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A Note on the Diophantine Equation $x^2 + y^6 = z^e$, $e \geq 4$

Konstantine Zelator

Abstract. We consider the diophantine equation $x^2 + y^6 = z^e$, $e \geq 4$. We show that, when e is a multiple of 4 or 6, this equation has no solutions in positive integers with x and y relatively prime. As a corollary, we show that there exists no primitive Pythagorean triangle one of whose leglengths is a perfect cube, while the hypotenuse length is an integer square.

1 Introduction

At an instructional conference on diophantine equations held at Leiden University in 2007 [1], some open questions were posed. One of them was the diophantine equation,

$$x^2 + y^6 = z^e, \quad e \geq 4. \tag{1.1}$$

This equation is the subject matter of this paper. We offer a purely elementary approach and an elementary proof of the the two main results of this work (Theorems 4.1 and 4.2).

In Theorem 4.1, we prove that if e is a multiple of 4, then the above diophantine equation has no solutions in positive integers x, y, z with $(x, y) = 1$.

Theorem 4.2 states that if e is a multiple of 6, then equation (1.1) has no such solutions either. To establish the two theorems, we make use of three well-known results in the literature of diophantine analysis. We also use Lemma 3.1 and Proposition 3.2, proved in Section 3.

At the end of the paper, we state two corollaries involving Pythagorean triangles (Corollaries 5.3 and 5.4).

2 Some Results from Number Theory

First we state the well-known parametric formulas that describe the entire family of Pythagorean triples.

Lemma 2.1 *All positive integer solutions (up to symmetry with respect to x and y) of the diophantine equation $x^2 + y^2 = z^2$ are given by,*

$$x = d(m^2 - n^2), \quad y = d(2mn), \quad z = d(m^2 + n^2), \tag{2.1}$$

with $m, n, d \in \mathbb{Z}^+$, $m > n$, $(m, n) = 1$, and $m + n \equiv 1 (\mathrm{mod}\, 2)$. When $d = 1$, all the primitive triples are obtained.

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The following result was first proved by Pocklington in 1914 [3]. For a quick reference and proof see [2].

Theorem 2.2 *All the positive integer solutions of the diophantine equation $x^4 - x^2y^2 + y^4 = z^2$, are given by $x = y = t$, $z = t^2$, t a positive integer.*
In particular, $x = y = z = 1$ is the only solution with $(x, y) = 1$.

Theorem 2.3 was established by Adrain [4, p. 636] in 1906.

Theorem 2.3 *The diophantine equation $x^4 + x^2y^2 + y^4 = z^2$ has no solutions in positive integers x, y, z.*

Theorem 2.4 can be found in [2].

Theorem 2.4 *All the integer solutions of the diophantine equation $2z^3 = x^3 + y^3$ with $z \neq 0$, are given by $x = y = z = t$, $t \neq 0$, $t \in \mathbb{Z}$, and the solutions with $z = 0$, are given by $z = 0$, $x = t$, $y = -t$, $t \in \mathbb{Z}$.*

3 A Lemma and a Proposition

Using the identities

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2) = (a \pm b)\left[(a \pm b)^2 \mp 3ab\right],$$

one establishes the following lemma.

Lemma 3.1 *Suppose that a and b are relatively prime integers, $(a, b) = 1$. Then $(a \pm b,\ a^2 \mp ab + b^2) = 1$ or 3, with 1 occurring when $a \pm b \not\equiv 0 (\operatorname{mod} 3)$, while the above greatest common divisor is equal to 3 in the case when $a \pm b \equiv 0 (\operatorname{mod} 3)$.*

Proposition 3.2 *All the positive integer solutions of the diophantine equation*

$$x^6 + y^6 = 2z^2 \tag{3.1}$$

are given by $x = y = t$, $z = t^3$, $t \in \mathbb{Z}^+$. In particular, the only such solution with $(x, y) = 1$ is $x = y = z = 1$.

Proof One direction is trivial, namely that if $(x, y, z) = (t, t, t^3)$, with t a positive integer. Then such a triple is a solution of (3.1). Below we establish the converse: every such solution must be of the above form. First, observe that if x, y, z are positive integers satisfying (3.1), then the highest power of 2 dividing x must be the same as the highest power of 2 dividing y. This is easy to see by virtue of the fact that the highest power of 2 dividing the right-hand side of (3.1) must be of the form 2^v, v being an odd integer. If x and y were exactly divisible by different powers of 2, then the left-hand side of (3.1) would be exactly divisible by 2^w, w being an even integer. So we would have a contradiction. Therefore, based on this observation we can put

$$\{x = \delta \cdot 2^r \cdot x_1,\ y = \delta \cdot 2^r \cdot y_1\}, \tag{3.2}$$

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where δ is an odd positive integer and x_1, y_1 are relatively prime odd positive integers, $(x_1, y_1) = 1$, and r a nonnegative integer. Accordingly, since $x_1^6 + y_1^6 \equiv 2 \pmod 4$, (3.1) implies $z = \delta^3 \cdot 2^{3r} \cdot z_1$, z_1 an odd positive integer.

Combining (3.1) and (3.2) we obtain,

$$x_1^6 + y_1^6 = 2z_1^6, \quad \text{or equivalently,} \quad (x_1^2 + y_1^2)(x_1^4 - x_1^2 y_1^2 + y_1^4) = 2z_1^2 \tag{3.3}$$

Since x_1 and y_1 are relatively prime, it follows by Lemma 3.1 that the two factors on the left-hand side of (3.3) are either co-prime or, their only factor in common is 3. The latter possibility is ruled out, since $(x_1, y_1) = 1$ easily implies that $x_1^2 + y_1^2 \not\equiv 0 \pmod 3$ (More generally, as is well known, a sum of two relatively prime squares cannot be divisible by a prime congruent to 3 modulo 4.) We conclude that the two factors on the left-hand side of (3.3) must be co-prime, and since x_1 and y_1 are both odd, equation (3.3) implies

$$x_1^2 + y_1^2 = 2k_1^2, \quad x_1^4 - x_1^2 y_1^2 + y_1^4 = k_2^2 \tag{3.4}$$

for some relatively prime integers k_1, k_2, with k_2 being odd and with $k_1 k_2 = z_1$.

The second equation in (3.4) combined with Theorem 2.2 implies that $x_1 = y_1 = \rho$, $k_2 = \rho^2$ for some odd positive integer ρ.

Combining this with the first equation in (3.4) and the formulas in (3.2) and $k_1 k_2 = z_1$ leads to $x = y = t$, $z = t^3$, where $t = \delta \cdot 2^r \cdot \rho$. The proof is complete. ∎

4 The Two Theorems

Theorem 4.1 *Let e be a positive integer that is a multiple of* 4*. Then the diophantine equation*

$$x^2 + y^6 = z^e \tag{4.1}$$

has no solutions in positive integers x, y, z such that $(x, y) = 1$.

Proof Suppose that x, y, z are positive integers satisfying equation (4.1), with x and y being relatively prime. Then x, y, z are pairwise relatively prime, and (x, y^3, z^{2k}) is a primitive Pythagorean triple, where $k = \frac{e}{4}$ and $e = 4k$. We apply the formulas in (2.1). Throughout the two cases in the proof, m and n will be assumed to be relatively prime integers, with one even while the other is odd, and with $m > n$.

Case 1: x odd, y even

We have

$$\left\{ x = m^2 - n^2, \quad y^3 = 2mn, \quad z^{2k} = m^2 + n^2 \right\}. \tag{4.2}$$

In the arguments to follow, only y and z are involved. Consequently, since the second and third equations in (4.2) are symmetric with respect to m and n, there is no loss of generality in assuming m to be even and n to be odd. With that in mind, the

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third equation in (4.2) shows that (m, n, z^k) is a primitive Pythagorean triple. Hence, since m is even and n is odd,

$$m = 2MN, \quad n = M^2 - N^2, \quad z^k = M^2 + N^2, \tag{4.3}$$

with positive integers M, N such that $M > N$, $(M, N) = 1$, and $M + N \equiv 1 \pmod 2$.

On the other hand, since m is even, n is odd, and $(m, n) = 1$, the second equation in (4.2) implies

$$m = 4a^3, \quad n = b^3, \tag{4.4}$$

for some relatively prime integers a and b, with b being odd.

Combining the first equations in (4.3) and (4.4) yields

$$4a^3 = 2MN, \quad MN = 2a^3 \tag{4.5}$$

Since M and N are relatively prime and have different parities, equation (4.5) implies either

$$M = 2c^3 \quad \text{and} \quad N = d^3 \tag{4.6}$$

or alternatively,

$$M = d^3 \quad \text{and} \quad N = 2c^3 \tag{4.7}$$

for positive integers c, d such that $(c, d) = 1$ and with d odd.

Combining (4.6) and (4.7) with the second equations in (4.4) and (4.3), we see that either $b^3 = 4c^6 - d^6$ or alternatively $b^3 = d^6 - 4c^6$.

Equivalently, we must have either

$$b^3 = (2c^3 - d^3)(2c^3 + d^3) \tag{4.8}$$

or

$$b^3 = (d^3 - 2c^3)(d^3 + 2c^3). \tag{4.9}$$

Due to the fact that $(c, d) = 1$ and d being odd, the two factors on the right-hand sides of equations (4.8) and (4.9) must be relatively prime. Thus, if (4.8) holds, then each of the two factors on the right-hand side of (4.8) must be an integer cube. In particular, $2c^3 + d^3 = g^3$ for some integer g. Obviously g must be positive, since c and d are. Moreover, $2c^3 = g^3 + (-d)^3$, which shows that the triple $(g, -d, c)$ is an integer solution of the diophantine equation $2Z^3 = X^3 + Y^3$.

By Result 2.4, and since $c \neq 0$, we must have $c = t$, $g = t$, $-d = t$; $d = -t$ for some nonzero integer t. But, this is impossible by virtue of the fact that c and d are both positive. The argument is identical if (4.9) holds.

Case 2: x even and y odd

In this case, we have (from equation (4.1))

$$\left\{ x = 2mn,\ y^3 = m^2 - n^2,\ z^{2k} = m^2 + n^2 \right\}. \tag{4.10}$$

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Because $(m,n) = 1$ and $m + n \equiv 1 \pmod 2$, we have $(m - n,\ m + n) = 1$, which, when combined with $y^3 = (m - n)(m + n)$, implies

$$\{m - n = \lambda_1^3,\ m + n = \lambda_2^3\} \tag{4.11}$$

for some odd relatively prime integers λ_1 and λ_2.

From (4.11) we obtain

$$\left\{m = \frac{\lambda_1^3 + \lambda_2^3}{2},\ n = \frac{\lambda_2^3 - \lambda_1^3}{2}\right\}. \tag{4.12}$$

From (4.12) and the third equation in (4.10) we obtain

$$2z^{2k} = \lambda_1^6 + \lambda_2^6. \tag{4.13}$$

According to (4.13), the triple $(\lambda_1, \lambda_2, z^k)$ is a positive integer solution to the diophantine equation $2Z^2 = X^2 + Y^6$, with $(\lambda_1, \lambda_2) = 1$. Hence, by Proposition 3.2, it follows that $\lambda_1 = \lambda_2 = z^k = 1$, which is a contradiction since, for example, n is a positive integer (and so nonzero). ■

Theorem 4.2 *Let e be a positive integer that is a multiple of 6. Then the diophantine equation*

$$x^2 + y^6 = z^e \tag{4.14}$$

has no solutions in positive integers x, y, z such that $(x, y) = 1$.

Proof We set $e = 6k$, k a positive integer and $x, y, z \in \mathbb{Z}^+$, satisfying (4.14) and with $(x, y) = 1$. Therefore, $(x, y) = 1 = (x, z) = (z, y)$. Moreover, since e is even, then according to (4.14), z must be odd, while x and y must have different parities. (This is clear when (4.14) is considered modulo 4.) Equation (4.14) is equivalent to

$$x^2 = \left(z^{2k} - y^2\right)\left(z^{4k} + z^{2k}y^2 + y^4\right). \tag{4.15}$$

Since z and y are relatively prime, by Lemma 3.1 the two factors on the right-hand side of (4.15) must be either co-prime or their greatest common divisor must equal 3. If they are relatively prime, then (4.15) implies that each of them must be a perfect square. In particular,

$$(z^k)^4 + (z^k)^2y^2 + y^4 = u^2$$

for some positive integer u, which is contrary to Theorem 2.3. Next, suppose that the greatest common divisor of the two factors on the right-hand side of (4.15) is 3. Equation (4.15) gives

$$\{z^{2k} - y^2 = 3x_1^2,\ \ z^{4k} + z^{2k}y^2 + y^4 = 3x_2^2\} \tag{4.16}$$

for some relatively prime positive integers x_1 and x_2, with $3x_1x_2 = x$. By combining the two equations in (4.16), a straightforward calculation establishes that

$$y^4 + 3x_1^2y^2 + 3x_1^4 = x_2^2. \tag{4.17}$$

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If x is odd and y is even, then both x_1, x_2 are odd. But in this case, (4.17) is rendered impossible modulo 4, since the left-hand side would be congruent to 3, while the right-hand side would be congruent to 1 modulo 4.

If x is even and y is odd, we go back to (4.14) with $e = 6k$. Then (x, y^3, z^{3k}) is a primitive Pythagorean triple. We must have

$$\left\{x = 2mn,\ \ y^3 = (m-n)(m+n),\ \ z^{3k} = m^2 + n^2\right\}. \tag{4.18}$$

Accordingly, $m - n = v_1^3$, $m + n = v_2^3$, for positive, relatively prime, odd integers v_1 and v_2. In combination with the third equation in (4.18), this then yields $2z^{3k} = v_1^6 + v_2^6$. Thus, by Theorem 2.4 it follows that $v_1^2 = v_2^2 = t = z^k$ for some $t \in \mathbb{Z}$. Since v_1 and v_2 are positive and co-prime, the only choice for t is $t = 1$, which a contradiction, since this implies $n = 0$. ■

5 Corollaries

Corollary 5.1 *The diophantine equation $x^2 + y^6 = z^6$ has no solutions in positive integers x, y, z.*

Proof Suppose to the contrary that x, y, z are positive integers satisfying the above equation. If $D = (y, z)$, then D^6 is a divisor of x^2, and so $D^3 \mid x$. We have $x = D^3 x_1$, $y = Dy_1$, $z = Dz_1$, where x_1, y_1, z_1 are positive integers such that $x_1^2 + y_1^6 = z_1^6$ and with $(y_1, z_1) = 1$; and thus also with $(x_1, y_1) = 1$. Clearly, $x_1^2 + y_1^6 = z_1^6$ and $(x_1, y_1) = 1$ contradict Theorem 4.2 (with $e = 6$). ■

Two other corollaries follow at once.

Corollary 5.2 *Let e be a multiple of 6. Then the diophantine equation $x^2 + y^6 = z^e$ has no solutions in positive integers x, y, z.*

Corollary 5.3 *There exists no Pythagorean triangle one of whose leg lengths is a perfect cube, while the hypotenuse length is also an integer cube.*

Finally, as a result of Theorem 4.1, we have the following corollary.

Corollary 5.4 *There exists no primitive Pythagorean triangle one of whose leg lengths is a perfect cube, while the hypotenuse length is an integer square.*

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Department of Mathematics and Computer Science, Rhode Island College, Providence, RI 02908, USA
e-mail: KZet159@pitt.edu