha))^2|\widehat{f}(\xi)|^2d\xi t^{2s-\nu-1}dt\\&\approx\int_{\mathbb{R}^n}\left(\int_0^\infty E_\beta(-\omega)^2\omega^{(2s-\nu-\alpha)/\alpha}du\right)|\xi|^\nu|\widehat{f}(\xi)|^2d\xi\\&\approx\int_{\mathbb{R}^n}|\xi|^\nu|\widehat{f}(\xi)|^2d\xi.\end{aligned}$$

Following the procedure of the proof of Theorem 3.1, we can prove Theorem 3.5 by using (3.9). We omit the details. ∎

Theorem 3.6 *Let $m\in\mathbb{Z}_+$, $\alpha>n$, $\beta\in(0,1]$, $f\in\dot{H}^{\nu/2}(\mathbb{R}^n)$ with $\nu\in(0,n)$ and $s\in(\nu/2,\alpha+\nu/2)$. For $(x,t)\in\mathbb{R}^n\times(0,\infty)$, denote by $u(x,t)=G_t^{\alpha,\beta}*f(x)$ the extension of f via the time-space fractional heat kernel. Then the following statements are true.*

(i) *There holds*

$$\left(\int_{\mathbb{R}^n}|f(x)|^{2n/(n-\nu)}\,dx\right)^{1-\nu/n}\lesssim\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t)\right|^2t^{\beta(2s-\nu)/\alpha-1}dxdt.$$

(ii) *If $\|f\|_{L^2(\mathbb{R}^n)}=1$, there holds*

$$\exp\left(\frac{\nu}{n}\int_{\mathbb{R}^n}|f(x)|^2\ln(|f(x)|^2)dx\right)\lesssim\int_{\mathbb{R}^{n+1}_+}\left|(-\Delta_x)^{s/2}u(x,t)\right|^2t^{\beta(2s-\nu)/\alpha-1}dxdt.$$

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(iii) *There holds*

$$\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^\nu} \lesssim \int_{\mathbb{R}^{n+1}_+} \left|(-\Delta_x)^{s/2} u(x,t)\right|^2 t^{\beta(2s-\nu)/\alpha-1} dx dt.$$

Proof Similarly to Theorem 3.5, we only need to prove

$$\int_{\mathbb{R}^{n+1}_+} \left|(-\Delta_x)^{s/2} u(x,t)\right|^2 t^{\beta(2s-\nu)/\alpha-1} dx dt \approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi.$$

It follows from (2.4) that

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} \left|(-\Delta_x)^{s/2} u(x,t)\right|^2 t^{\beta(2s-\nu)/\alpha-1} dx dt &\approx \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}(\xi,t)|^2 t^{\beta(2s-\nu)/\alpha-1} d\xi dt \\
&\approx \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2s} (E_\beta(-|\xi|^\alpha t^\beta))^2 |\widehat{f}(\xi)|^2 t^{\beta(2s-\nu)/\alpha-1} d\xi dt \\
&\approx \int_{\mathbb{R}^n} \left(\int_0^\infty E_\beta(-\omega)^2 \omega^{(2s-\nu-\alpha)/\alpha} du \right) |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi \\
&\approx \int_{\mathbb{R}^n} |\xi|^\nu |\widehat{f}(\xi)|^2 d\xi.
\end{aligned}$$

■

3.3 The general case $p > 1$

In Theorem 3.1, we consider the scope of (p, ν) when $p = 2$ and $\nu \in (0, \min\{n, 2\})$. We can generalize inequalities in Theorem 3.1 to the general case $p \in (1, \infty)$.

To obtain the Sobolev-type trace inequalities for the general index p, we need a characterization of the Sobolev spaces $\dot{W}^{\nu/2}_p(\mathbb{R}^n)$ via the time-space fractional heat kernel.

Definition 3.7 For $\tau = (\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{N}^n$, denote $|\tau| = \sum_{i=1}^n \tau_i$ and $\partial^\tau := \partial^{\tau_1}_{\xi_1} \dots \partial^{\tau_n}_{\xi_n}$.

(C1) (Cancellation) Let $\widehat{\Phi} \in C^{n+1+[\Lambda]}(\mathbb{R}^n \backslash \{0\})$ such that for every $|\tau| \le n + 1 + [\Lambda]$, we have

$$\partial^\tau \widehat{\Phi} = O(|\xi|^{r-|\tau|}) \quad \text{as } |\xi| \to 0.$$

(C2) For every $\xi \in \mathbb{S}^{n-1}$, there exist $a_1, a_2 \in \mathbb{R}$ (depending on ξ) with $0 < 2a_1 \le a_2$ such that for every $a_1 < t < a_2$, $|\widehat{\Phi}(t\xi)| > 0$.

(C3) Take $\widehat{\Phi} \in C^{n+1+[\Lambda]}(\mathbb{R}^n \backslash \{0\})$ such that for every $|\tau| \le n + 1 + [\Lambda]$, we have

$$\partial^\tau \widehat{\Phi} = O(|\xi|^{-n-b}) \quad \text{as } |\xi| \to \infty.$$

Lemma 3.8 [6, Theorem 1.1(i)] *Let $\nu \in \mathbb{R}$, $0 < p \le \infty$ and $\zeta \ge 0$ with $\zeta > \nu/2 - n/p$. Assume that $(1 + |\cdot|)^\zeta \Phi \in L^1(\mathbb{R}^n)$ and Φ satisfies (C1) and (C3) for $\Lambda = n/p$, $r > \nu/2$ and $b > \Lambda - \nu/2$. If $f \in \dot{W}^{\nu/2}_p(\mathbb{R}^n)$, there exists a polynomial g such that $f - g$ is a distribution of growth ζ and we have inequalities*

$$\left(\sum_{j \in \mathbb{Z}} (2^{j\nu/2} \|\Phi_j * (f-g)\|_{L^p(\mathbb{R}^n)})^p \right)^{1/p} \lesssim \|f\|_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)}.$$

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Lemma 3.9 [6, Theorem 5.3] *Let $0 < p \le \infty$ and $\nu \in \mathbb{R}$. Let $\Lambda \ge 0$ and $b_0 > \Lambda - \nu/2$. Assume $(1+|\cdot|)^{-\zeta} f \in L^\infty$ with $\zeta \ge 0$. Suppose $\Phi \in L^1(\mathbb{R}^n)$ satisfies the (C2) with $(1+|\cdot|)^\zeta \Phi(\cdot) \in L^1(\mathbb{R}^n)$. Furthermore, assume that $\widehat{\Phi} \in C^{n+1+\max\{[\zeta],[\Lambda]\}}(\mathbb{R}^n \backslash \{0\})$ with*

$$\partial^\tau \widehat{\Phi}(\xi) = O(|\xi|^{-\max\{b_0,0\}}) \qquad \text{as } |\xi| \to \infty$$

for $|\tau| \le \max\{[\Lambda],[\zeta]\}+1$. If $\Lambda = n/p$, then

$$\|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)} \lesssim \left(\sum_{j\in\mathbb{Z}} (2^{j\nu/2} \|\Phi_j * f\|_{L^p(\mathbb{R}^n)})^p \right)^{1/p}.$$

Theorem 3.10 *Let $m \in \mathbb{Z}_+$, $\alpha > n+m+n/p$, $\beta \in (0,1]$, $f \in \dot{W}_p^{\nu/2}(\mathbb{R}^n)$ with $p>1$, $\nu \in (0,2)$ and $(1+|\cdot|)^\zeta f \in L^1(\mathbb{R}^n)$ with $\zeta > \max\{0, \nu/2 - n/p\}$. For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, denote by $u(x,t) = G_t^{\alpha,\beta} * f(x)$ the time-space fractional extension of f. Then*

$$\left(\int_{\mathbb{R}_+^{n+1}} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^p t^{pm-p\nu/2-1} dx dt \right)^{1/p} \approx \|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)} \tag{3.10}$$

and

$$\left(\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m u(x, t^{\alpha/\beta})}{\partial t^m} \right|^p t^{pm-p\nu/2-1} dx dt \right)^{1/p} \approx \|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}. \tag{3.11}$$

Proof Denote

$$\begin{cases} \Phi_{i,1,t}(x) = \partial_{x_i}^m G_{t^{\alpha/\beta}}^{\alpha,\beta}(x), \\ \widehat{\Phi}_{i,1} = \xi_i^m E_\beta(-|\xi|^\alpha), \end{cases}$$

where $\Phi_{1,t}(x) = t^{-n}\Phi_1(x/t)$. Without loss of generality, for every $|\tau| \le n+1+[\gamma]$ and all $\xi_j \in \{\xi_1, \xi_2, \ldots, \xi_n\}$, we just prove that $\xi_i^m E_\beta(-|\xi|^\alpha)$ satisfies

$$\begin{cases} \frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} = O(|\xi|^{m/2+\nu/4-|\tau|}) \qquad \text{as } |\xi| \to 0, \\ \frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} = O(|\xi|^{-n-n/p}) \qquad \text{as } |\xi| \to \infty. \end{cases} \tag{3.12}$$

Case 1: $i \ne j$.

$$\begin{aligned} \frac{\partial^{|\tau|}(\xi_i^m(E_\beta(-|\xi|^\alpha)))}{\partial \xi_j^{|\tau|}} &= \xi_i^m \frac{\partial^{|\tau|}(E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} \\ &= \xi_i^m \left(\sum_{k=0}^{|\tau|} \left(\sum_{s=0}^{[(|\tau|-k)/2]+1} C_{s,k}^{|\tau|} \xi_j^{|\tau|-2s} |\xi|^{k\alpha-2|\tau|+2s} \right) \frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha} \right) \\ &\approx \sum_{k=0}^{|\tau|} |\xi|^{k\alpha-|\tau|+m} \frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha}. \end{aligned}$$

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It follows from (2.6) that for $|\xi|$ in a neighborhood of 0, we know

$$\frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha} < \infty.$$

Since $\alpha > n + n/p + m$, we get

$$\lim_{|\xi|\to 0}|\xi|^{|\tau|-m/2-\nu/4}\frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} \approx \lim_{|\xi|\to 0}|\xi|^{|\tau|-\alpha}\left(\sum_{k=0}^{|\tau|}|\xi|^{k\alpha-|\tau|+m}\frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha}\right) = 0.$$

Then (2.5) and $\alpha > n + n/p + m$ imply

$$\lim_{|\xi|\to\infty}|\xi|^{n+n/p}\frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} \approx \lim_{|\xi|\to\infty}\sum_{k=0}^{|\tau|}|\xi|^{k\alpha-|\tau|+m+n+n/p}\frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha} = 0.$$

Case 2: $i = j$. When $m \geq |\tau|$, we have

$$\begin{aligned}
&\frac{\partial^{|\tau|}(\xi_i^m(E_\beta(-|\xi|^\alpha)))}{\partial \xi_i^{|\tau|}}\\
&\quad = C_0\xi_i^{m-|\tau|}E_\beta(-|\xi|^\alpha) + \sum_{w=1}^{|\tau|}C_w\xi_i^{m-|\tau|+w}\frac{\partial^w(E_\beta(-|\xi|^\alpha))}{\partial \xi_i^w}\\
&\quad = C_0\xi_i^{m-|\tau|}E_\beta(-|\xi|^\alpha)\\
&\qquad + \sum_{w=0}^{|\tau|}C_w\xi_i^{m-|\tau|+w}\left(\sum_{k=0}^{w}\left(\sum_{s=0}^{[(w-k)/2]+1}C_{s,k}^w\xi_j^{w-2s}|\xi|^{k\alpha-2w+2s}\right)\frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha}\right)\\
&\quad \approx |\xi|^{m-|\tau|}E_\beta(-|\xi|^\alpha) + \sum_{w=0}^{|\tau|}\sum_{k=0}^{w}|\xi|^{k\alpha-|\tau|+m}\frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha}\\
&\quad \approx \sum_{w=0}^{|\tau|}|\xi|^{w\alpha-|\tau|+m}\frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha}.
\end{aligned}$$

Then we have

$$\lim_{|\xi|\to 0}|\xi|^{|\tau|-\nu/4-m/2}\frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_i^{|\tau|}} \approx \lim_{|\xi|\to 0}\sum_{w=0}^{|\tau|}|\xi|^{w\alpha+m-\nu/4-m/2}\frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha} = 0.$$

Since $\alpha > n + m + n/p$ and (2.5), we get

$$\lim_{|\xi|\to\infty}|\xi|^{n+n/p}\frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_i^{|\tau|}} \approx \lim_{|\xi|\to\infty}\sum_{w=0}^{|\tau|}|\xi|^{w\alpha-|\tau|+m+n+n/p}\frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha} = 0.$$

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On the other hand, when $m < |\tau|$, we can obtain

$$\begin{aligned}
\frac{\partial^{|\tau|}(\xi_i^m(E_\beta(-|\xi|^\alpha)))}{\partial \xi_i^{|\tau|}} &= C_w \sum_{w=|\tau|-m}^{|\tau|} \xi_i^{m-|\tau|+w} \frac{\partial^w(E_\beta(-|\xi|^\alpha))}{\partial \xi_i^w} \\
&= C_w \sum_{w=|\tau|-m}^{|\tau|} \xi_i^{m-|\tau|+w} \left(\sum_{k=1}^{w} \left(\sum_{s=0}^{[(w-k)/2]+1} C_{s,k}^w \xi_j^{w-2s} |\xi|^{k\alpha-2w+2s} \right) \frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha} \right) \\
&\approx \sum_{w=|\tau|-m}^{|\tau|} \sum_{k=1}^{w} |\xi|^{k\alpha-|\tau|+m} \frac{\partial^k(E_\beta(-u))}{\partial u^k}\Big|_{u=|\xi|^\alpha} \\
&\approx \sum_{w=|\tau|-m}^{|\tau|} |\xi|^{w\alpha-|\tau|+m} \frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha}.
\end{aligned}$$

Moreover, we can get

$$\lim_{|\xi|\to 0} |\xi|^{|\tau|-v/4-m/2} \frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_i^{|\tau|}} \approx \lim_{|\xi|\to 0} \sum_{w=|\tau|-m}^{|\tau|} |\xi|^{w\alpha+m-v/4-m/2} \frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha} = 0$$

and

$$\lim_{|\xi|\to \infty} |\xi|^{n+n/p} \frac{\partial^{|\tau|}(\xi_i^m E_\beta(-|\xi|^\alpha))}{\partial \xi_i^{|\tau|}} \approx \lim_{|\xi|\to \infty} \sum_{w=|\tau|-m}^{|\tau|} |\xi|^{w\alpha-|\tau|+m+n+n/p} \frac{\partial^w(E_\beta(-u))}{\partial u^w}\Big|_{u=|\xi|^\alpha} = 0.$$

By (3.12), when $r = m/2 + v/4$, $b = n/p$ and $b_0 = n/p$, $|\widehat{\Phi}_{1,i}|$ satisfies (C1), (C2), and (C3) in Lemmas 3.8 and 3.9. Thus, we know

$$\begin{aligned}
\left(\int_{\mathbb{R}_+^{n+1}} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^p t^{pm-pv/2-1} dxdt \right)^{1/p} &\approx \sum_{i=1}^{n} \left(\int_{\mathbb{R}_+^{n+1}} \left| \Phi_{1,i} * f(x, t^{\alpha/\beta}) \right|^p t^{pm-pv/2-1} dxdt \right)^{1/p} \\
&\approx \|f\|_{\dot{W}_p^{v/2}(\mathbb{R}^n)}.
\end{aligned}$$

Below, we prove (3.11). For $\Phi_{2,t}(x) = t^{-n}\Phi_2(x/t)$, we have

$$\begin{cases} \Phi_{2,t}(x) = \frac{\partial^m (G_{t^{\alpha/\beta}}^{\alpha,\beta}(x))}{\partial t^m}, \\ \widehat{\Phi}_2 = \sum_{i=1}^m C_{i,\alpha} |\xi|^{i\alpha} E_\beta^i(-|\xi|^\alpha), \end{cases}$$

where $E_\beta^i(-t) = \frac{d^i}{dt^i}(E_\beta(-t))$. Similarly to $\widehat{\Phi}_1$, we just prove

$$\begin{cases} \frac{\partial^{|\tau|}(\widehat{\Phi}_2)}{\partial \xi_j^{|\tau|}} = O(|\xi|^{m/2+v/2-|\tau|}) & \text{as } |\xi| \to 0, \\ \frac{\partial^{|\tau|}(\widehat{\Phi}_2)}{\partial \xi_j^{|\tau|}} = O(|\xi|^{-n-n/p}) & \text{as } |\xi| \to \infty. \end{cases}$$

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By a direct calculation, we can obtain

$$
\begin{aligned}
\frac{\partial^{|\tau|}(\widehat{\Phi}_2)}{\partial \xi_j^{|\tau|}} &\approx \sum_{i=1}^{m} \frac{\partial^{|\tau|}(|\xi|^{i\alpha} E_\beta^i(-|\xi|))}{\partial \xi_j^{|\tau|}} \\
&\approx \sum_{i=1}^{m} \left(\sum_{w=1}^{|\tau|} \left(\sum_{j=1}^{w} |\xi|^{i\alpha + j\alpha - |\tau|} \frac{\partial^j E_\beta^i(-u)}{\partial u^j}\Big|_{u=|\xi|^\alpha} \right) \right) \\
&\approx \sum_{i=1}^{m} \left(\sum_{w=1}^{|\tau|} |\xi|^{(i+w)\alpha - |\tau|} \frac{\partial^w E_\beta^i(-u)}{\partial u^w}\Big|_{u=|\xi|^\alpha} \right).
\end{aligned} \tag{3.13}
$$

Then we can obtain

$$
\begin{cases}
\lim\limits_{|\xi|\to 0} |\xi|^{|\tau|-\alpha} \frac{\partial^{|\tau|}(\widehat{\Phi}_2)}{\partial \xi_j^{|\tau|}} = 0, \\
\lim\limits_{|\xi|\to \infty} |\xi|^{n+n/p} \frac{\partial^{|\tau|}(\widehat{\Phi}_2)}{\partial \xi_j^{|\tau|}} = 0.
\end{cases}
$$

Similarly, when $r = m/2 + \nu/4$, $b = n/p$ and $b_0 = n/p$, $|\widehat{\Phi}_2|$ satisfies (C1), (C2), and (C3) in Lemmas 3.8 and 3.9. Hence, we have

$$
\begin{cases}
\sum_{i=1}^{m} C_{i,\alpha} |\xi|^{i\alpha} E_\beta^i(-|\xi|^\alpha) = O(|\xi|^{m/2+\nu/4-|\tau|}) & \text{as } |\xi| \to 0, \\
\sum_{i=1}^{m} C_{i,\alpha} |\xi|^{i\alpha} E_\beta^i(-|\xi|^\alpha) = O(|\xi|^{-n-n/p}) & \text{as } |\xi| \to \infty.
\end{cases}
$$

Similarly, by Lemmas 3.8 and 3.9, we know

$$
\left(\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial^m (u(x, t^{\alpha/\beta}))}{\partial t^m} \right|^p t^{pm - p\nu/2 - 1} dx dt \right)^{1/p} \approx \|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}.
$$

■

For a compact set $K \subset \mathbb{R}^n$, the fractional-Sobolev capacity $C_\nu^p(K)$ is defined as

$$
C_\nu^p(K) := \inf \left\{ \|f\|_{\dot{W}_p^\nu(\mathbb{R}^n)}^p : f \in C_0^\infty(\mathbb{R}^n) \text{ and } f \geq 1_K \right\}
$$

and for any set $E \subset \mathbb{R}^n$, one defines

$$
C_\nu^p(E) := \inf_{\text{open } O \supseteq E} \ \sup_{\text{compact } K \subseteq O} \left\{ C_\nu^p(K) \right\},
$$

where 1_E denotes the characteristic function of E. Let 1_O denote the characteristic function of the set O. In [28], Li, Hu, and Zhai obtained the following results.

Lemma 3.11 [28, Theorem 3.2] *Let $\nu \in (0, 2)$, $1 \leq p < 2n/\nu$ and $f \in C_0^\infty(\mathbb{R}^n)$. The following statements are equivalent.*

(i) *The analytic inequality:*

$$
\left(\int_0^\infty (V(O_t(f)))^{(2n-p\nu)/(2n)} dt^p \right)^{1/p} \lesssim \|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}, \tag{3.14}
$$

where $O_t(f) := \{x \in \mathbb{R}^n : |f(x)| > t\}$ and $V(O_t) := \int_{\mathbb{R}^n} 1_{O_t} dx$.

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(ii) *The fractional Sobolev inequality:*

$$\left(\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x|^{p\nu/2}}dx\right)^{1/p}\lesssim\|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}. \tag{3.15}$$

(iii) *The fractional Hardy inequality:*

$$\left(\int_{\mathbb{R}^n}|f(x)|^{np/(n-p\nu/2)}dx\right)^{(n-p\nu/2)/np}\lesssim\|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}. \tag{3.16}$$

(iv) *For any bounded domain $O\subset\mathbb{R}^n$ with C^∞ boundary ∂O, the iso-capacitary inequalities:*

$$(V(O))^{1-p\nu/(2n)}\lesssim C^p_{\nu/2}(\overline{O}). \tag{3.17}$$

Moreover, (3.14), (3.15), (3.16), and (3.17) are all true.

Notice that the proof of (3.16) can be found in [13, 31]. For (3.15), readers can see [31] and the references therein.

Similarly to the inequalities involving fractional Laplacian in [21, Theorem 2.5 and Corollary 2.6], we will prove fractional logarithmic Gagliardo–Nirenberg inequalities which imply the L^p-logarithmic Sobolev inequalities for $\dot{W}_p^{\nu/2}(\mathbb{R}^n)$.

Theorem 3.12 *Let $1<q<\infty$, $0<\nu<2n$, $1<p<2n/\nu$ and $f\in\dot{W}_p^{\nu/2}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ with $\|f\|_{L^q(\mathbb{R}^n)}>0$. Then the following inequality holds.*

$$\exp\left(\left(\frac{1}{q}+\frac{\nu}{2n}-\frac{1}{p}\right)\int_{\mathbb{R}^n}\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\ln\left(\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right)dx\right)\lesssim\frac{\|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}}{\|f\|_{L^q(\mathbb{R}^n)}}.$$

Proof Let

$$g(h):=h\ln\left(\int_{\mathbb{R}^n}|f(x)|^{1/h}dx\right),$$

where $g(\cdot)$ is a convex function. For $h>h_1\geq0$, we can obtain

$$g'(h)=\ln\left(\int_{\mathbb{R}^n}|f(x)|^{1/h}dx\right)-h^{-1}\frac{\int_{\mathbb{R}^n}|u(x)|^{1/h}\ln|f(x)|dx}{\int_{\mathbb{R}^n}|f(x)|^{1/h}dx}\geq\frac{g(h_1)-g(h)}{h_1-h}.$$

Taking $h=1/q$, $h_1=1/p_1$ and $0<q<p_1\leq\infty$, by [33, Lemma 1], we have

$$\exp\left(\int_{\mathbb{R}^n}\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\ln\left(\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right)dx\right)\leq\frac{p_1}{p_1-q}\frac{\|f\|^q_{L^{p_1}(\mathbb{R}^n)}}{\|f\|^q_{L^q(\mathbb{R}^n)}}.$$

For $\gamma>0$, Hölder's inequality implies

$$\begin{aligned}\|f\|_{L^{p_1}(\mathbb{R}^n)}&=\left(\int_{\mathbb{R}^n}|f(x)|^\gamma|f(x)|^{p_1-\gamma}\right)^{1/p_1}dx\\&\leq\|f\|^{\gamma/p_1}_{L^{\gamma p_2'}(\mathbb{R}^n)}\|f\|^{(p_1-\gamma)/p_1}_{L^{(p_1-\gamma)/p_3'}(\mathbb{R}^n)}\\&=\|f\|^{\gamma/p_1}_{L^{p_2}(\mathbb{R}^n)}\|f\|^{(p_1-\gamma)/p_1}_{L^{p_3}(\mathbb{R}^n)},\end{aligned}$$

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where $1/p_2' + 1/p_3' = 1$, $p_2 := \gamma p_2'$ and $p_3 := (p_1 - \gamma)/p_3'$. By (3.16), for $1/p + (n - v/2)/n = 1 + 1/p_2$, we get

$$\|f\|_{L^{p_1}(\mathbb{R}^n)} \lesssim \|f\|^{\gamma/p_1}_{\dot{W}^{v/2}_p(\mathbb{R}^n)} \|f\|^{(p_1-\gamma)/p_1}_{L^{p_3}(\mathbb{R}^n)}.$$

Then we can choose $p_1 = nq/(n - qv/2) \in (q, \infty)$ for $p_3 = q$, γ and p satisfying

$$\gamma(1/p - v/(2n)) + (p_1 - \gamma)/p_3 = 1.$$

Hence,

$$\int_{\mathbb{R}^n} \frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}} \ln\left(\frac{|f(x)|^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right) dx \lesssim \frac{p_1}{p_1 - q} \ln\left(\frac{\left(\|f\|^{\gamma/p_1}_{\dot{W}^{v/2}_p(\mathbb{R}^n)} \|f\|^{(p_1-\gamma)/p_1}_{L^{p_3}(\mathbb{R}^n)}\right)^q}{\|f\|^q_{L^q(\mathbb{R}^n)}}\right)$$
$$\lesssim \frac{q\gamma}{p_1 - q} \ln\left(\frac{\|f\|_{\dot{W}^{v/2}_p(\mathbb{R}^n)}}{\|f\|_{L^q(\mathbb{R}^n)}}\right).$$

We can get $\frac{q\gamma}{p_1 - q} = \frac{1}{1/q + v/(2n) - 1/p}$. ∎

When $p = q$ and $\|f\|_{L^q(\mathbb{R}^n)} = 1$, there holds the L^p-logarithmic-type Sobolev inequality.

Corollary 3.13 *Let* $0 < v < 2n$, $1 < p < 2n/v$, $f \in \dot{W}^{v/2}_p(\mathbb{R}^n)$ *with* $\|f\|_{L^p(\mathbb{R}^n)} = 1$. *Then*

$$\exp\left(\frac{v}{2n} \int_{\mathbb{R}^n} |f(x)|^p \ln\left(|f(x)|^p\right) dx\right) \lesssim \|f\|_{\dot{W}^{v/2}_p(\mathbb{R}^n)}.$$

Let $f \in \dot{W}^{v/2}_p(\mathbb{R}^n)$ with $v \in (0, 2)$ and $(1 + |\cdot|)^\zeta f \in L^1(\mathbb{R}^n)$. Using (iii) of Lemma 3.11, we can obtain the following Sobolev-type trace inequality: for $f \in \dot{W}^{v/2}_p(\mathbb{R}^n)$ with $p > 1$ and $u(x,t) = G^{\alpha,\beta}_t * f(x)$ the time-space fractional extension of f,

$$\|f\|_{L^{np/(n-pv/2)}(\mathbb{R}^n)} \lesssim \left(\int_{\mathbb{R}^{n+1}_+} \left|\widetilde{\nabla}^m_x u(x, t^{\alpha/\beta})\right|^p t^{pm-pv/2-1} dx dt\right)^{1/p}, \quad v \in (0, \min\{2, 2n/p\}),$$

and

$$\|f\|_{L^{np/(n-pv/2)}(\mathbb{R}^n)} \lesssim \left(\int_{\mathbb{R}^{n+1}_+} \left|\frac{\partial^m}{\partial t^m}(u(x, t^{\alpha/\beta}))\right|^p t^{mp-1-pv/2} dx dt\right)^{1/p}, \quad v \in (0, \min\{2, 2n/p\}).$$

For $s = t^{\alpha/\beta}$ and (3.10), we can obtain

$$\left(\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}^m_x u(x,s)|^p s^{\beta pm/\alpha - \beta pv/(2\alpha) - 1} dx ds\right)^{1/p} \approx \|f\|_{\dot{W}^{v/2}_p(\mathbb{R}^n)}. \tag{3.18}$$

Notice that

$$\frac{\partial^m u(x, t^{\alpha/\beta})}{\partial t^m} \approx \sum_{i=1}^m t^{i\alpha/\beta - m} \frac{\partial^i u(x,s)}{\partial s^i} = \sum_{i=1}^m s^{i - m\beta/\alpha} \frac{\partial^i u(x,s)}{\partial s^i}. \tag{3.19}$$

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Then

$$\left(\int_{\mathbb{R}_+^{n+1}}\left|\sum_{i=1}^{m}s^i\frac{\partial^i u(x,s)}{\partial s^i}\right|^p s^{-1-\beta p\nu/(2\alpha)}dxds\right)^{1/p}\approx\left(\int_{\mathbb{R}_+^{n+1}}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}.$$

By (3.11), we can obtain

$$\left(\int_{\mathbb{R}_+^{n+1}}\left|\sum_{i=1}^{m}s^i\frac{\partial^i u(x,s)}{\partial s^i}\right|^p s^{-1-\beta p\nu/(2\alpha)}dxds\right)^{1/p}\approx\|f\|_{\dot{W}_p^{\nu/2}(\mathbb{R}^n)}. \tag{3.20}$$

Moreover, based on Corollary 3.13 and (ii) of Lemma 3.11, applying (3.10) and (3.11), we can establish the logarithmic Sobolev trace inequalities for $\|f\|_{L^p(\mathbb{R}^n)}=1$:

(3.21)

$$\exp\left(\frac{\nu}{2n}\int_{\mathbb{R}^n}|f(x)|^p\ln\left(|f(x)|^p\right)dx\right)\lesssim\left(\int_{\mathbb{R}_+^{n+1}}|\widetilde{\nabla}_x^m u(x,t^{\alpha/\beta})|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p},$$

(3.22)

$$\exp\left(\frac{\nu}{2n}\int_{\mathbb{R}^n}|f(x)|^p\ln\left(|f(x)|^p\right)dx\right)\lesssim\left(\int_{\mathbb{R}_+^{n+1}}|\widetilde{\nabla}_x^m u(x,s)|^p s^{\beta pm/\alpha-\beta p\nu/(2\alpha)-1}dxds\right)^{1/p},$$

and the Hardy-type trace inequalities:

$$\left(\int_{\mathbb{R}^n}|f(x)|^p\frac{dx}{|x|^{p\nu/2}}\right)^{1/p}\lesssim\left(\int_{\mathbb{R}_+^{n+1}}|\widetilde{\nabla}_x^m u(x,t^{\alpha/\beta})|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}, \tag{3.23}$$

$$\left(\int_{\mathbb{R}^n}|f(x)|^p\frac{dx}{|x|^{p\nu/2}}\right)^{1/p}\lesssim\left(\int_{\mathbb{R}_+^{n+1}}|\widetilde{\nabla}_x^m u(x,s)|^p s^{\beta pm/\alpha-\beta p\nu/(2\alpha)-1}dxds\right)^{1/p}. \tag{3.24}$$

Furthermore, the right-hand side of (3.21) and (3.23) can be replaced by

$$\left(\int_{\mathbb{R}_+^{n+1}}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}.$$

The right-hand side of (3.22) and (3.24) can be replaced by

$$\left(\int_{\mathbb{R}_+^{n+1}}\left|\sum_{i=1}^{m}s^i\frac{\partial^i u(x,s)}{\partial s^i}\right|^p s^{-1-\beta p\nu/(2\alpha)}dxds\right)^{1/p}.$$

Therefore, Theorems 3.1 and 3.3 can be generalized to $p>1$.

As a direct consequence of Lemma 3.11 and Corollary 3.13, we can use (3.10) and (3.11) to deduce the following results.

Corollary 3.14 *Let $m\in\mathbb{Z}_+$, $\alpha>n+m+n/p$, $\beta\in(0,1]$, $\nu\in(0,2)$, $1<p<2n/\nu$ and $f\in C_0^\infty(\mathbb{R}^n)$. For $(x,t)\in\mathbb{R}^n\times(0,\infty)$, denote by $u(x,t)=G_t^{\alpha,\beta}*f(x)$ the time-space fractional extension of f. The following statements are equivalent.*

(i) *The analytic inequality:*

(3.25)

$$\left(\int_0^\infty(V(O_t(f)))^{(2n-p\nu)/(2n)}dt^p\right)^{1/p}\lesssim\left(\int_{\mathbb{R}_+^{n+1}}|\widetilde{\nabla}_x^m u(x,t^{\alpha/\beta})|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}.$$

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(ii) *The fractional Sobolev trace inequality:*

(3.26)
$$\left(\int_{\mathbb{R}^n}|f(x)|^{np/(n-p\nu/2)}dx\right)^{(n-p\nu/2)/np}\lesssim\left(\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}.$$

(iii) *The fractional Hardy inequality:*

$$\left(\int_{\mathbb{R}^n}|f(x)|^p\frac{dx}{|x|^{p\nu/2}}\right)^{1/p}\lesssim\left(\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}. \tag{3.27}$$

(iv) *For any bounded domain $O\subset\mathbb{R}^n$ with C^∞ boundary ∂O, the iso-capacitary inequalities:*

$$(V(O))^{1-p\nu/(2n)}\lesssim C^p_{\nu/2}(\overline{O}).$$

Moreover, (3.25), (3.26), and (3.27) are all true, and the right-hand side of (3.25), (3.26), and (3.27) can be replaced by

$$\left(\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}.$$

Proof If $f\in C_0^\infty(\mathbb{R}^n)$, we know $(1+|\cdot|)^\zeta f\in L^1(\mathbb{R}^n)$. By (3.10) and (3.11), we have

$$\begin{aligned}\left(\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,t^{\alpha/\beta})|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}&\approx\left(\int_{\mathbb{R}^{n+1}_+}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^p t^{pm-p\nu/2-1}dxdt\right)^{1/p}\\&\approx\|f\|_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)}.\end{aligned}$$

Then Corollary 3.14 follows from Lemma 3.11 and Corollary 3.13. ∎

Corollary 3.15 *Let $m\in\mathbb{Z}_+$, $\alpha>n+m+n/p$, $\beta\in(0,1]$, $\nu\in(0,2)$, $1<p<2n/\nu$ and $f\in C_0^\infty(\mathbb{R}^n)$. For $(x,t)\in\mathbb{R}^n\times(0,\infty)$, denote by $u(x,t)=G^{\alpha,\beta}_t*f(x)$ the time-space fractional extension of f. The following statements are equivalent.*

(i) *The analytic inequality:*

(3.28)
$$\left(\int_0^\infty(V(O_t(f)))^{(2n-p\nu)/(2n)}dt^p\right)^{1/p}\lesssim\left(\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,s)|^p s^{\beta pm/\alpha-p\beta\nu/(2\alpha)-1}dxds\right)^{1/p}.$$

(ii) *The fractional Sobolev inequality:*

(3.29)
$$\left(\int_{\mathbb{R}^n}|f(x)|^{np/(n-p\nu/2)}dx\right)^{(n-p\nu/2)/np}\lesssim\left(\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,s)|^p s^{\beta pm/\alpha-p\beta\nu/(2\alpha)-1}dxds\right)^{1/p}.$$

(iii) *The fractional Hardy inequality:*

(3.30)
$$\left(\int_{\mathbb{R}^n}|f(x)|^p\frac{dx}{|x|^{p\nu/2}}\right)^{1/p}\lesssim\left(\int_{\mathbb{R}^{n+1}_+}|\widetilde{\nabla}^m_x u(x,s)|^p s^{\beta pm/\alpha-p\beta\nu/(2\alpha)-1}dxds\right)^{1/p}.$$

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(iv) *For any bounded domain $O \subset \mathbb{R}^n$ with C^∞ boundary ∂O, the iso-capacitary inequalities:*

$$(V(O))^{1-p\nu/(2n)} \lesssim C^p_{\nu/2}(\overline{O}).$$

Moreover, (3.28), (3.29), and (3.30) are all true, and the right-hand side of (3.28), (3.29), and (3.30) can be replaced by

$$\left(\int_{\mathbb{R}^{n+1}_+} \left|\sum_{i=1}^m s^i \frac{\partial^i u(x,s)}{\partial s^i}\right|^p s^{-1-\beta p\nu/(2\alpha)} dx ds\right)^{1/p}.$$

Proof By (3.18) and (3.20), we have

$$\begin{aligned}\left(\int_{\mathbb{R}^{n+1}_+} \left|\sum_{i=1}^m s^i \frac{\partial^i u(x,s)}{\partial s^i}\right|^p s^{-1-\beta p\nu/(2\alpha)} dx ds\right)^{1/p} &\approx \left(\int_{\mathbb{R}^{n+1}_+} |\widetilde{\nabla}^m_x u(x,s)|^p s^{\beta pm/\alpha - p\beta\nu/(2\alpha)-1} dx ds\right)^{1/p} \\ &\approx \|f\|_{\dot{W}^{\nu/2}_p(\mathbb{R}^n)}.\end{aligned}$$

Then Lemma 3.11 and Corollary 3.13 imply Corollary 3.15. ∎

4 Sobolev affine trace inequalities

Definition 4.1 Assume that $\sigma\colon \mathbb{R}^{n+1}_+ \to \mathbb{R}_+$ is a positive measurable function. Denote by $L^p(\mathbb{R}^{n+1}_+, \sigma)$ the weighted Lebesgue space of all measurable functions $f : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ with

$$\|f\|_{L^p(\mathbb{R}^{n+1}_+,\sigma)} := \left(\int_{\mathbb{R}^{n+1}_+} |f(x,t)|^p \sigma(x,t) dx dt\right)^{1/p} < \infty.$$

Define

$$\Theta_p(f,\sigma) := A_{n,p}\left(\int_{\mathbb{S}^{n-1}} \|\nabla_\xi f\|^{-n}_{L^p(\mathbb{R}^{n+1}_+,\sigma)} d\xi\right)^{-1/n},$$

where $A_{n,p}$ is a constant depending on n, p.

The following affine Sobolev-type inequality was obtained in Haddad, Jiménez, and Montenegro [20].

Theorem 4.2 [20, Theorem 1.1] *Define a function σ on $\mathbb{R}^{n+1}_+$ as $\sigma(x,t) := t^\gamma \ \forall\, (x,t) \in \mathbb{R}^{n+1}_+$. Let $\gamma \geq 0$, $1 \leq p < n+\gamma+1$ and $p^*_\gamma = p(n+\gamma+1)/(n+\gamma+1-p)$. There exists a sharp constant $J(n,p,\gamma)$ such that*

$$\|g(\cdot,\cdot)\|_{L^{p^*_\gamma}(\mathbb{R}^{n+1}_+,\sigma)} \leq J(n,p,\gamma)(\Theta_p(g,\sigma))^{n/(n+\gamma+1)} \left\|\frac{\partial g}{\partial t}(\cdot,\cdot)\right\|^{(\gamma+1)/(n+\gamma+1)}_{L^p(\mathbb{R}^{n+1}_+,\sigma)}. \tag{4.1}$$

Moreover, in (4.1), the equality holds if

$$g(x,t) := \begin{cases} \frac{c}{\left(1+|\Delta_x t|^{(1+1/p)}+|A(x-x_0)|^{(1+1/p)}\right)^{(1+n+\gamma-p)/p}}, & p > 1, \\ c1_{\mathbb{B}^{n+1}}(\Delta_x t, A(x-x_0)), & p = 1, \end{cases}$$

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where $(c, |\Delta_x|, x_0, A) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n \times GL_n$. *and* $1_{\mathbb{B}^{n+1}}$ *is the characteristic function of the unit ball in* $\mathbb{R}^{n+1}$ *and* GL_n *denotes the set of all invertible real* $n \times n$*-matrices.*

Motivated by Lombardi and Xiao [30], we establish the following affine trace inequality.

Theorem 4.3 *Let* $m \in \mathbb{Z}_+$, $\alpha > n$, $\beta \in (0,1]$, $f \in C_0^\infty(\mathbb{R}^n)$ *and its time-space fractional extension* $u(x,t) := G_t^{\alpha,\beta} * f(x)\ \forall\ (x,t) \in \mathbb{R}_+^{n+1}$ *when* $\nu \geq 1$. *For* $p = \frac{2(n+\nu+2m)}{n+\nu+2+2m}$ *and* $\nu \geq 1 - 2m$, *there holds*

$$\|f\|_{\dot{H}^{-\nu/2}(\mathbb{R}^n)} \lesssim \left(\Theta_p\left(\frac{\partial^m u}{\partial t^m}, t^{\nu+2m-1}\right)\right)^{n/(n+\nu+2m)} \left\|\frac{\partial^{m+1}u}{\partial t^{m+1}}(\cdot,\cdot)\right\|_{L^p(\mathbb{R}_+^{n+1}, t^{\nu+2m-1})}^{\nu/(n+\nu+2m)}.$$

Proof From (ii) and (iii) of Lemma 2.5, we know

$$\int_{\mathbb{R}^n} |\xi|^{-(\gamma+1-2m)} |\widehat{f}(\xi)|^2 d\xi \approx \|u(\cdot,\cdot)\|_{L^2(\mathbb{R}_+^{n+1}, t^\gamma)}^2$$

when $2m-1 < \gamma < 2m + 2\alpha - 1$. Let $\gamma = \nu + 2m - 1 \in (2m-1, 2m+2\alpha-1)$. By Theorem 4.2, we can obtain $\sigma := t^\gamma = t^{\nu+2m-1}$ and

$$\begin{aligned}
\|f\|_{\dot{H}^{-\nu/2}(\mathbb{R}^n)} &= \|(-\Delta_x)^{-\nu/4} f\|_{L^2(\mathbb{R}^n)} \\
&\approx \int_{\mathbb{R}^n} |\xi|^{-\nu} |\widehat{f}(\xi)|^2 d\xi \\
&\approx \left\|\frac{\partial^m}{\partial t^m} u(\cdot,\cdot)\right\|_{L^2(\mathbb{R}_+^{n+1}, t^\gamma)}^2 \\
&\lesssim \left(\Theta_p\left(\frac{\partial^m}{\partial t^m} u, t^{\nu+2m-1}\right)\right)^{n/(n+\nu+2m)} \left\|\frac{\partial^{m+1}u}{\partial t^{m+1}}(\cdot,\cdot)\right\|_{L^p(\mathbb{R}_+^{n+1}, t^{\nu+2m-1})}^{\nu/(n+\nu+2m)}.
\end{aligned}$$

■

Theorem 4.3 suggests

$$\|f\|_{\dot{H}^{-\nu/2}(\mathbb{R}^n)} \approx \|u(x, t^{\alpha/\beta})\|_{L^2(\mathbb{R}_+^{n+1}, \sigma)},$$

which can be generalized as follows.

Theorem 4.4 *Let* $m \in \mathbb{Z}_+$, $\alpha > n + m + n/p$, $p > 1$, $\beta \in (0,1]$, $t > 0$, $f \in \dot{W}_p^{-\nu/2}(\mathbb{R}^n)$ *with* $\nu \in (0,2)$ *and* $(1+|\cdot|)^\zeta f \in L^1(\mathbb{R}^n)$ *with* $\zeta > \max\{0, \nu/2 - n/p\}$. *For* $(x,t) \in \mathbb{R}^n \times (0,\infty)$, *denote by* $u(x,t) = G_t^{\alpha,\beta} * f(x)$ *the extension of* f *via the time-space fractional heat kernel. Then*

$$\left(\int_{\mathbb{R}_+^{n+1}} |u(x, t^{\alpha/\beta})|^p t^{2p/\nu - 1} dx dt\right)^{1/p} \approx \|f\|_{\dot{W}_p^{-\nu/2}(\mathbb{R}^n)}. \tag{4.2}$$

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Proof Let

$$\Phi_{3,t}(\cdot) := G^{\alpha,\beta}_{t^{\alpha/\beta}}(\cdot) \quad \text{and} \quad \widehat{\Phi}_3(\xi) := E_\beta(-|\xi|^\alpha),$$

$\Phi_{3,t}(x) = t^{-n}\Phi_3(x/t)$. Similarly to the proof of Theorem 3.10, for every $\xi_j \in \{\xi_1, \xi_2, \dots, \xi_n\}$, we just prove

$$\begin{cases} \dfrac{\partial^{|\tau|}(\widehat{\Phi}_3)}{\partial \xi_j^{|\tau|}} = O(|\xi|^{r-|\tau|}) & \text{as } |\xi| \to 0, \\ \dfrac{\partial^{|\tau|}(\widehat{\Phi}_3)}{\partial \xi_j^{|\tau|}} = O(|\xi|^{-n-b}) & \text{as } |\xi| \to \infty. \end{cases}$$

By (3.13), we can obtain

$$\begin{aligned} \frac{\partial^{|\tau|}\left(\widehat{\Phi}_3\right)}{\partial \xi_j^{|\tau|}} &= \left(\sum_{k=1}^{w} \left(\sum_{s=0}^{[(w-k)/2]+1} C^{w}_{s,k} \xi_j^{w-2s} |\xi|^{k\alpha-2w+2s} \right) \frac{\partial^k E_\beta(-u)}{\partial u^k}\Big|_{u=|\xi|^\alpha} \right) \\ &= \left(\sum_{k=1}^{|\tau|} \left(\sum_{s=0}^{[(|\tau|-k)/2]+1} C^{|\tau|}_{s,k} \xi_j^{|\tau|-2s} |\xi|^{k\alpha-2|\tau|+2s} \right) \frac{\partial^k E_\beta(-u)}{\partial u^k}\Big|_{u=|\xi|^\alpha} \right) \\ &\approx \sum_{k=1}^{|\tau|} |\xi|^{k\alpha-|\tau|} \frac{\partial^k E_\beta(-u)}{\partial u^k}\Big|_{u=|\xi|^\alpha}. \end{aligned}$$

Using (2.6), for $|\xi|$ in a neighborhood of 0, we know

$$\frac{\partial^k E_\beta(-u)}{\partial u^k}\Big|_{u=|\xi|^\alpha} < \infty.$$

Take $r = \alpha$. This indicates that

$$\frac{\partial^{|\tau|}(E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} = O(|\xi|^{\alpha-|\tau|}) \quad \text{as } |\xi| \to 0.$$

By (2.5), for $M > 0$, we know

$$\frac{d^j(E_\beta(-t))}{dt^j} \approx (-1)^{l+j+1} \sum_{l=1}^{M} \frac{1}{\Gamma(1-\beta l)} \frac{1}{t^{l+j}} + (-1)^{M+1+j} O\left[\frac{1}{t^{M+1+j}}\right].$$

So, for $b_0 = b = n/p$,

$$\frac{\partial^{|\tau|}(E_\beta(-|\xi|^\alpha))}{\partial \xi_j^{|\tau|}} = O(|\xi|^{-n-n/p}) \quad \text{as } |\xi| \to \infty.$$

If $|\tau| = 0$, for $r = -\nu/4 > -\nu/2$ and $b = b_0 = n/p$, we have

$$\begin{cases} E_\beta(-|\xi|^\alpha) = O(|\xi|^{r-|\tau|}), & \text{as } |\xi| \to 0, \\ E_\beta(-|\xi|^\alpha) = O(|\xi|^{-n-b}), & \text{as } |\xi| \to \infty. \end{cases}$$

Hence, by Lemmas 3.8 and 3.9, we can obtain

$$\left(\int_{\mathbb{R}^{n+1}_+} |u(x, t^{\alpha/\beta})|^p t^{2p/\nu-1} dx dt \right)^{1/p} \approx \|f\|_{\dot{W}_p^{-\nu/2}(\mathbb{R}^n)}. \qquad \blacksquare$$

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Then (4.2) and Theorem 4.2 imply that

$$\|f\|_{\dot{W}^{-\nu/2}_{p^*}(\mathbb{R}^n)} \lesssim \left(\Theta_p\left(g, t^{p^*\nu/2-1}\right)\right)^{2n/(2n+p^*\nu)} \left\|\frac{\partial u}{\partial t}(\cdot,\cdot)\right\|^{p^*\nu/(2n+p^*\nu)}_{L^p(\mathbb{R}^{n+1}_+, t^{p^*\nu-1})},$$

where $\nu \in (2/p^*, 2n)$, $1 \le p < n + p^*\nu/2$ with p^* satisfying $p^* \ge \max\{2/\nu, 1\}$ and

$$1/p = 1/p^* + 2/(2n + p^*\nu).$$

5 Local Sobolev-type trace inequalities

In this section, we prove (1.7) and (1.8) by the Carleson measure characterization of Q-type spaces $Q_\kappa(\mathbb{R}^n)$ introduced in [11].

Definition 5.1 For $0 \le \kappa < 1$, $Q_\kappa(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f such that

$$\|f\|^2_{Q_\kappa(\mathbb{R}^n)} := \sup_I (\ell(I))^{2\kappa-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\kappa}} dx dy < \infty,$$

where the symbol $\sup_I$ denotes the supremum taken over all cubes I with the edge length $\ell(I)$ and the edges parallel to the coordinate axes in $\mathbb{R}^n$.

In the literature, Q-type spaces were introduced as a new class of function spaces between $W^{1,n}(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. In 1995, Aulaskari, Xiao, and Zhao [3] first introduced a class of Möbius invariant analytic function space $Q_p(\mathbb{D})$ for $p \in (0,1)$ on the unit disk $\mathbb{D}$ of the complex plane. The class $Q_p(\mathbb{D})$, $p \in (0,1)$ can be seen as subspaces and subsets of $BMOA$ and UBC on $\mathbb{D}$ and were investigated extensively (see Aulaskari, Stegenga, and Xiao [2], Aulaskari, Xiao, and Zhao [3], Xiao [37, 40] and the references therein). As a class of analytic function spaces, the boundary of $Q_p(\mathbb{D})$ is $Q_p(\partial\mathbb{D})$ which was introduced by Nicolau and Xiao in [35], where $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. Correspondingly, in the setting of Euclidean spaces, the real-variable Q-type spaces $Q_\kappa(\mathbb{R}^n)$ were first introduced by Essén, Janson, Peng, and Xiao [16]. Since then, various characterizations of Q-type spaces have been established (see Cui and Yang [10], Dafni and Xiao [11, 12], Yang and Yuan [44, 45] and the references therein).

By the aid of Hausdorff capacities and tent spaces, Dafni and Xiao [11] proved the following equivalent characterization of $Q_\kappa(\mathbb{R}^n)$.

Theorem 5.2 [11, Theorems 3.3 and 7.0] *Given a C^∞ real-valued function ψ on $\mathbb{R}^n$ with*

$$\psi \in L^1(\mathbb{R}^n), \quad |\psi(x)| \lesssim (1+|x|)^{-(n+1)}, \quad \int_{\mathbb{R}^n} \psi(x) dx = 0. \tag{5.1}$$

Let $\psi_t(x) := t^{-n}\psi(x/t)$. Then $f \in Q_{\nu/2}(\mathbb{R}^n)$ with $\nu \in (0, \min\{2, n\})$ if and only if

$$\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n} \int_0^r \int_{|y-x_0|<r} |\psi_t * f(x)|^2 \, t^{-1-\nu} dx dt\right)^{1/2} < \infty.$$

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Remark 5.3 In [11], ψ is defined as a function satisfying

$$\begin{cases} (1)\ \operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le 1\}; \\ (2)\ \psi \text{ is radial;} \\ (3)\ \psi \in C^\infty(\mathbb{R}^n); \\ (4)\ \int_{\mathbb{R}^n} x^\gamma \psi(x)dx = 0, |\gamma| \le \mathbb{N},\ \gamma \in \mathbb{N}^n,\ x^\gamma = x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n},\ |\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n; \\ (5)\ \int_0^\infty (\widehat{\psi}(t\xi))^2 \frac{dt}{t}, \quad \xi \in \mathbb{R}^n \setminus \{0\} \end{cases}$$

(see [11, Lemma 3.1]). In fact, Theorem 5.2 also holds for the functions ψ which satisfy (5.1) (see [41, p. 228]).

As an application of Theorem 5.2, we can obtain the following result.

Theorem 5.4 *Let $m \in \mathbb{Z}_+$, $\alpha > n + m$, $\beta \in (0,1]$, $t > 0$, $f \in Q_{\nu/2}(\mathbb{R}^n)$ and $\nu \in (0, \min\{2, n\})$. For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, denote by $u(x,t) = G_t^{\alpha,\beta} * f(x)$ the time-space fractional extension of f. There hold the following local Sobolev-type trace inequalities*

$$\sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I|^{2n/(n-\nu)}\, dx \right)^{(n-\nu)/(2n)} \lesssim \left(\sup_{x_0 \in \mathbb{R}^n, r \in (0,\infty)} r^{\nu-n} \int_0^r \int_{|y-x_0|<r} \left| \frac{\partial^m u(x, t^{\alpha/\beta})}{\partial t^m} \right|^2 t^{2m-1-\nu} dx dt \right)^{1/2} \tag{5.2}$$

and

$$\sup_I \left(\frac{1}{|I|} \int_I |f(x) - f_I|^{2n/(n-\nu)}\, dx \right)^{(n-\nu)/(2n)} \lesssim \left(\sup_{x_0 \in \mathbb{R}^n, r \in (0,\infty)} r^{\nu-n} \int_0^r \int_{|y-x_0|<r} |\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})|^2 t^{2m-1-\nu} dx dt \right)^{1/2}. \tag{5.3}$$

Proof Let

$$\psi_{1,t}(x) = t^{-n}\psi_1(x/t) := t^m \frac{\partial^m G_{t^{\alpha/\beta}}^{\alpha,\beta}(x)}{\partial t^m}$$

and $C^i_{\alpha,\beta,m}$ is a constant depend on i, α, m. It can be deduced from the Fourier transform that

$$\begin{aligned} \widehat{\psi_{1,t}}(\xi) &= \widehat{\psi_1}(t|\xi|) \\ &= t^m \frac{\partial^m \widehat{G_{t^{\alpha/\beta}}^{\alpha,\beta}}(\xi)}{\partial t^m} \\ &= t^m \frac{\partial^m E_\beta(-|t\xi|^\alpha)}{\partial t^m} \\ &= |t\xi|^m \sum_{i=1}^m C^i_{\alpha,\beta,m} |t\xi|^{i\alpha - m} \frac{\partial^i E_\beta(-u)}{\partial u^i}\Big|_{u=|t\xi|^\alpha}, \end{aligned}$$

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which implies that

$$\psi_1(x) = \int_{\mathbb{R}^n} \widehat{\psi_1}(\xi)e^{ix\cdot\xi}d\xi,$$

where

$$\widehat{\psi_1}(\xi) := |\xi|^m \sum_{i=1}^{m} C^i_{\alpha,\beta,m}|\xi|^{i\alpha-m}\frac{\partial^i E_\beta(-u)}{\partial u^i}\Big|_{u=|\xi|^\alpha}. \tag{5.4}$$

By (5.4), we can get

$$\int_{\mathbb{R}^n}\psi_1(x)dx = \widehat{\psi_1}(0) = 0.$$

It follows from Proposition 2.10 that for $\alpha > n+m$,

$$|\psi_{1,t}(x)| = t^{-n}|\psi_1(x/t)| = \left|t^m\frac{\partial^m G^{\alpha,\beta}_{t^{\alpha/\beta}}(x)}{\partial t^m}\right| \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}} \lesssim \frac{1}{t^n}\frac{1}{(1+|x|/t)^{n+1}},$$

which gives $|\psi_1(x)| \lesssim (1+|x|)^{-n-1}$ and $\psi_1 \in L^1(\mathbb{R}^n)$. Then Theorem 5.2 implies

$$\|f\|_{Q_{\nu/2}(\mathbb{R}^n)} \approx \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r\int_{|y-x_0|<r}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt\right)^{1/2}.$$

Since $Q_{\nu/2}(\mathbb{R}^n)$ is a subspace of $BMO(\mathbb{R}^n)$, we have $\|f\|_{BMO(\mathbb{R}^n)} \le \|f\|_{Q_{\nu/2}(\mathbb{R}^n)}$. We can deduce from the equivalent norm:

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_I\left(\frac{1}{|I|}\int_I|f(x)-f_I|\,dx\right) \simeq \sup_I\left(\frac{1}{|I|}\int_I|f(x)-f_I|^{2n/(n-\nu)}\,dx\right)^{(n-\nu)/(2n)} \tag{5.5}$$

that

$$\begin{aligned}&\sup_I\left(\frac{1}{|I|}\int_I|f(x)-f_I|^{2n/(n-\nu)}\,dx\right)^{(n-\nu)/(2n)}\\ &\lesssim \|f\|_{BMO(\mathbb{R}^n)} \le \|f\|_{Q_{\nu/2}(\mathbb{R}^n)}\\ &\lesssim \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r\int_{|y-x_0|<r}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt\right)^{1/2},\end{aligned}$$

which proves (5.2).

Now, we prove (5.3). Denote

$$\psi_{2,t}(x) = t^{-n}\psi_2(x/t) := t^m\widetilde{\nabla}^m_x G^{\alpha,\beta}_{t^{\alpha/\beta}}(x).$$

It can be deduced from the Fourier transform that

$$\widehat{\psi_{2,t}}(\xi) = \widehat{\psi_2}(t|\xi|) = t^m\left(\widehat{\widetilde{\nabla}^m_x G^{\alpha,\beta}_{t^{\alpha/\beta}}}(\xi)\right) = |t\xi|^m\left(E_\beta(-|t\xi|^\alpha)\right),$$

which implies that

$$\psi_2(x) = \int_{\mathbb{R}^n}\widehat{\psi_2}(\xi)e^{ix\cdot\xi}d\xi,$$

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where

$$\widehat{\psi_2}(\xi) := |t\xi|^m E_\beta(-|t\xi|^\alpha). \tag{5.6}$$

By (5.6), it is easy to verify that

$$\int_{\mathbb{R}^n} \psi_2(x)dx = \widehat{\psi_2}(0) = 0.$$

Also, it follows from Proposition 2.11 that for $\alpha > n + m$,

$$|\psi_{2,t}(x)| = \frac{1}{t^n}|\psi_2(x/t)| = \left|t^m \widetilde{\nabla}_x^m G_{t^{\alpha/\beta}}^{\alpha,\beta}(x)\right| \lesssim \frac{t^\alpha}{(t+|x|)^{n+\alpha}} \lesssim \frac{1}{t^n}\frac{1}{(1+|x|/t)^{n+1}},$$

which gives $|\psi_2(x)| \lesssim (1+|x|)^{-n-1}$ and $\psi_2 \in L^1(\mathbb{R}^n)$. Theorem 5.2 implies

$$\|f\|_{Q_{\nu/2}(\mathbb{R}^n)} \approx \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r \int_{|y-x_0|<r} \left|\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})\right|^2 t^{2m-1-\nu} dxdt\right)^{1/2}.$$

Similarly, using (5.5), we have

$$\begin{aligned}
&\sup_I \left(\frac{1}{|I|}\int_I |f(x)-f_I|^{2n/(n-\nu)}\,dx\right)^{(n-\nu)/(2n)} \\
&\lesssim \|f\|_{BMO(\mathbb{R}^n)} \le \|f\|_{Q_{\nu/2}(\mathbb{R}^n)} \\
&\lesssim \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r \int_{|y-x_0|<r} \left|\widetilde{\nabla}_x^m u(x, t^{\alpha/\beta})\right|^2 t^{2m-1-\nu} dxdt\right)^{1/2},
\end{aligned}$$

which proves (5.3). ∎

Remark 5.5 Let $m \in \mathbb{Z}_+$, $\alpha > n+m$, $\beta \in (0,1]$, $s > 0$, $f \in Q_{\nu/2}(\mathbb{R}^n)$ and $\nu \in (0, \min\{2, n\})$. For $(x,t) \in \mathbb{R}^n \times (0,\infty)$, denote by $u(x,t) = G_t^{\alpha,\beta} * f(x)$ the time-space fractional extension of f. There hold the local Sobolev-type trace inequalities

$$\begin{aligned}
&\sup_I \left(\frac{1}{|I|}\int_I |f(x)-f_I|^{2n/(n-\nu)}\,dx\right)^{(n-\nu)/(2n)} \\
&\lesssim \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^{r^{\alpha/\beta}} \int_{|y-x_0|<r} \left|\widetilde{\nabla}_x^m u(x, s)\right|^2 s^{\beta(2m-\nu)/\alpha-1} dxds\right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_I \left(\frac{1}{|I|}\int_I |f(x)-f_I|^{2n/(n-\nu)}\,dx\right)^{(n-\nu)/(2n)} \\
&\lesssim \left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^{r^{\alpha/\beta}} \int_{|y-x_0|<r} \left|\sum_{i=1}^m s^i \frac{\partial^i u(x,s)}{\partial s^i}\right|^2 s^{-1-\nu\beta/\alpha} dxds\right)^{1/2}.
\end{aligned}$$

In fact, by $s = t^{\alpha/\beta}$ and (3.19), we get

$$\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r\int_{|y-x_0|<r}\left|\frac{\partial^m u(x,t^{\alpha/\beta})}{\partial t^m}\right|^2 t^{2m-1-\nu}dxdt\right)^{1/2}$$
$$\approx\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^{r^{\alpha/\beta}}\int_{|y-x_0|<r}\left|\sum_{i=1}^m s^i\frac{\partial^i u(x,s)}{\partial s^i}\right|^2 s^{-1-\nu\beta/\alpha}dxds\right)^{1/2}$$

and

$$\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^r\int_{|y-x_0|<r}\left|\widetilde{\nabla}_x^m u(x,t^{\alpha/\beta})\right|^2 t^{2m-1-\nu}dxdt\right)^{1/2}$$
$$=(\beta/\alpha)\left(\sup_{x_0\in\mathbb{R}^n, r\in(0,\infty)} r^{\nu-n}\int_0^{r^{\alpha/\beta}}\int_{|y-x_0|<r}\left|\widetilde{\nabla}_x^m u(x,s)\right|^2 s^{\beta(2m-\nu)/\alpha-1}dxds\right)^{1/2}.$$

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