Bull. Aust. Math. Soc. **106** (2022), 89–101
doi:10.1017/S0004972721001064

THE NUMBER OF SET ORBITS OF PERMUTATION GROUPS AND THE GROUP ORDER

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(Received 22 September 2021; accepted 5 November 2021; first published online 27 January 2022)

Abstract

If G is permutation group acting on a finite set Ω, then this action induces a natural action of G on the power set $\mathscr{P}(\Omega)$. The number $s(G)$ of orbits in this action is an important parameter that has been used in bounding numbers of conjugacy classes in finite groups. In this context, $\inf(\log_2 s(G)/\log_2 |G|)$ plays a role, but the precise value of this constant was unknown. We determine it where G runs over all permutation groups not containing any $A_l, l > 4$, as a composition factor.

2020 *Mathematics subject classification*: primary 20B05.

Keywords and phrases: permutation groups, set orbits, group order.

1. Introduction

Let G be a permutation group acting on a finite set Ω of size n. Then G induces a natural action on the power set $\mathscr{P}(\Omega)$. The orbits of this action are called set orbits and we let $s(G)$ denote the total number of set orbits in this action. This number was studied by Babai and Pyber in [1]; in particular, they proved that if G is a permutation group of degree n with no composition factor isomorphic to A_k for $k > t$ (where $t \geq 4$) then $s(G) \geq 2^{c_1 n/t}$ for some absolute constant $c_1 > 0$. Clearly c_1 depends on t, but no value or bound for c_1 was given. As a corollary, they obtained $s(G) \geq |G|^{c_2/(t \log_2 t)}$ for a constant c_2 which depends only on t but was also unspecified. This latter bound plays a crucial role in finding lower bounds for the number of conjugacy classes of finite groups. The best such bounds are currently obtained via Pyber's approach [6], which relies on the bound on set orbits. It is therefore desirable to have an idea of the size of c_2, or even its exact value. It turns out that with today's computational power, it is possible to determine c_2 in some situations. We focus on the important case when $t = 4$, that is, avoiding any simple alternating composition factor. For $t = 4$ we can restate the above bound by saying that there is an absolute positive constant c_3

This research was supported by NSF-REU grant DMS-1757233 and NSA grant H98230-21-1-0333. Y. Yang was also partially supported by a grant from the Simons Foundation (#499532).

such that for any permutation group G with no simple alternating composition factor we have $s(G) \geq |G|^{c_3}$ and the best possible value for c_3 is $\inf(\log_2 s(G)/\log_2 |G|)$. We determine this value and also the corresponding permutation groups that attain it. Our main result is the following theorem.

THEOREM 1.1. *We have*

$$\inf\left(\frac{\log_2 s(G)}{\log_2 |G|}\right) = \lim_{k\to\infty} \frac{\log_2 s(M_{12} \wr M_{12} \wr \overbrace{S_4 \wr \cdots \wr S_4}^{k \text{ terms}})}{\log_2 |M_{12} \wr M_{12} \wr \underbrace{S_4 \wr \cdots \wr S_4}_{k \text{ terms}}|} =: M,$$

where the infimum is taken over all permutation groups G not containing any $\mathrm{A}_l, l > 4$*, as a composition factor. (Here M_{12} acts naturally on* 12 *elements.)*

We will give a good estimate for the value M in Theorem 2.11.

While we believe that the result is nice, its proof, admittedly, is not. By its nature the proof requires some subtle, but tedious, estimates and lots of calculations, The clean end result justifies the effort.

For solvable groups, $\inf(\log_2 s(G)/\log_2 |G|)$ has already been determined in [3] to be ≈ 0.18939, which is obtained by the group $G = S_4 \wr \cdots \wr S_4$. The value of c_1 has also been studied. In [9], the ratio $\log_2 s(G)/n$ was considered for solvable G; here $G = A\Gamma(2^3) \wr S_4 \wr \cdots \wr S_4$ gave the minimum. In [10], the same ratio for arbitrary G with no simple alternating composition factors was determined and the group that yields the minimum is $G = M_{24} \wr M_{12} \wr S_4 \wr \cdots \wr S_4$. Some more general situations have recently been studied in [8].

The main difficulty of this and the previous papers is to determine a sequence of groups that achieves the infimum. The experience gained in the previous work allows us to narrow this down to a few candidates with some thorough calculations. The other major challenge of this paper is that much tighter estimates than before are needed to eliminate candidate groups that give sequences very close to the one we ultimately prove to yield the infimum.

2. Main results

We use $H \wr S$ to denote the wreath product of H with S where H is a group and S is a permutation group.

Let G be a permutation group and $s(G)$ denote the number of set orbits of G. Following the notation in [3], we let $ds(G) = \log_2 s(G)/\log_2 |G|$.

We provide some preliminary facts about transitive groups. Let G be a transitive permutation group on a set Ω where $|\Omega| = n$. A system of imprimitivity is a partition of Ω which is invariant under the action of G. A transitive group is primitive if the only systems of imprimitivity are 1-sets and Ω itself. Let $(\Omega_1, \ldots, \Omega_m)$ denote a system of imprimitivity with maximal block-size b (where $1 \leq b < n, bm = n$ and $b = 1$ if and

https://doi.org/10.1017/S0004972721001064 Published online by Cambridge University Press

only if G is primitive). Let N be the intersection of the stabilisers of the blocks. Then G/N is a primitive group acting on the set of blocks Ω_i.

Let G be a transitive permutation group of degree n that is not primitive. If we have a system of imprimitivity with $m \geq 2$ blocks of size b with b maximal, then $G \lesssim K \wr P_1$ where K is a permutation group of degree n/m and P_1 is the primitive group acting on the m blocks. We may keep doing this and, after reindexing for convenience, we have $G \lesssim H \wr P_1 \wr \cdots \wr P_j$ where each P_i is primitive and H is a permutation group. In this case, we say that G is induced from H.

The following two results are from [1].

LEMMA 2.1. *If $L \leq G \leq Sym(\Omega)$, then $s(G) \leq s(L) \leq s(G) \cdot |G : L|$.*

LEMMA 2.2. *Assume that G is intransitive on Ω and has orbits $\Omega_1, \ldots, \Omega_m$. Let G_i be the restriction of G to Ω_i. Then $s(G) \geq s(G_1) \times \cdots \times s(G_m)$.*

Suppose the action of G on Ω is not transitive. Since the number of set orbits will increase with the number of orbits on Ω, we may assume that G has two orbits Ω_1 and Ω_2. Let G_i denote the restriction of G to Ω_i. Then $G \leq G_1 \times G_2$ and $|G| \leq |G_1| \cdot |G_2|$. By Lemma 2.2, $s(G) \geq s(G_1) \cdot s(G_2)$. Then

$$ds(G) = \frac{\log_2 s(G)}{\log_2 |G|} \geq \frac{\log_2(s(G_1) \cdot s(G_2))}{\log_2(|G_1| \cdot |G_2|)} = \frac{\log_2 s(G_1) + \log_2 s(G_2)}{\log_2 |G_1| + \log_2 |G_2|}$$
$$\geq \min\{ds(G_1), ds(G_2)\}$$

(since $(a+b)/(c+d) \geq \min\{a/c, b/d\}$ for positive numbers a, b, c and d). Thus, in order to prove Theorem 1.1, we need only consider transitive groups.

LEMMA 2.3. *Let $G = H \wr P$ be a permutation group of degree nm where H is a permutation group of degree n and P has degree m. Then $ds(H) \geq ds(G)$.*

PROOF. Let $F = H \times H \times \cdots \times H$ with m terms. Note that $F \trianglelefteq G$ and $s(F) = s(H)^m$ by Lemma 2.2. Also $s(F) \geq s(G)$ by Lemma 2.1 and $|H|^m = |F| \leq |G|$. Then

$$ds(H) = \frac{m \log_2 s(H)}{m \log_2 |H|} = \frac{\log_2 s(H)^m}{\log_2 |H|^m} = \frac{\log_2 s(F)}{\log_2 |F|} \geq \frac{\log_2 s(G)}{\log_2 |G|} = ds(G). \qquad \square$$

We also make use of Tables 1 and 2 in [10]. Table 1 provides lower bounds for the number of set orbits of a primitive group and Table 2 provides upper bounds on the orders of primitive groups not containing A_l, $l > 4$, as a composition factor.

We introduce several sequences to assist with our proof. Define $\{a_k\}_{k \geq -1}$ by $a_{-1} = s(M_{12}) = 14$ and $a_0 = s(M_{12} \wr M_{12}) = 604576714$. Let $a_k = \binom{a_{k-1}+3}{4}$ for $k \geq 1$. The value of a_{-1} may be easily verified in GAP [7], and we explain a_0 shortly. Define the sequence $\{b_k\}_{k \geq 0}$ by $b_0 = 0$ and $b_{k+1} = 4 \cdot b_k + 1$. The explicit formula is $b_k = (4^k - 1)/3$, which may be easily checked by induction. Lastly, define

$$c_k = \frac{\log_2(a_k)}{\log_2(95040^{13 \cdot 4^k} \cdot 24^{b_k})} = \frac{\log_2(a_k)}{13 \cdot 4^k \cdot \log_2 95040 + b_k \cdot \log_2 24}.$$

The following calculation shows that c_k is decreasing:

$$\begin{aligned} c_{k+1} &= \frac{\log_2(a_{k+1})}{\log_2(95040^{13\cdot 4^{k+1}} \cdot 24^{b_{k+1}})} = \frac{\log_2 \binom{a_k+3}{4}}{4^{k+1}\log_2 95040^{13} + (4b_k+1)\cdot \log_2 24} \\ &\le \frac{4\log_2 a_k}{4^{k+1}\cdot \log_2 95040^{13} + 4\cdot b_k \cdot \log_2 24 + \log_2 24} \\ &< \frac{\log_2 a_k}{4^{k}\cdot \log_2 95040^{13} + b_k \cdot \log_2 24} = c_k. \end{aligned}$$

We may obtain using Maple that $c_1 \approx 0.129675$, $c_2 \approx 0.128179$, $c_3 \approx 0.127806$ and $c_4 \approx 0.127712$. Also $c_8 < 0.1276818245$.

To calculate $s(M_{12} \wr M_{12})$, we consider the structure of the group action of M_{12} and provide a method for calculating $s(G \wr M_{12})$ in general. A *partition* of a positive integer n expresses n as the sum of a sequence of strictly positive integers. Let Π denote the set of all partitions of 12 and suppose that $\pi \in \Pi$ is a partition of 12. Let $B(\pi)$ denote the number of terms in the partition, say $B(\pi) = n_1 + n_2 + \cdots + n_j$, where n_1 is the number of occurrences of the largest term in the partition, n_2 is the number of occurrences of the second largest term and so on. Define $F(\pi) = n_1!\, n_2! \cdots n_j!$. For example, if $\pi = (4, 2, 2, 1, 1, 1, 1)$, then $n_1 = 1$, $n_2 = 2$ and $n_3 = 4$, which gives $B(\pi) = 7$ and $F(\pi) = 1!\,2!\,4!$. Let $P(n, k)$ denote the number of ways to permute k objects out of n, so $P(s(G), B(\pi))/F(\pi)$ gives the number of ways of choosing $B(\pi)$ orbits from the set orbits of G with repetitions described by $n_1, n_2, \ldots, n_k$. Finally, we define $N(\pi)$ to be the number of orbits of M_{12} on all the multiset permutations (permutations with repetitions) of a set of 12 elements with the partition π as multiset structure. Table 3 in [10] provides a summary of this information. Thus we can calculate $s(G \wr M_{12})$ using

$$s(G \wr M_{12}) = \sum_{\pi\in\Pi} N(\pi) \cdot \frac{P(s(G), B(\pi))}{F(\pi)}$$

and Table 3 in [10] and verify that $a_0 = 604576714$. The GAP code for these calculations is available at https://www.math.txstate.edu/research-conferences/summerreu/yang_documents.html.

LEMMA 2.4 [10, Lemma 2.3]. *Let G be a transitive permutation group acting on a set Ω, where $|\Omega| = n$. Let $(\Omega_1, \ldots, \Omega_m)$ denote a system of imprimitivity of maximal block-size b. Let N denote the normal subgroup of G stabilising each of the blocks Ω_i. Let $G_i = \mathrm{Stab}_G(\Omega_i)$ and $s = s(G_1)$. Then*

(1) $s(G) \ge s^m/|G/N|$,
(2) $s(G) \ge \binom{s+m-1}{s-1}$ *and equality holds if $G/N \cong S_m$.*

LEMMA 2.5. *Let G be a permutation group that does not contain any alternating group A_l with $l > 4$ as a composition factor and suppose G is induced from H. If*

$$\frac{\log_2(s(H))}{\log_2|H|} - \frac{\log_2(24^{\alpha})}{\log_2|H|} \ge \beta, \tag{2.1}$$

where $\alpha = 23/60$ and $0 \le \beta \le 3/20$, then $\log_2(s(G))/\log_2|G| \ge \beta$.

https://doi.org/10.1017/S0004972721001064 Published online by Cambridge University Press

PROOF. We may assume $G \lesssim H \wr P_1 \wr \cdots \wr P_j$ where each P_i is primitive and $\deg(P_i) = m_i$. By (2.1), $s(H) \geq 24^{\alpha} \cdot |H|^{\beta}$. Also $|P_1| \leq (24^{1/3})^{m_1-1}$ by [5, Corollary 1.5]. By Lemma 2.4,

$$s(H \wr P_1) \geq s(H)^{m_1}/|P_1| \geq \frac{24^{\alpha m_1} \cdot |H|^{\beta m_1}}{24^{(m_1-1)/3}} = 24^{\alpha} \cdot |H|^{\beta \cdot m_1} \cdot (24^{(1/3)(m_1-1)})^{3/20}$$
$$\geq 24^{\alpha} \cdot |H|^{\beta \cdot m_1} \cdot (24^{(1/3)(m_1-1)})^{\beta} \geq 24^{\alpha} \cdot |H|^{\beta \cdot m_1} \cdot |P_1|^{\beta}.$$

Thus

$$\frac{\log_2 s(H \wr P_1)}{\log_2 |H \wr P_1|} - \frac{\log_2 24^{\alpha}}{\log_2 |H \wr P_1|} \geq \beta.$$

By induction,

$$\frac{\log_2 s(H \wr P_1 \wr \cdots \wr P_j)}{\log_2 |H \wr P_1 \wr \cdots \wr P_j|} - \frac{\log_2 24^{\alpha}}{\log_2 |H \wr P_1 \wr \cdots \wr P_j|} \geq \beta,$$

from which $ds(G) \geq \log_2 s(H \wr P_1 \wr \cdots \wr P_j)/\log_2 |H \wr P_1 \wr \cdots \wr P_j|$ by Lemma 2.1. □

LEMMA 2.6 [5]. *If H be a primitive group of degree n where H does not contain A_n, then:*

(1) $|H| < 50 \cdot n^{\sqrt{n}}$;
(2) $|H| < 3^n$ *and* $|H| < 2^n$ *if* $n > 24$;
(3) $|H| < 2^{0.76n}$ *when* $n \geq 25$ *and* $n \neq 32$.

PROPOSITION 2.7. *Let G be a primitive permutation group of degree n, not containing A_l with $l > 4$ as a composition factor. Then $ds(G) > c_8$.*

PROOF. Let G be a primitive permutation group of degree n not containing A_l, for $l > 4$, as a composition factor. If $n \geq 25$ and $n \neq 32$, then $|G| \leq 2^{0.76n}$ by Lemma 2.6, so $s(G) \geq 2^n/|G| \geq 2^{0.24n}$ and

$$ds(G) = \frac{\log_2 s(G)}{\log_2 |G|} \geq \frac{\log_2 2^{0.24n}}{\log_2 2^{0.76n}} = \frac{24}{76} > 0.3157 > c_8.$$

If $n = 32$, then $s(G) \geq 361$ by Table 2 of [10]. Also $|G| < 2^{32}$ by Lemma 2.6. So

$$ds(G) = \frac{\log_2 s(G)}{\log_2 |G|} > \frac{\log_2 361}{32 \cdot \log_2 2} > 0.2654 > c_8.$$

If $n < 25$, we note that $s(G) \geq n + 1$ and $|G| < 3^n$ by Lemma 2.6. For $2 \leq n \leq 20$, direct calculation shows that

$$ds(G) = \frac{\log_2 s(G)}{\log_2 |G|} > \frac{\log_2 (n+1)}{n \log_2 3} > c_8$$

for each n. For $n = 21, 22, 23$ and 24, we use the upper bounds for $|G|$ in Table 2 of [10]. In all cases, $ds(G) > 1.66 > c_8$. □

THEOREM 2.8. *Let G be a transitive permutation group not containing any composition factors $A_l, l > 4$. Let G be induced from H where H is a primitive permutation group of degree n. If H is different from M_{12}, then $ds(G) > c_8$.*

PROOF. By Lemma 2.5, it suffices to show that

$$\frac{\log_2 s(H)}{\log_2 |H|} - \frac{\frac{23}{60}\log_2(24)}{\log_2 |H|} > c_8$$

for all n. Suppose $n \geq 25$ and $n \neq 32$. Then $|H| \leq 2^{0.76n}$ and $s(H) \geq 2^n/|H| \geq 2^{0.24n}$ by Lemma 2.6. Then

$$\frac{\log_2 s(H) - \frac{23}{60}\log_2 24}{\log_2 |H|} \geq \frac{0.24n - \log_2(24^{23/60})}{0.76n} = \frac{24}{76} - \frac{\frac{23}{60}\log_2 24}{0.76 \cdot 25} > 0.223 > c_8.$$

Suppose $n = 32$. Then $s(H) \geq 361$ and $|H| \leq 2^{32}$ by Table 2 of [10] and Lemma 2.6 and so

$$\frac{\log_2 361 - \frac{23}{60}\log_2 24}{\log_2 2^{32}} > 0.21 > c_8.$$

For $21 \leq n \leq 24$ and $n = 14, 15, 16$ and 17 we refer to Tables 1 and 2 in [10] to find bounds for $s(H)$ and $|P_1|$. Direct calculation shows that the inequality holds.

For $3 \leq n \leq 13$ and $n = 18, 19$ and 20, we note that $s(H) \geq n + 1$. We use the upper bounds for $|P_1|$ from [10] and direct calculation shows that the inequality holds in all cases excluding M_{12}.

We need to consider $n = 2, 3$ and 4 differently. We note that by Lemma 2.7, G is not primitive and so we may assume that $G \lesssim H \wr P_1 \wr \cdots \wr P_j$ where each P_i is primitive of degree m_1. Let $K = H \wr P_1$. We show that

$$\frac{\log_2 s(K)}{\log_2 |K|} - \frac{\frac{23}{60}\log_2 24}{\log_2 |K|} > c_8. \tag{$\star$}$$

Suppose that $n = 4$. Then $s(H) \geq 5$ and $s(K) \geq 5^{m_1}/|P_1|$ by Lemma 2.4. Also $|H| \leq 24$. If $m_1 \geq 25$ and $m_1 \neq 32$, then $|P_1| \leq 2^{0.76m_1}$ by Lemma 2.6. Then

$$\frac{\log_2(5^{m_1}/2^{0.76m_1})}{\log_2(24^{m_1} \cdot 2^{0.76m_1})} - \frac{\frac{23}{60}\log_2 24}{25 \cdot \log_2 24 \cdot 2^{0.76}} > 0.279 > c_8.$$

If $m_1 = 32$, then $|P_1| \leq 319979520$ by [10]. So $s(K) \geq 5^{32}/319979520$. We verify that ($\star$) is satisfied.

For $5 \leq m_1 \leq 24$ we use the bounds for $|P_1|$ in [10] and the estimate $s(K) \geq 5^{m_1}/|P_1|$ and direct calculation shows that ($\star$) is satisfied in all cases except when $P_1 \cong M_{12}$. If $P_1 \cong M_{12}$, we calculate $s(S_4 \wr M_{12}) = 5825$ by the method outlined above. Since $|M_{12}| = 95040$, direct calculation shows that ($\star$) is satisfied.

For $2 \leq m_1 \leq 4$, we note that $s(K) \geq \binom{s(H)-1+m_1}{s(H)-1} = \binom{4+m_1}{4}$ by Lemma 2.6. If $m_1 = 4$, then $|P_1| \leq 24$ and $s(K) \geq \binom{8}{4} = 70$. It is easy to see that ($\star$) is satisfied. Similarly, direct calculation shows that ($\star$) is satisfied when $m_1 = 2$ or 3.

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If $n = 3$, then $s(H) \geq 4$ and $s(K) \geq 4^{m_1}/|P_1|$ by Lemma 2.4. If $m_1 \geq 25$ and $m_1 \neq 32$, then $|P_1| \leq 2^{0.76m_1}$ by Lemma 2.6 and $|K| = |H|^{m_1}|P_1| \leq 6^{m_1} \cdot 2^{0.76m_1}$ since $|H| \leq 6$. Thus, we see that

$$\frac{\log_2 s(K)}{\log_2 |K|} - \frac{\frac{23}{60}\log_2 24}{\log_2 |K|} \geq \frac{\log_2(4^{m_1}/2^{0.76m_1})}{\log_2 6^{m_1} \cdot 2^{0.76m_1}} - \frac{\frac{23}{60}\log_2 24}{\log_2 6^{25} \cdot 2^{0.76 \cdot 25}} = 0.349 > c_8.$$

For $m_1 = 32$ and $5 \leq m_1 \leq 24$ with $P_1 \not\cong M_{12}$ and $m_1 \neq 8$, the bounds in [10] and $s(K) \geq 4^{m_1}/|P_1|$ show that ($\star$) is satisfied. If $P_1 \cong M_{12}$, then $s(S_3 \wr M_{12}) = 862$ by the method outlined above. Direct calculation shows that ($\star$) is satisfied.

For $2 \leq m_1 \leq 4$ and $m_1 = 8$, the bounds $s(K) \geq \binom{s(H)-1+m_1}{s(H)-1} = \binom{3+m_1}{3}$ and the bounds for $|P_1|$ in [10] show that ($\star$) is satisfied by direct computation.

Suppose that $n = 2$. Then $|H| = 2$ and $s(H) \geq 3$. If $m_1 \geq 25$ and $m_1 \neq 32$, then $|P_1| \leq 2^{0.76m_1}$. Also $s(H \wr P_1) \geq 3^{m_1}/|P_1|$. Then

$$\frac{\log_2 s(K)}{\log_2 |K|} - \frac{\frac{23}{60}\log_2 24}{\log_2 |K|} \geq \frac{\log_2(3^{m_1}/2^{0.76m_1})}{\log_2 2^{m_1} \cdot 2^{0.76m_1}} - \frac{\frac{23}{60}\log_2 24}{\log_2 2^{25} \cdot 2^{0.76 \cdot 25}} = 0.664 > c_8.$$

For $m_1 = 32$ and $12 \leq m_1 \leq 24$ and $P_1 \not\cong M_{12}$, we use the bound $s(K) \geq 3^{m_1}/|P_1|$ and the bounds in [10]. Direct calculation shows that ($\star$) is satisfied. If $P_1 \cong M_{12}$, then $s(S_2 \wr M_{12}) = 120$ by the method described above. We verify that ($\star$) is satisfied.

For $2 \leq m_1 \leq 11$, we use $s(K) \geq \binom{s(H)-1+m_1}{s(H)-1} = \binom{2+m_1}{2}$. Direct calculation shows that ($\star$) is satisfied.

Therefore, $ds(G) > c_8$ provided H is different from M_{12}. □

THEOREM 2.9. *Let G be a transitive permutation group where G does not contain any alternating group A_l, $l > 4$, as a composition factor. Let $G \cong M_{12} \wr P_1 \wr \cdots \wr P_j$ where each P_i is a primitive group. If P_1 is different from M_{12}, then $ds(G) > c_8$.*

PROOF. Note that $s(M_{12}) = 14$ and $|M_{12}| = 95040$. Let $L = M_{12} \wr P_1$, where $\deg(P_1) = m_1 \geq 2$. By Lemma 2.4, $s(L) \geq 14^{m_1}/|P_1|$. Also $|L| = 95040^{m_1} \cdot |P_1|$. By Lemma 2.5, it suffices to show that for all $m_1 \geq 2$,

$$\star(L) = \frac{\log_2 s(L)}{\log_2 |L|} - \frac{\frac{23}{60}\log_2 24}{\log_2 |L|} > c_8.$$

If $m_1 \geq 25$ and $m_1 \neq 32$, then $|P_1| \leq 2^{0.76m_1}$ by Lemma 2.6. Then $s(L) \geq 14^{m_1}/2^{0.76m_1}$ and

$$\star(L) \geq \frac{\log_2(14/2^{0.76})}{\log_2(95040 \cdot 2^{0.76})} - \frac{\frac{23}{60}\log_2 24}{25 \cdot \log_2(95040 \cdot 2^{0.76})} > 0.172 > c_8.$$

If $m_1 = 32$, then $|P_1| \leq 319979520$ and so $s(L) \geq 14^{32}/319979520$. Thus we see that $\star(L) > 0.164 > c_8$.

For $5 \leq m_1 \leq 24$, bounds for $|P_1|$ are obtained from [10] and $\star(L) > c_8$ is verified by direct computation.

For the remaining computations, we use Lemma 2.4 to get a better bound:

$$s(L) \geq \binom{s(M_{12}) - 1 + m_1}{s(M_{12}) - 1} = \binom{s(M_{12}) - 1 + m_1}{13}.$$

If $m_1 = 4$, then $|P_1| \leq 24$ and $s(L) \geq \binom{17}{13} = 2380$, so $\star(L) > 0.133 > c_8$. If $m_1 = 3$, then $|P_1| \leq 6$ and $s(L) \geq \binom{16}{13} = 560$, so $\star(L) > 0.141 > c_8$. If $m_1 = 2$, then $|P_1| \leq 2$ and $s(L) \geq \binom{15}{13} = 105$, so $\star(L) > 0.145 > c_8$.

Therefore, $\deg(P_1) = 12$ and $P_1 \cong M_{12}$. □

THEOREM 2.10. *Let H be a transitive permutation group where H does not contain any alternating group A_l, $l > 4$, as a composition factor. Let $H \cong M_{12} \wr M_{12} \wr \underbrace{S_4 \wr \cdots \wr S_4}_{t\ terms}$, where $t \geq 0$. Then $ds(H \wr S_4 \wr S_4 \wr S_4 \wr S_4 \wr S_4) \leq ds(H \wr P_1 \wr \cdots \wr P_j)$ where each P_i is a primitive group if* $\deg(P_1) \neq 4$.

PROOF. As calculated before, $s(M_{12} \wr M_{12}) = 604576714$ and $|M_{12} \wr M_{12}| = 95040^{13}$. If K is an arbitrary group, $|K \wr S_4| = |K|^4 \cdot 24$, and so one can easily verify by induction that $|H| = 95040^{13\cdot 4^t} \cdot 24^{b_t} = 95040^{13\cdot 4^t} \cdot 24^{(4^t-1)/3}$.

Next we need some bounds on $s(H \wr S_4 \wr S_4 \wr S_4 \wr S_4 \wr S_4)$ and $s(H \wr P_1 \wr \cdots \wr P_j)$. We handle the first group by defining the sequence $A_0 = s(H)$, $A_1 = s(H \wr S_4)$, $A_2 = s(H \wr S_4 \wr S_4)$, $A_3 = s(H \wr S_4 \wr S_4 \wr S_4)$, $A_4 = s(H \wr S_4 \wr S_4 \wr S_4 \wr S_4)$ and $A_5 = s(H \wr S_4 \wr S_4 \wr S_4 \wr S_4 \wr S_4)$. By Lemma 2.4(2), $A_{i+1} = \binom{A_i+3}{4}$ for $0 \leq i \leq 4$. Hence $A_{i+1} = (A_i + 3)(A_i + 2)(A_i + 1)(A_i)/24$ and $A_{i+1} + 3 \leq (A_i + 3)^4/24$ since $A_0 \geq a_0 = 604576714$. For simplicity, we set $A_0 = A$. Then

$$A_5 = \binom{A_4 + 3}{4} \leq \frac{(A_4 + 3)^4}{24} \leq \frac{((A_3 + 3)^4/24)^4}{24} = \frac{(A_3 + 3)^{16}}{24^5} \leq \cdots \leq \frac{(A + 3)^{1024}}{24^{341}}.$$

Also $|H \wr S_4 \wr S_4 \wr S_4 \wr S_4 \wr S_4| = |H|^{1024} \cdot 24^{341} > |H|^{1024} \cdot 2.88^{1024}$. Consequently,

$$\begin{aligned} ds(H \wr S_4 \wr S_4 \wr S_4 \wr S_4 \wr S_4) &\leq \frac{\log_2((A + 3)^{1024}/24^{341})}{\log_2 |H|^{1024} \cdot 24^{341}} \\ &\leq \frac{\log_2(A + 3)^{1024} - \log_2 24^{341}}{\log_2 |H|^{1024} \cdot 2.88^{1024}} \\ &= \frac{\log_2(A + 3)}{\log_2 2.88|H|} - \frac{341 \cdot \log_2 24}{1024 \log_2 2.88|H|}. \end{aligned}$$

Next we obtain a similar bound for $s(H \wr P_1 \wr \cdots \wr P_j)$. Consider $s(H \wr P_1)$ where $\deg(P_1) = m_1 \neq 4$. Note that $s(H \wr P_1) \geq A^{m_1}/|P_1|$ by Lemma 2.4(1). From the proof of Lemma 2.5,

$$ds(H \wr P_1 \wr \cdots \wr P_j) \geq \frac{\log_2 A^{m_1}/|P_1|}{\log_2 |H|^{m_1} \cdot |P_1|} - \frac{\frac{23}{60} \log_2 24}{\log_2 |H|^{m_1} \cdot |P_1|}.$$

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Since $|P_1| < 3^{m_1}$ by Lemma 2.6, we have

$$ds(H \wr P_1 \wr \cdots \wr P_j) > \frac{\log_2 A}{\log_2 3|H|} - \frac{\log_2 |P_1|}{m_1 \log_2 3|H|} - \frac{\frac{23}{60}\log_2 24}{m_1 \log_2 3|H|}$$
$$> \frac{\log_2 A}{\log_2 3|H|} - \frac{\log_2 |P_1|}{m_1 \log_2 2|H|} - \frac{\frac{23}{60}\log_2 24}{m_1 \log_2 2|H|}.$$

It suffices to show that

$$\frac{\log_2 A}{\log_2 3|H|} - \frac{\log_2 |P_1|}{m_1 \log_2 2|H|} - \frac{\frac{23}{60}\log_2 24}{m_1 \log_2 2|H|} \geq \frac{\log_2(A+3)}{\log_2 2|H|} - \frac{341 \cdot \log_2 24}{1024 \log_2 2|H|}.$$

By rearranging these terms, we can write this inequality as

$$\frac{\log_2 2|H|}{\log_2 3|H|} \cdot \log_2 A - \log_2(A+3) \geq \frac{\log_2 |P_1|}{m_1} + \frac{23 \cdot \log_2 24}{60 m_1} - \frac{341 \cdot \log_2 24}{1024}.$$

Note that $|H| \geq 95040^{13}$, so $\log_2 2|H|/\log_2 3|H| \geq 0.99729879$. Further, we have $A \geq 604576714$, so

$$\frac{\log_2 2|H|}{\log_2 3|H|} \cdot \log_2 A - \log_2(A+3) + \frac{341 \cdot \log_2 24}{1024}$$
$$\geq 0.99729879 \cdot \log_2 A - \log_2(A+3) + \frac{341 \cdot \log_2 24}{1024} \geq 1.4480303828.$$

Thus it suffices to show that

$$\star(P_1) = \frac{\log_2 |P_1|}{m_1} + \frac{23 \log_2 24}{60 m_1} \leq 1.4480303828 = \gamma.$$

If $m_1 \geq 25$, then $|P_1| \leq 2^{m_1}$ by Lemma 2.6 and $\star(P_1) \leq 1.08 < \gamma$. Similar computations hold for $2 \leq m_1 \leq 24$ by reading off the bounds for $|P_1|$ from Table 2 of [10], and one obtains $\star(P_1) < \gamma$ in all cases except when $P_1 \cong M_{12}$ and $P_1 \cong \mathrm{ASL}(3,2)$, a primitive group of degree 8. We now take care of these two cases.

Suppose $P_1 \cong M_{12}$. Since $|M_{12}| = 95040 < 2.6^{12}$,

$$ds(H \wr P_1 \wr \cdots \wr P_j) \geq \frac{\log_2(A^{12}) - \log_2 95040 - \frac{23}{60}\log_2 24}{\log_2(95040|H|^{12})}$$
$$> \frac{12 \cdot \log_2(A) - \log_2 95040 - \frac{23}{60}\log_2 24}{12 \cdot \log_2(2.6|H|)}$$
$$= \frac{\log_2(A)}{\log_2(2.6|H|)} - \frac{\log_2 95040}{12 \cdot \log_2(2.6|H|)} - \frac{\frac{23}{60}\log_2 24}{12 \cdot \log_2(2.6|H|)}.$$

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Given that $A \geq 604576714$, it suffices to show that,

$$\frac{\log_2(A)}{\log_2(2.6|H|)} - \frac{\log_2 95040}{12 \cdot \log_2(2.6|H|)} - \frac{\frac{23}{60}\log_2 24}{12 \cdot \log_2(2.6|H|)} \geq \frac{\log_2(A+3)}{\log_2(2.88|H|)} - \frac{341 \cdot \log_2 24}{1024 \cdot \log_2(2.88|H|)}.$$

By rearranging these terms, we can write this inequality as

$$\frac{\log_2(2.88|H|) \cdot \log_2(A)}{\log_2(2.6|H|)} - \log_2(A+3) + \frac{341 \cdot \log_2 24}{1024} \geq \frac{\log_2(2.88|H|)}{\log_2(2.6|H|)}\Big(\frac{\log_2 95040}{12} + \frac{\frac{23}{60}\log_2 24}{12}\Big).$$

Here $|H| = 95040^{13 \cdot 4^t} \cdot 24^{(4^t-1)/3}$, where t is the number of terms of S_4 in H. Define

$$h_1(t) = \frac{\log_2(2.88|H|)}{\log_2(2.6|H|)} = \frac{\log_2 2.88 + 13 \cdot 4^t \cdot \log_2 95040 + ((4^t-1)/3)\log_2 24}{\log_2 2.6 + 13 \cdot 4^t \cdot \log_2 95040 + ((4^t-1)/3)\log_2 24},$$

$$a_1(x,t) = h_1(t) \cdot \log_2 x - \log_2(x+3) + \frac{341 \cdot \log_2 24}{1024},$$

$$s_1(t) = h_1(t) \cdot \Big(\frac{\log_2 95040 + \frac{23}{60}\log_2 24}{12}\Big).$$

It suffices to show that the inequality above, now translated as $a_1(x,t) > s_1(t)$, holds for all $x \geq A = 604576714$ and $t \geq 0$.

Clearly $h_1(t)$ is a decreasing function and $h_1(t) \to 1$ as $t \to \infty$. The function $a_1(x,t)$ is increasing for a fixed value of $t \geq 0$, and achieves a minimum when $t \to \infty$ and $x = 604576714$. Thus, $a_1(x,t) \geq 1.5268$. Since $h_1(t)$ is decreasing, $s_1(t)$ is decreasing. So $s_1(t)$ achieves its maximum value for $t = 1$ and $s_1(1) < 1.5248$. Thus, $a_1(x,t) > s_1(t)$ for all x and t.

Finally, consider $P_1 \cong \mathrm{ASL}(3,2)$, where $|\mathrm{ASL}(3,2)| = 1344 < 2.47^8$. Thus,

$$\begin{aligned} ds(H \wr P_1 \wr \cdots \wr P_j) &\geq \frac{\log_2(A^8) - \log_2 1344 - \frac{23}{60}\log_2 24}{\log_2(|H|^8 \cdot 1344)} \\ &> \frac{8 \cdot \log_2(A) - \log_2 1344 - \frac{23}{60}\log_2 24}{8 \cdot \log_2(2.47 \cdot |H|)} \\ &= \frac{\log_2 A}{\log_2(2.47|H|)} - \frac{\log_2 1344}{8 \cdot \log_2(2.47|H|)} - \frac{\frac{23}{60}\log_2 24}{8 \cdot \log_2(2.47|H|)}. \end{aligned}$$

It suffices to show that

$$\frac{\log_2 A}{\log_2(2.47|H|)} - \frac{\log_2 1344}{8 \cdot \log_2(2.47|H|)} - \frac{\frac{23}{60}\log_2 24}{8 \cdot \log_2(2.47|H|)} \geq \frac{\log_2(A+3)}{\log_2(2.88|H|)} - \frac{341 \cdot \log_2 24}{1024 \cdot \log_2(2.88|H|)}.$$

By rearranging the terms, we can write this inequality as

$$\frac{\log_2(2.88|H|)\cdot\log_2 A}{\log_2(2.47|H|)} - \log_2(A+3) + \frac{341\cdot\log_2 24}{1024} \geq \frac{\log_2(2.88|H|)}{\log_2(2.47|H|)}\Big(\frac{\log_2(1344)+\frac{23}{60}\log_2 24}{8}\Big).$$

As before, $|H| = 95040^{13\cdot 4^t}\cdot 24^{(4^t-1)/3}$, where t is the number of terms of S_4 in H. Define

$$h_2(t) = \frac{\log_2 2.88 + 13\cdot 4^t\cdot\log_2 95040 + ((4^t-1)/3)\log_2 24}{\log_2 2.47 + 13\cdot 4^t\cdot\log_2 95040 + ((4^t-1)/3)\log_2 24},$$

$$a_2(x,t) = h_2(t)\cdot\log_2 x - \log_2(x+3) + \frac{341\cdot\log_2 24}{1024},$$

$$s_2(t) = h_2(t)\cdot\Big(\frac{\log_2(1344)+\frac{23}{60}\log_2 24}{8}\Big).$$

Again, $h_2(t)$ is a decreasing function with $h_2(t)\to 1$ as $t\to\infty$ and $a_2(x,t)$ is increasing for a fixed value of t. We still have $a_2(x,t)\geq 1.526$ and $s_2(t)$ achieves its maximum value for $t=1$ and $s_2(1) < 1.520$. Thus, $a_2(x,t) > s_2(t)$ for all x and t. □

THEOREM 2.11. *We have*

$$\inf\Big(\frac{\log_2 s(G)}{\log_2|G|}\Big) = \lim_{k\to\infty}\frac{\log_2 s(M_{12}\wr M_{12}\wr \overbrace{S_4\wr\cdots\wr S_4}^{k\ terms})}{\log_2|M_{12}\wr M_{12}\wr \underbrace{S_4\wr\cdots\wr S_4}_{k\ terms}|} = \lim_{k\to\infty} c_k,$$

where the infimum is taken over all permutation groups G not containing any A_l, $l>4$, as a composition factor.

PROOF. Let G be a permutation group. Let $M=\lim_{k\to\infty}c_k$ so that $M<c_8$. By our earlier remarks, G is transitive. By Proposition 2.7, if G is primitive, $ds(G) > c_8 > M$. So we assume that G is imprimitive.

By Theorem 2.8, G is induced from M_{12} and, by Theorem 2.9, G is induced from $M_{12}\wr M_{12}$. By Theorem 2.10, $\inf(\log_2 s(G)/\log_2|G|) = \lim_{k\to\infty}c_k$. □

REMARK 2.12. Again we set $M=\lim_{k\to\infty}c_k$. By Lemma 2.5,

$$M > c_8 - \frac{\frac{23}{60}\log_2(24)}{(13\cdot 4^8)\log_2(95040) + ((4^8-1)/3)\log_2(24)} \approx 0.1276817008.$$

Since c_k is a strictly decreasing sequence with $c_8 < 0.1276818247$, this gives

$$0.1276817008 < M < 0.1276818247.$$

This estimate will give an explicit bound for [1, Corollary 1] when $t=4$. Since [1, Corollary 1] is an important ingredient in Pyber's proof of a lower bound for the

https://doi.org/10.1017/S0004972721001064 Published online by Cambridge University Press

number of conjugacy classes $k(G)$ of a finite group G in terms of its order [6] (which has undetermined constants in it), it is likely that our result will help to find an explicit scalar constant in this result. We note that Pyber's result has been improved in [2, 4]. We also note that [4, Theorem 3.1] now has an (albeit extremely small) bound thanks to the main result of [3] (which is generalised by our result here). In particular, the currently best lower bound for $k(G)$ in terms of $|G|$ for solvable groups, as stated in [4, Corollary 3.2], is explicit.

Acknowledgements

This research was conducted by Gintz, Kortje and Wang during the summer of 2021 under the supervision of Keller and Yang. The authors thank Texas State University for providing a great working environment and support. The authors are grateful to the referee for the valuable suggestions which greatly improved the manuscript.

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