

Invariant escaping Fatou components with two rank-one limit functions for automorphisms of \mathbb{C}^2

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Abstract. We construct automorphisms of \mathbb{C}^2 , and more precisely transcendental Hénon maps, with an invariant escaping Fatou component which has exactly two distinct limit functions, both of (generic) rank one. We also prove a general growth lemma for the norm of points in orbits belonging to invariant escaping Fatou components for automorphisms of the form $F(z, w) = (g(z, w), z)$ with $g(z, w) : \mathbb{C}^2 \rightarrow \mathbb{C}$ holomorphic.

Key words: transcendental Henon maps, Baker domains, Fatou components with rank one limit manifolds, escaping Fatou components

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1. Introduction

We consider the dynamical system generated by the iteration of a holomorphic automorphism $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. A *Fatou component* is a maximal connected open set U on which the family of iterates $\{F^n\}$ is normal, that is, every sequence has a subsequence which converges uniformly on compact sets to a holomorphic function $g : U \rightarrow \mathbb{P}^2$, where g may depend on the subsequence itself (see [ABFP19a] for a discussion about the definition of normality). Such a function g is called a *limit function*, and its image $g(U)$ is called a *limit set*. If a limit set intersects the line at infinity, then it is in fact contained in it (see [ABFP19a, Lemmas 2.4 and 4.3]).

It is natural to classify invariant Fatou components both from the point of view of a dynamical characterization (that is, to which model map the iterates are conjugate to) and from the point of view of a geometric characterization (that is, to which model manifold the Fatou component is biholomorphic). The first characterization strongly influences the

latter; for example, for polynomial automorphisms of \mathbb{C}^2 , any invariant Fatou component on which the iterates converge to a fixed point is biholomorphic to \mathbb{C}^2 [PVW08, RR88, Ued86]. The dynamical characterization is also very related to which types of limit functions there can be in the Fatou component, for example, their rank, and whether the limit sets are in the boundary of the Fatou component or in its interior.

In this paper we consider invariant escaping Fatou components. A Fatou component U is called *escaping* if for any of its limit functions g we have $g(U) \subset \ell^\infty$, where ℓ^∞ is the line at infinity in the projective space \mathbb{P}^2 used to compactify \mathbb{C}^2 .

In the past three decades, the investigation of the dynamics of holomorphic maps from \mathbb{C}^2 to \mathbb{C}^2 has concentrated on studying polynomial automorphisms, and in particular (polynomial) Hénon maps, that is, automorphisms with constant Jacobian of the form

$$F(z, w) = (P(z) + \delta w, z)$$

with $P : \mathbb{C} \rightarrow \mathbb{C}$ polynomial of degree $d \geq 2$. Indeed, by results of Friedland and Milnor [FM89], any polynomial automorphism with non-trivial dynamical behavior can be written as a finite composition of polynomial Hénon maps. In this view, studying polynomial Hénon maps gives a relatively complete picture of the dynamics of polynomial automorphisms of \mathbb{C}^2 . For polynomial Hénon maps it is not difficult to see [BS91] that unbounded forward orbits belong to the Fatou set and converge to the point $[1 : 0 : 0] \in \ell^\infty$. So, in this case, there is always exactly one escaping Fatou component, which can be seen as the attracting basin of $[1 : 0 : 0]$, and whose structure has been studied for example in [BS99, HOV94, Mum07]. So for polynomial automorphisms, the matter of existence and properties of escaping Fatou components is essentially settled.

On the other hand, one-dimensional transcendental dynamics shows that periodic Fatou components on which the iterates tend to infinity (called *Baker domains* in this setting) are nowadays an active research topic (see for example the most recent papers [BFJK15, BFJK19, BZ12, Rem21, RS18]). One may be tempted to think of Baker domains as parabolic basins whose parabolic fixed point has been moved to infinity, but in fact, there can be different dynamical behaviors [Cow81, FH06], only some of which relate to parabolic dynamics. On the other hand, from the geometric point of view, all Baker domains for entire functions are simply connected and hence, because of the Riemann uniformization theorem, biholomorphic to the unit disk \mathbb{D} . Inspired by the one-dimensional examples, a transcendental Hénon map featuring an escaping Fatou component with a constant limit function and which is not an attracting basin has been constructed in [ABFP19a, §5].

Our first preliminary result is that orbits in escaping Fatou components cannot grow too fast under appropriate conditions. This is in analogy with results obtained by Baker [Bak88, Theorem 1] for Baker domain in one variable, and in contrast to the escaping points constructed in [ABFP19b], whose orbits converge to infinity faster than any polynomial. The proof uses methods similar to [ABFP19a, Lemma 5.9].

PROPOSITION 1.1. (Slow growth in escaping components) *Let F be an automorphism of \mathbb{C}^2 of the form $F(z, w) = (g(z, w), z)$ with an escaping Fatou component U on which the iterates converge to a function $h : U \rightarrow \ell^\infty$ uniformly on compact subsets.*

Let K be a compact subset of U , such that h does not take the values $[0 : 1 : 0]$, $[1 : 0 : 0]$ on K , and fix $0 < \varepsilon < \min_K |h|$. Then there exists $C = C(K)$ such that for n large enough and for any $P \in K$ we have

$$\frac{(\min_K |h| - \varepsilon)^n}{C} \leq \|F^n(P)\| \leq C \left(\max_K |h| + \varepsilon \right)^n. \quad (1.1)$$

The main result of this paper is the construction of examples of transcendental Hénon maps with an escaping Fatou component which has exactly two limit functions, both of (generic) rank one. Transcendental Hénon maps are automorphisms with constant Jacobian of the form

$$F(z, w) = (f(z) + \delta w, z)$$

with $f: \mathbb{C} \rightarrow \mathbb{C}$ entire transcendental. They have been introduced in [Duj04] to construct automorphisms with infinite entropy, and have been studied in [ABFP19a, ABFP19b, ABFP21]. Transcendental Hénon maps always have both escaping and periodic points (hence non-empty Julia set), infinite entropy, a pseudoconvex Fatou set, and can exhibit a variety of dynamical behavior ranging from having various types of wandering domains to the possibility that the Julia set is all of \mathbb{C}^2 .

THEOREM 1.2. (Escaping components with distinct rank-one limit functions) *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a transcendental entire function which is bounded in a right half plane, and $a > 1$. Let $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the transcendental Hénon map defined by*

$$F(z, w) = (f(z) + aw, z).$$

Then we have the following.

- (1) F has an invariant escaping Fatou component U with exactly two distinct limit functions $h_1, h_2: U \rightarrow \ell^\infty$, both of which have (generic) rank one.
- (2) $h_1(U), h_2(U) \supset \ell^\infty \setminus \{[1 : 0 : 0], [0 : 1 : 0]\}$.
- (3) F is conjugate to the linear map $L(z, w) = (aw, z)$ on an appropriate subset of U .
- (4) If $f(z) = e^{-z}$, then F is conjugate to L on all of U and U is biholomorphic to $\mathbb{H} \times \mathbb{H}$.

Several ideas in the proof are taken from [ABFP19a, §5], modified to apply to this different setting. The escaping component constructed in [ABFP19a] differs from ours both from the dynamical and from the geometric point of view: indeed, it is biholomorphic to $\mathbb{H} \times \mathbb{C}$ and the map is conjugate to the linear map $G(z, w) = (2z - w, w)$.

In general, it is very unclear under which conditions and for which types of automorphisms it is possible to have invariant Fatou components with limit sets of dimension one in the boundary. While [LP14] gives conditions under which this cannot happen for polynomial Hénon maps, there are a few examples of automorphisms sporting a Fatou component with a rank-one limit manifold in the boundary: see [BTBP21, JL04, Rep21]. All examples are of non-escaping Fatou components, for automorphisms with non-constant Jacobian, and wherever this has been computed the map in question is conjugate to the linear map $G(z, w) = (z + 1, w)$, so their dynamics can be considered parabolic. On the other hand, their complex structures are different: the Fatou components in [BTBP21] are biholomorphic to \mathbb{C}^2 while the ones in [Rep21] are biholomorphic to $\mathbb{C}^* \times \mathbb{C}$ (compare with the construction in [BRS21]).

Let us finish by remarking that from both the dynamical and the geometric point of view, the richness of possibilities in $2D$ as compared to $1D$ is striking. In one variable, all periodic and preperiodic Fatou components for entire and meromorphic functions are fully classified: on each such component, including Baker domains, the dynamics is semi-conjugate to an appropriate linear map, and, in the entire case, all periodic components are simply connected hence biholomorphic to the unit disk.

In several variables, recurrent Fatou components for polynomial automorphisms have been classified in [BS91] (see also [ABFP19a]), but it is currently unknown whether such components can be biholomorphic to an annulus times \mathbb{C} (for convincing evidence that this may indeed happen see [Bed18]). Several additional geometric possibilities are open in the transcendental Hénon case: a priori, the rotation surface may be biholomorphic also to the punctured disk, the punctured plane, or even the plane itself. Non-recurrent Fatou components have been classified in [LP14] for polynomial automorphisms under the assumption that the Jacobian is small; however, by removing this assumption, it is not known what other dynamical behaviors may appear and what would be the geometry of the limit sets and of the Fatou component.

1.1. *Notation.* We denote by \mathbb{C} the complex plane, by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ its one-point compactification (the Riemann sphere), by \mathbb{H} the right half plane $\{\operatorname{Re} z > 0\}$, and by \mathbb{D} the Euclidean unit disk. The complex projective space is denoted by \mathbb{P}^2 and the line at infinity by ℓ^∞ .

The complex line ℓ^∞ is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$, via the biholomorphism φ which sends $[p : q : 0]$ to p/q . Given a holomorphic map h from a domain of \mathbb{C}^2 to ℓ^∞ we identify it with a holomorphic map h to the Riemann sphere.

2. Slow growth in escaping components

Using hyperbolic geometry, Baker [Bak88, Theorem 1] proved that, if z is in a Baker domain for an entire transcendental function $f : \mathbb{C} \rightarrow \mathbb{C}$, then $\log |f^n(z)| = O(n)$ as $n \rightarrow \infty$. We show an analogous result for periodic escaping components for transcendental Hénon maps, establishing Proposition 1.1. This is in contrast to the escaping points constructed in [ABFP19b], whose orbits converge to infinity faster than any polynomial. This result, which we believe to be of independent interest, is used in §3.3.

We restate Proposition 1.1 for convenience. The proof uses methods similar to [ABFP19a, Lemma 5.9].

PROPOSITION 2.1. (Slow growth in escaping components) *Let F be an automorphism of \mathbb{C}^2 of the form $F(z, w) = (g(z, w), z)$ with an escaping Fatou component U on which the iterates converge to a function $h : U \rightarrow \ell^\infty$ uniformly on compact sets.*

Let K be a compact subset of U , such that h does not take the values $0, \infty$ on K , and fix $0 < \varepsilon < \min_K |h|$. Then there exists $C = C(K)$ such that for n large enough and for any $P \in K$ we have

$$\frac{(\min_K |h| - \varepsilon)^n}{C} \leq \|F^n(P)\| \leq C \left(\max_K |h| + \varepsilon \right)^n. \quad (2.1)$$

Proof. Let $P_n = (z_n, w_n) = (z_n, z_{n-1})$ by the special form of F . Since $F^n(P) \rightarrow \ell^\infty$ and h does not take the values $0, \infty$, we can assume that $|z_n|, |w_n| \neq 0$ for all n large enough. Since $F^n \rightarrow h$ uniformly on K , there exists n_ε such that for all $n \geq n_\varepsilon$ and all $P \in K$ we have

$$\left| \frac{z_n}{w_n} - h(P) \right| = \left| \frac{z_n}{z_{n-1}} - h(P) \right| < \varepsilon,$$

and hence, using the triangular inequality,

$$\min_K |h| - \varepsilon \leq \left| \frac{z_n}{z_{n-1}} \right| \leq \max_K |h| + \varepsilon,$$

from which it follows (adding the multiplicative factor $c^{\pm 1}$ to account for $n \leq n_\varepsilon$)

$$\frac{(\min_K |h| - \varepsilon)^n}{c} \leq |z_n| \leq c \left(\max_K |h| + \varepsilon \right)^n.$$

Since $w_n = z_{n-1}$, the analogous inequality holds for $|w_n| = |z_{n-1}|$ and the claim for $\|P_n\|$ follows for some constant C . □

The proof is easily generalized to obtain the following.

PROPOSITION 2.2. (Slow growth general version) *Let F be as in Proposition 2.1 with an escaping Fatou component U with finitely many limit functions $h_i : U \rightarrow \ell^\infty$. Suppose that there exists a partition of \mathbb{N} into finitely many subsequences N_i such that for each i the iterates of F converge to the limit function h_i along the subsequence N_i . Let K be a compact subset of U such that none of the h_i attains the value $0, \infty$ on K , and let $\varepsilon < \min_{K,i} |h_i|$. Then there exists C such that for any $P \in K$ and n large enough we have*

$$\frac{(\min_{K,i} |h_i| - \varepsilon)^n}{C} \leq \|F^n(P)\| \leq C \left(\max_{K,i} |h_i| + \varepsilon \right)^n. \tag{2.2}$$

We note the following corollary that we will use in §3.3, and which follows from the proof of Proposition 2.1 and Proposition 2.2.

COROLLARY 2.3. *Let U be an escaping Fatou component for F as in Proposition 2.1, such that $F^{2n} \rightarrow h_1, F^{2n+1} \rightarrow h_2$. Then for any K compact subset of U with $h_i \neq 0, \infty$ on K there exists C such that for n large enough and every $P \in K$ we have*

$$\frac{(\min_K |h_i| - \varepsilon)^n}{C} \leq |z_n| \leq C \left(\max_K |h_i| + \varepsilon \right)^n, \tag{2.3}$$

where $(z_n, w_n) := F^n(P)$.

3. A transcendental Hénon map with an invariant escaping Fatou component with two distinct limit functions of rank one

In this section we construct a family of transcendental Hénon maps, each of which has an invariant escaping Fatou component with exactly two limit functions, both of which have (generic) rank one. Recall that the *rank* of a holomorphic functions h at a point P is the rank of its differential at P .

PROPOSITION 3.1. *Let $a > 1$ and let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a nonlinear entire function which is bounded in a right half plane. Let $F: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the transcendental Hénon map defined by*

$$F(z, w) = (aw + f(z), z).$$

Then F has an invariant escaping Fatou component U with exactly two distinct limit functions $h_1, h_2: U \rightarrow \ell^\infty$, both of which have generic rank one and such that $F^{2n}(z, w) \rightarrow h_1(z, w)$, $F^{2n+1}(z, w) \rightarrow h_2(z, w)$ as $n \rightarrow \infty$, uniformly on compact subsets of U .

Observe that the condition that f is nonlinear, entire and bounded in a right half plane implies that f is transcendental, and that the function $f(z) = Ae^{-kz}$ satisfies the hypothesis of the proposition for every $A \in \mathbb{R}$, $k > 0$, as well as any finite linear combination of such function. Many more examples, even with the stronger assumption that $|f| \rightarrow 0$ as $\text{Re } z \rightarrow \infty$, can be constructed using tangential approximation, for example, the following relatively elementary result (see [Gai87, Theorem 2', pp. 142 and 153]).

THEOREM 3.2. (Approximation) *Let $S \subset \mathbb{C}$ be a closed set such that $\hat{\mathbb{C}} \setminus S$ is connected and locally connected at infinity. Let h be holomorphic in the interior of S and continuous on \bar{S} (the closure of S in \mathbb{C}). Let $\varepsilon > 0$. Then there exists g entire such that*

$$|g - h| < \varepsilon \quad \text{on } S \quad \text{and} \quad |g(z) - h(z)| < \frac{1}{|z|} \quad \text{as } |z| \rightarrow \infty \text{ on } S.$$

Indeed, if we set $h = 0$ on the right half plane and anything you like in, say, a finite collection of topological disks with pairwise disjoint closure (which do not intersect the right half plane), then the approximating g will satisfy the assumptions of Proposition 3.1.

From the perspective of the identification of ℓ^∞ with $\hat{\mathbb{C}}$ that associates to the point $[p : q : 0] \in \ell^\infty$ the point $(p/q) \in \hat{\mathbb{C}}$, with $p, q \in \mathbb{C}$, the limit functions h_i are holomorphic functions from U to $\hat{\mathbb{C}}$.

Proof of Proposition 3.1. Given (z_0, w_0) in \mathbb{C} we define $(z_n, w_n) := F^n(z_0, w_0)$. For $R > 0$ we define the set

$$W_R := \{(z, w) \in \mathbb{C}^2 : \text{Re } z, \text{Re } w > R\}. \tag{3.1}$$

Fix $\varepsilon > 0$. Since f is bounded in a right half plane, for any R sufficiently large we have that $|f(z)| < (a - 1)R - \varepsilon$ for all z with $\text{Re } z > R$. Let $W = W_R$ for any R which satisfies this condition. Then for $(z_0, w_0) \in W$ we have

$$\begin{aligned} \text{Re } z_1 &= a \text{Re } w_0 + \text{Re } f(z_0) > aR - R + R - |f(z_0)| > R + \varepsilon, \\ \text{Re } w_1 &= \text{Re } z_0 > R, \end{aligned}$$

and hence W is forward invariant and $\text{Re } z_n, \text{Re } w_n \rightarrow \infty$ if $z_0, w_0 \in W$. It follows that $F^n(z_0, w_0) \rightarrow \ell^\infty$.

We now show convergence of the subsequences F^{2n} and F^{2n+1} on W , implying that W is contained in an escaping Fatou component.

A recursive computation gives that

$$F^{2n}(z_0, w_0) = \left(a^n z_0 + a^n \sum_{j=1}^n a^{-j} f(z_{2j-1}), a^n w_0 + a^n \sum_{j=1}^n a^{-j} f(z_{2j-2}) \right), \tag{3.2}$$

$$F^{2n+1}(z_0, w_0) = \left(a^{n+1} w_0 + a^{n+1} \sum_{j=1}^{n+1} a^{-j} f(z_{2j-2}), a^n z_0 + a^n \sum_{j=1}^n a^{-j} f(z_{2j-1}) \right). \tag{3.3}$$

Consider the ratio

$$\frac{z_{2n}}{w_{2n}} = \frac{a^n z_0 + \sum_{j=1}^n a^{n-j} f(z_{2j-1})}{a^n w_0 + \sum_{j=1}^n a^{n-j} f(z_{2j-2})} = \frac{z_0 + \sum_{j=1}^n a^{-j} f(z_{2j-1})}{w_0 + \sum_{j=1}^n a^{-j} f(z_{2j-2})}, \tag{3.4}$$

and the ratio

$$\frac{z_{2n+1}}{w_{2n+1}} = \frac{a^{n+1} w_0 + a^{n+1} \sum_{j=1}^{n+1} a^{-j} f(z_{2j-2})}{a^n z_0 + a^n \sum_{j=1}^n a^{-j} f(z_{2j-1})} = \frac{a(w_0 + \sum_{j=1}^{n+1} a^{-j} f(z_{2j-2}))}{z_0 + \sum_{j=1}^n a^{-j} f(z_{2j-1})}. \tag{3.5}$$

Set

$$\Delta := \max \left(\left| \sum_{j=1}^{\infty} a^{-j} f(z_{2j-1}) \right|, \left| \sum_{j=1}^{\infty} a^{-j} f(z_{2j-2}) \right| \right). \tag{3.6}$$

Using the assumption that $|f(z)|$ is bounded for $\text{Re } z > R$ we get that

$$\Delta \leq \sum_{j=1}^{\infty} |a^{-j/2} f(z_j)| < \sup_{\text{Re } z > R} |f(z)| \sum_{j=1}^{\infty} |a^{-j/2}| < \infty. \tag{3.7}$$

Hence we can take the limit as $n \rightarrow \infty$ in (3.4) and (3.5) to obtain

$$h_1(z_0, w_0) := \lim_{n \rightarrow \infty} \frac{z_{2n}}{w_{2n}} = \frac{z_0 + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-1})}{w_0 + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-2})}, \tag{3.8}$$

$$h_2(z_0, w_0) := \lim_{n \rightarrow \infty} \frac{z_{2n+1}}{w_{2n+1}} = \frac{aw_0 + a \sum_{j=1}^{\infty} a^{-j} f(z_{2j-2})}{z_0 + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-1})} = \frac{a}{h_1}. \tag{3.9}$$

Both the numerator and the denominator in h_1, h_2 are non-constant holomorphic functions from W to \mathbb{C} ; indeed, by taking two points $(z_0, w_0), (z'_0, w'_0) \in W$ with $|z_0 - z'_0|, |w_0 - w'_0| > 2\Delta$ we have that $h_i(z, w) \neq h_i(z', w')$. So h_1 and h_2 are holomorphic functions from W to $\hat{\mathbb{C}}$.

We now show that h_1, h_2 are non-constant. By Sard's Theorem, and since $h_i(W)$ is contained in the line at infinity, it follows that h_1, h_2 have generic rank one. Since $h_1 = a/h_2$, this also implies that $h_1 \neq h_2$. Suppose for a contradiction that $|h_1| = c$ is constant. Then one has

$$|z_0| - \Delta \leq \left| z_0 + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-1}) \right| = c \left| w_0 + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-2}) \right| \leq c|w_0| + c\Delta,$$

and hence

$$|z_0| \leq c|w_0| + (c + 1)\Delta,$$

contradicting the fact that (z_0, w_0) could be any point in W , which is unbounded in the z direction for any choice of w . □

3.1. *Image of the limit functions h_1, h_2 .* In this section we show that the image of the limit functions h_1, h_2 contains the line at infinity minus $0, \infty$.

PROPOSITION 3.3. *Let F, U, h_1, h_2 be as in Proposition 3.1. Then*

$$h_1(U), h_2(U) \supset \ell^\infty \setminus \{[0 : 1 : 0], [1 : 0 : 0]\}.$$

The idea of the proof is to show that h_1 is close enough to the model function $h_0(z, w) := z/w$ on suitable disks contained in W , and then to use the fact that $h_0(W)$ satisfies the claim together with Rouché’s Theorem to deduce the claim for h_1 . The claim for h_2 follows because $h_2 = a/h_1$.

THEOREM 3.4. (Rouché’s Theorem) *Let $D \subset \mathbb{C}$ be a Jordan domain, and f, g be holomorphic in a neighborhood of \overline{D} . Assume that $c \in g(D)$ and that*

$$|f - g| < \text{dist}(c, g(\partial D)) \quad \text{on } \partial D.$$

Then $c \in f(D)$.

Observe that, for $c \in \mathbb{C}$, the complex line $L_c = \{(cw, w) : w \in \mathbb{C}\}$ passing through the origin is mapped to the point c under the map $h_0(z, w) = z/w$. Similarly, the preimage of $c = \infty$ under h_0 is the line $L_\infty := \{(z, 0) : z \in \mathbb{C}\}$. We first need a lemma about the size of disks contained in W whose center is a point $(cw_0, w_0) \in L_c$ and which is contained in a line orthogonal to L_c , that is, a line of the form $\{(cw_0, w_0) + (-w, \bar{c}w)\}$ with $w \in \mathbb{C}$. Let \mathbb{D}_δ denote the Euclidean disk of radius δ centered at the origin.

LEMMA 3.5. *For $c \in \mathbb{C}, \delta > 0$, and $(cw_0, w_0) \in W$, let $D_{c,\delta}(w_0)$ be the disk of radius $\sqrt{1 + |c|^2}\delta$ defined as*

$$D_{c,\delta}(w_0) = \{(z, w) \in \mathbb{C}^2 : (z, w) = (cw_0, w_0) + t(-1, \bar{c}), t \in \mathbb{D}_\delta\}.$$

Then $D_{c,\delta}(w_0) \subset W$ for

$$\delta = \min \left(\frac{|\text{Re } w_0 - R|}{|c|}, |\text{Re } c \text{ Re } w_0 - \text{Im } c \text{ Im } w_0 - R| \right).$$

Proof. A point $(z, w) \in \partial W$ satisfies either $\text{Re } w = R$ or $\text{Re } z = R$, so to find the maximal δ such that $D_{c,\delta}(w_0) \subset W$ we impose the conditions

$$\text{Re}(cw_0 - t) = R,$$

$$\text{Re}(w_0 + t\bar{c}) = R$$

and find the minimal δ for which one is verified for some $|t| = \delta$. The above equations are equivalent to

$$\begin{aligned} \operatorname{Re} c \operatorname{Re} w_0 - \operatorname{Im} c \operatorname{Im} w_0 - \operatorname{Re} t &= R, \\ \operatorname{Re} w_0 + \operatorname{Re} c \operatorname{Re} t + \operatorname{Im} c \operatorname{Im} t &= R. \end{aligned}$$

By setting $\operatorname{Re} t = x$, $\operatorname{Im} t = y$, finding the minimal δ is equivalent to finding the distance in \mathbb{R}^2 from the origin of two lines of the form

$$Ax + By + C = 0.$$

For the first equation $A = -1$, $B = 0$, $C = \operatorname{Re} c \operatorname{Re} w_0 - \operatorname{Im} c \operatorname{Im} w_0 - R$. The distance of such a line from the origin is given by

$$\frac{|C|}{\sqrt{A^2 + B^2}} = |\operatorname{Re} c \operatorname{Re} w_0 - \operatorname{Im} c \operatorname{Im} w_0 - R|.$$

For the second equation $A = \operatorname{Re} c$, $B = \operatorname{Im} c$, $C = \operatorname{Re} w_0 - R$, and the distance of such a line from the origin is

$$\frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|\operatorname{Re} w_0 - R|}{|c|},$$

hence the theses. □

Proof of Proposition 3.3. We will show that $h_1(W), h_2(W) \supset \ell^\infty \setminus \{0, \infty\}$, which implies the claim. Let $h_0(z, w) := z/w$. It is easy to check that $h_0(W) = \ell^\infty$. We use this fact to show that for any $c \in \hat{\mathbb{C}} \setminus \{0, \infty\}$, $c \in h_1(W)$.

In view of Rouché’s Theorem, it is enough to find $r > 0$ and a one-dimensional disk $D \subset W$ such that:

- $h_0(D)$ contains a disk of radius r centered at c in $\ell^\infty \setminus \{0, \infty\}$;
- $|h_1 - h_0| < r$ on ∂D .

Let

$$w_0 = M + R + i \frac{(M + R) \operatorname{Re} c - 2M - R}{\operatorname{Im} c} \quad \text{if } \operatorname{Im} c \neq 0$$

or

$$w_0 = 2M + R \quad \text{if } \operatorname{Im} c = 0.$$

We claim that for $M > 0$ sufficiently large, the disk $D := D_{c,\delta}(w_0)$ centered in (cw_0, w_0) with $\delta = (M - 1) \min(|c|, 1/|c|)$ is contained in W and satisfies the requirements.

We first check that $D \subset W$. In view of Lemma 3.5, we only need to check that

$$\delta \leq \min \left(\frac{|\operatorname{Re} w_0 - R|}{|c|}, |\operatorname{Re} c \operatorname{Re} w_0 - \operatorname{Im} c \operatorname{Im} w_0 - R| \right).$$

If $\operatorname{Im} c \neq 0$, $|\operatorname{Re} w_0 - R|/|c| = (M/|c|) > \delta$ and $|\operatorname{Re} c \operatorname{Re} w_0 - \operatorname{Im} c \operatorname{Im} w_0 - R| = 2M > \delta$ for all choices of $M > 0$.

If $\operatorname{Im} c = 0$, $|\operatorname{Re} w_0 - R|/|c| = (2M/|c|) > \delta$ and $|\operatorname{Re} c \operatorname{Re} w_0 - \operatorname{Im} c \operatorname{Im} w_0 - R| = |(2M + R) \operatorname{Re} c - R| \geq (2M + R)|\operatorname{Re} c| - R \geq \delta$ for M large enough.

From now on, it is no longer necessary to divide the two cases. We now compute the distance $|h_0(\partial D) - c|$. Let $t \in \mathbb{C}$, $|t| = \delta$ and $(z, w) = (cw_0, w_0) + t(-1, \bar{c}) \in \partial D$. Then

$$|h_0(z, w) - c| = \left| \frac{cw_0 - t}{w_0 + \bar{c}t} - c \frac{w_0 + \bar{c}t}{w_0 + \bar{c}t} \right| = \frac{|t|(1 + |c|^2)}{|w_0 + \bar{c}t|} =: A.$$

We want to compare this with $|h_0 - h_1|$ on ∂D . Let us define

$$k_1 = k_1(z, w) := \sum_{j=1}^{\infty} a^{-j} f(z_{2j-1}), \quad k_2 = k_2(z, w) := \sum_{j=1}^{\infty} a^{-j} f(z_{2j-2}),$$

and note that $|k_1|, |k_2|$ are bounded uniformly in W (see (3.7)). Let $(z, w) \in \partial D$ as before. Then

$$\begin{aligned} |(h_1 - h_0)(z, w)| &= \left| \frac{z + k_1}{w + k_2} - \frac{z}{w} \right| = \left| \frac{k_1 w - k_2 z}{w(w + k_2)} \right| \\ &= \frac{|(k_1 - ck_2)w_0 + (\bar{c}k_1 - k_2)t|}{|w_0 + \bar{c}t||w_0 + \bar{c}t + k_2|} =: B. \end{aligned}$$

By calculating the ratio A/B we see that

$$\frac{A}{B} = \frac{|t|(1 + |c|^2)|w_0 + \bar{c}t + k_2|}{|(k_1 - ck_2)w_0 + (\bar{c}k_1 - k_2)t|} \rightarrow \infty \quad \text{as } M \rightarrow \infty,$$

since the numerator is a polynomial of degree two in M and the denominator is a polynomial of degree 1 in M (indeed, both $|t|$ and $\text{Re } w_0$ grow linearly in M). This implies that for M large enough, D satisfies the requirement for Rouché’s Theorem, and hence $h_1(W) \supset \ell^\infty \setminus \{[1 : 0 : 0], [0 : 1 : 0]\}$. Since $h_2 = a/h_1$, the same holds for h_2 . \square

Remark 3.6. If $R > \sup_{W_R} |\Delta|$, then we have precisely that $h_i(W_R) = \ell^\infty \setminus \{[0 : 1 : 0], [1 : 0 : 0]\}$, because the numerator and the denominator in (3.8) cannot attain the exact value 0. If W_R is an absorbing domain for U as in §3.3, then $h_i(U) = \ell^\infty \setminus \{[0 : 1 : 0], [1 : 0 : 0]\}$.

3.2. Conjugacy of F to a linear map.

PROPOSITION 3.7. *Let F be as in Proposition 3.1. Then for R sufficiently large, F is conjugate to the linear map $L(z, w) = (aw, z)$ on the set $\bigcup_{n \geq 0} F^{-n}(W)$, where $W = W(R) = \{(z, w) \in \mathbb{C}^2 : \text{Re } z, \text{Re } w > R\}$.*

Proof. Recall that $|f(z)|$ is bounded by some constant, say M , for R large enough and $\text{Re } z > R$. Let $W := W(R)$ for such R . It is easy to check that $L^{-n}(z, w) = (z/a^{n/2}, w/a^{n/2})$ if n is even and $L^{-n}(z, w) = (w/(a^{(n-1)/2}), z/(a^{(n+1)/2}))$ if n is odd; hence by a direct computation, using the fact that $a > 1$, for any $n \in \mathbb{N}$ and for any $P \in \mathbb{C}^2$ we have that $\|L^{-n}(P)\| \leq a^{-((n-1)/2)} \|P\|$.

Let $\varphi_n : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the automorphisms defined as

$$\varphi_n := L^{-n} \circ F^n.$$

We will show that the φ_n converge to a map $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ uniformly on W . Since the φ_n satisfy the functional equation $\varphi_{n+1} = L^{-1} \circ \varphi_n \circ F$, the map φ is a conjugacy between F and L .

Using the explicit expressions for the iterates of F given by (3.2) and (3.3) we compute

$$\varphi_{2k}(z, w) = \left(z + \sum_{j=1}^k a^{-j} f(z_{2j-1}), w + \sum_{j=1}^k a^{-j} f(z_{2j-2}) \right), \tag{3.10}$$

$$\varphi_{2k+1}(z, w) = \left(z + \sum_{j=1}^k a^{-j} f(z_{2j-1}), w + \sum_{j=1}^{k+1} a^{-j} f(z_{2j-2}) \right), \tag{3.11}$$

and taking the limit we obtain

$$\varphi(z, w) = \left(z + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-1}), w + \sum_{j=1}^{\infty} a^{-j} f(z_{2j-2}) \right),$$

which is a biholomorphism between W and $\varphi(W)$ since both series converge because f is bounded in a right half plane. It is injective by the Hurwitz theorem because the maps φ_n are injective and their limit has rank two (see [Kra01, Exercise 3, p. 310]).

For any $P \in F^{-k}(W)$ we extend φ as $\varphi(P) = L^{-k} \circ \varphi \circ F^k(P)$. Since F is an automorphism and since $\varphi \circ F = L \circ \varphi$, the extension of φ (which we still denote by φ) is well defined as a biholomorphism from $\bigcup_{n \geq 0} F^{-n}(W)$ to $\bigcup_{n \geq 0} L^{-n}(\varphi(W))$. \square

Remark 3.8. If $|f(z)| \rightarrow 0$ as $\operatorname{Re} z \rightarrow \infty$, instead of just being bounded, and since the real parts of z, w are increasing under iteration, we have that $\varphi(z, w)$ tends to the identity as $\operatorname{Re} z, \operatorname{Re} w \rightarrow \infty$. However, in general, this may not be the case.

3.3. Geometric structure of U for $f(z) = e^{-z}$. In this section we prove that, in the special case that $f(z) = e^{-z}$, the Fatou component U is the union of the backwards images of W . As a corollary, using the linearization results from §3.2 we obtain that U is biholomorphic to $\mathbb{H} \times \mathbb{H}$. In fact, the proof holds for any f satisfying the hypothesis of Theorem 3.9 as long as $|f(z)|$ grows fast enough for $\operatorname{Re} z \rightarrow -\infty$. It is based on a modification of the plurisubharmonic method used in [ABFP19a, §5].

PROPOSITION 3.9. *Let $F(z, w) = (aw + e^{-z}, z)$, and U be as in Proposition 3.1. For R sufficiently large and $W = \{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > R\}$, the set W is an absorbing domain for U , that is*

$$U = A := \bigcup_{n \in \mathbb{N}} F^{-n}(W).$$

In view of Remark 3.6, we obtain the following.

COROLLARY 3.10. *If $F(z, w) = (aw + e^{-z}, z)$, then $h_1(U) = h_2(U) = \ell^\infty \setminus \{[1 : 0 : 0], [0 : 1 : 0]\}$.*

The proof of Proposition 3.9 goes by contradiction, by assuming that there is a point $P \in U \setminus A$. We first show that we can assume that $P \in U \cap \partial A$ and that $h_1(P) \neq 0, \infty$.

LEMMA 3.11. *If $U \neq A$, there exists $P \in \partial A \cap U$ such that $h_1(P) \neq 0, \infty$.*

Proof. If $U \neq A$, since $U \supset A$, and also U is connected and open, we have that $\partial A \cap U \neq \emptyset$. Since U is a Fatou component, the function h_1 is well defined on all of U (though the numerator and the denominator in expression (3.8) may not necessarily converge independently). Let \mathcal{Z} be the subset of U such that h_1 takes the value 0, and \mathcal{P} the subset of U such that h_1 takes the value ∞ . Suppose for the sake of contradiction that $\partial A \cap U$ is a subset of \mathcal{X} . For $P \in \partial A \cap U$ consider a neighborhood $V \subset U$ of P . Since \mathcal{X} is an analytic set we have that $V \setminus \mathcal{X}$ is connected (see for example Proposition 7.4 in [KK83]). Since $P \in \partial A$ and A is open, we have that $A \cap V \neq \emptyset$. Since $V \setminus \mathcal{X}$ is connected and $\partial A \cap U \subset \mathcal{X}$ by the contradiction assumption we get that $V \setminus \mathcal{X} \subset A$. Since h_1 is not constant, $\mathcal{X} := \mathcal{Z} \cup \mathcal{P}$ is locally a finite union of complex curves and of finitely many points (see e.g. [Chi85, §§5.1 and 5.2]). It follows that there are infinitely many directions such that a sufficiently small Euclidean disk D tangent to that direction satisfies $D \setminus \{P\} \subset A$.

We now show that the existence of such D implies that $P \in A$. Indeed, the sequence of harmonic functions $g_n : (z, w) \rightarrow \operatorname{Re} z_n = \operatorname{Re} \pi_z(F^n(z, w))$ converges to infinity on compact subsets of A , hence, since $D \setminus \{P\} \subset A$, it converges to infinity uniformly on the boundary of a subdisk of D , and therefore converges to infinity on its center P by Cauchy’s formula. Hence the real parts of the first coordinate of iterates of P converge to infinity, as well as the real parts of the second coordinate (since $w_n = z_{n-1}$), which implies $P \in A$. This contradicts the fact that $P \in U \setminus A$, and hence the assumption that $(\partial A \cap U) \subset \mathcal{X}$ is false and there is a point P as in the claim. □

From now on we consider P with the properties of Lemma 3.11. Since $h_1 \neq 0, \infty$ in P , the same is true for $h_2 = a/h_1$, and hence by continuity h_1, h_2 do not take the values 0, ∞ in some small closed ball B centered at P .

Hence we can define the quantities

$$M := \max_B(\max(|h_1|, |h_2|)) < \infty,$$

$$m := \min_B(\min(|h_1|, |h_2|)) > 0.$$

Note that $M > 1$ because $h_2 = a/h_1$ and $a > 1$. By Corollary 2.3, if $0 < \varepsilon < m$ there exists a constant C such that for every $P = (z_0, w_0) \in B$,

$$|z_n| \leq C(M + \varepsilon)^n. \tag{3.12}$$

Recall that $w_n = z_{n-1}$ to get

$$|w_n| \leq C(M + \varepsilon)^{n-1}. \tag{3.13}$$

The proof of Proposition 3.9 relies on the following technical lemma.

LEMMA 3.12. *Define the sequence of harmonic functions u_n from a neighborhood of B to \mathbb{R} as $u_n(z) := (-\operatorname{Re} z_n)/n$. Then we have the following.*

- (1) $u_n \leq \log M$ in U for n large enough.
- (2) $u_n \rightarrow -\infty$ uniformly on compact subsets of A .

- (3) Let $P \in U \setminus A$. Then for every $\varepsilon > 0$ there exists a subsequence n_k such that $u_{n_k}(P) > -\varepsilon$.

Proof. (1) Suppose that there is a subsequence (n_k) and points $(z^k, w^k) \in B$ such that

$$\frac{-\operatorname{Re} z_{n_k}^k}{n_k} > \beta$$

for some β and let us show that $\log M$ is an upper bound for β . It follows that $\operatorname{Re} z_{n_k}^k < -n_k\beta$. Since $f(z) = e^{-z}$ we have (using the triangular inequality in the second step and (3.12) for $w_{n_k}^k$ in the third step) that

$$|z_{n_k+1}^k| = |e^{-z_{n_k}^k} + aw_{n_k}^k| \geq |e^{-z_{n_k}^k}| - |aw_{n_k}^k| > e^{+n_k\beta} - aC(M + \varepsilon)^{n_k-1}.$$

On the other hand, again using (3.13), we have that

$$|z_{n_k+1}^k| \leq C(M + \varepsilon)^{n_k},$$

hence

$$e^{n_k\beta} - aC(M + \varepsilon)^{n_k-1} < C(M + \varepsilon)^{n_k},$$

which gives (using $M > 1$)

$$e^{n_k\beta} < (a + 1)C(M + \varepsilon)^{n_k},$$

from which (using $n_k \rightarrow \infty$ and $\varepsilon \rightarrow 0$) we obtain $\beta \leq \log M$.

(2) Let K be a compact subset of A . Then there exists n such that $F^n(K) \subset W$, and so it is enough to show the claim for a compact subset of W . By the explicit expression for z_{2n}, z_{2n+1} given by (3.2), (3.3) we get that $\operatorname{Re} z_n \geq a^n(R - \Delta)$ as defined in (3.6). Hence by assuming that R is chosen large enough so that $R - \Delta > 0$, we get that

$$u_n \leq \frac{-a^n(R - \Delta)}{n} \rightarrow -\infty.$$

- (3) If not, there exists $\varepsilon > 0, N \in \mathbb{N}$ such that

$$u_n(P) \leq -\varepsilon \quad \text{for all } n \geq N.$$

Hence if $F^n(P) = (z_n, w_n)$ we have that $((-\operatorname{Re} z_n)/n) \leq -\varepsilon$ for all $n \geq N$, so $\operatorname{Re} z_n \geq n\varepsilon > R$ for n large since $\varepsilon > 0$. Since $w_n = z_{n-1}$, $(z_n, w_n) \in W$ and $(z_0, w_0) \in F^{-n}(W) \subset A$. □

Remark 3.13. We only use the assumption $f(z) = e^{-z}$ to prove (1), that is, that the u_n are bounded from above. In fact, it is enough to assume that $|f(z)|$ grows sufficiently fast as $\operatorname{Re} z \rightarrow -\infty$.

Proof of Proposition 3.9. Let P as in Lemma 3.11, B be a ball centered in P as described above, and let D be a one-dimensional Euclidean disk compactly contained

in B , intersecting A , and passing through P . Consider the real one-dimensional Lebesgue measure on ∂D . Let K be a compact subset of A such that the measure in ∂D of $K \cap \partial D$ is strictly positive. This can be done because A is open, and hence $A \cap \partial D$ is open in the topology of ∂D . Let $\mu_{\text{good}} > 0$ be the measure of the set $\partial D \cap K$ and μ_{bad} be the measure of the set $\partial D \cap (U \setminus K)$. Since U contains B , $\partial D = (\partial D \cap K) \cup (\partial D \cap (U \setminus K))$, and, since K is compact and U is open, the sets in question are measurable.

By Lemma 3.12 for any given $M > 0$ there exists N such that $u_N \leq -M$ on K , $u_N(P) \geq -\varepsilon$ for ε arbitrarily small since $P \in U \setminus A$, and $u_N(P) \leq \log M$ on U . By the mean value property for u_N we have

$$\begin{aligned} -\varepsilon \leq u_N(P) &= \int_{\partial D} u_N(\zeta) d\zeta = \int_{\partial D \cap K} u_N(\zeta) d\zeta + \int_{\partial D \cap (U \setminus K)} u_N(\zeta) d\zeta \\ &\leq -M\mu_{\text{good}} + \log M\mu_{\text{bad}}. \end{aligned}$$

Since M is arbitrarily large, this gives a contradiction. □

COROLLARY 3.14. *Let $F(z, w) = (aw + e^{-z}, z)$, and U be as in Proposition 3.1. Then U is biholomorphic to $\{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > 0\}$.*

Proof. Let $W_R = \{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > R\}$ with R large enough so that Proposition 3.9 holds and so that $R - \Delta > 0$ with Δ defined as in (3.6), with $f(z) = e^{-z}$. Then, by Proposition 3.9, W_R is an absorbing domain for U , and, by the explicit form of φ ,

$$\begin{aligned} W_{R+\Delta} &= \{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > R + \Delta\} \subset \varphi(W) \\ &\subset \{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > R - \Delta\} = W_{R-\Delta}. \end{aligned}$$

Since $R \pm \Delta > 0$ we have that

$$\bigcup_n L^{-n}(W_{R+\Delta}) = \bigcup_n L^{-n}(W_{R-\Delta}) = \{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > 0\}.$$

It follows that

$$\bigcup_n L^{-n}(\varphi(W_R)) = \{(z, w) \in \mathbb{C} : \operatorname{Re} z, \operatorname{Re} w > 0\}.$$

Since φ is a biholomorphism between $\bigcup_n L^{-n}(\varphi(W_R))$ and $\bigcup_n F^{-n}(W_R) = U$, the claim follows. □

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