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Abstract

We classify log-canonical pairs (X, Δ) of dimension two such that $K_X + \Delta$ is an ample Cartier divisor with $(K_X + \Delta)^2 = 1$, giving some applications to stable surfaces with $K^2 = 1$. A rough classification is also given in the case where $\Delta = 0$.

1. Introduction

The study of stable curves and, more generally, stable pointed curves is by now a classical subject. Stable surfaces were introduced by Kollár and Shepherd-Barron in [KSB88], and it was subsequently realized (see, for instance, [Ale06, Kol12, Kol14] and references therein) that this definition can be extended to higher-dimensional varieties and pairs. So the study of (semi-)log-canonical pairs became an important topic in the theory of singular higher-dimensional varieties.

Here we consider two-dimensional log-canonical pairs in which the log-canonical divisor is Cartier and has self-intersection equal to 1, and we give some applications to Gorenstein stable surfaces.

First we study the case with non-empty boundary.

Theorem 1.1. *Let (X, Δ) be a log-canonical pair of dimension two with $\Delta > 0$, $K_X + \Delta$ Cartier and ample, and $(K_X + \Delta)^2 = 1$.*

Then (X, Δ) belongs to one of the types (P), (dP), (E_+) or (E_-) described in List 2.2.

In particular, Theorem 1.1 implies that X is either the projective plane, a del Pezzo surface of degree 1, the symmetric product S^2E of an elliptic curve, or a projective bundle $\mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(x))$ over an elliptic curve with the section of square -1 contracted. It came as rather a surprise to us that the list is so short and that in each case the underlying surface is itself Gorenstein.

The case where $\Delta = 0$ cannot be described so precisely, since it includes, for instance, all smooth surfaces of general type with $K^2 = 1$; however, in § 4 we give a rough classification, according to the Kodaira dimension of a smooth model of X (see Theorem 4.1). Using this result, one can show that all the possible Kodaira dimensions occur in the normal case (cf. [FPR15b] and [FPR15a]), thus answering a question posed by Kollár during his lecture at the conference on 'Compact moduli and vector bundles' held at the University of Georgia in October 2010.

Although log-canonical pairs are interesting in their own right, our main motivation for proving the above results is that, by a result of Kollár, a non-normal Gorenstein stable surface gives rise to a pair as in Theorem 1.1 via normalization (see Corollary 3.4). In § 3, we explain how the above pairs can be used to construct stable surfaces and which pairs can occur as

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normalizations of stable surfaces for given invariants K^2 and χ. In particular, we show that $\chi(X) \geqslant 0$ for a Gorenstein stable surface X with $K_X^2 = 1$, improving upon results in [LR13].

We will study the geometry and moduli of Gorenstein stable surfaces with $K^2 = 1$ in more detail in a subsequent paper, building on the classification results proven here.

Notation and conventions

We work over the complex numbers; all varieties are assumed to be projective and irreducible unless otherwise stated. A canonical divisor is a Weil divisor K_X which is Cartier outside codimension two and such that the associated divisorial sheaf $\mathcal{O}_X(K_X)$ is isomorphic to the dualizing sheaf ω_X; this makes sense since all schemes we consider are demi-normal; see [Kol13, § 5.6]. We do not distinguish between Cartier divisors and invertible sheaves in our notation. For a variety X, we denote by $\chi(X)$ the holomorphic Euler characteristic.

2. Classification of pairs

Let (X, Δ) be a log-canonical (lc) pair of dimension two (cf. [KM98, Definition 2.34] for the definition).

Definition 2.1. We say that (X, Δ) is stable if $K_X + \Delta$ is ample, and Gorenstein if $K_X + \Delta$ is Cartier.

The aim of this section is the classification of Gorenstein stable lc pairs with $(K_X + \Delta)^2 = 1$ and $\Delta > 0$. We start by listing and describing quickly the cases that occur in our classification.

List 2.2.

- (P) $X = \mathbb{P}^2$ and Δ is a nodal quartic. Here $p_a(\Delta) = 3$ and $K_X + \Delta = \mathcal{O}_{\mathbb{P}^2}(1)$.
- (dP) X is a (possibly singular) del Pezzo surface of degree 1, i.e. X has at most canonical singularities, $-K_X$ is ample and $K_X^2 = 1$. The curve Δ belongs to the system $|-2K_X|$; hence $K_X + \Delta = -K_X$ and $p_a(\Delta) = 2$.
- (E_-) Let E be an elliptic curve and let $a : \widetilde{X} \to E$ be a geometrically ruled surface that contains an irreducible section C_0 with $C_0^2 = -1$; that is, $\widetilde{X} = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}_E(-x))$ where $x \in E$ is a point and C_0 is the only curve in the system $|\mathcal{O}_X(1)|$. Set $F = a^{-1}(x)$; then the normal surface X is obtained from $\widetilde{X}$ by contracting C_0 to an elliptic Gorenstein singularity of degree 1 and Δ is the image of a curve $\Delta_0 \in |2(C_0 + F)|$ disjoint from C_0, so $p_a(\Delta) = 2$. The line bundle $K_X + \Delta$ pulls back to $C_0 + F$ on $\widetilde{X}$.
- (E_+) $X = S^2E$, where E is an elliptic curve. Let $a : X \to E$ be the Albanese map, which is induced by the addition map $E \times E \to E$. Denote by F the class of a fiber of a and by C_0 the image in X of the curve $\{0\} \times E + E \times \{0\}$, where $0 \in E$ is the origin, so that $C_0F = C_0^2 = 1$. Then Δ is a divisor numerically equivalent to $3C_0 - F$, $p_a(\Delta) = 2$, and $K_X + \Delta$ is numerically equivalent to C_0.
  An equivalent description of X is as follows (cf. [CC93, § 1]). Denote by $\mathcal{E}$ the only indecomposable extension of the form $0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E(0) \to 0$, and set $X = \mathbb{P}(\mathcal{E})$; then C_0 is the only effective divisor in $|\mathcal{O}_X(1)|$.

For completeness, we give in Table 1 the numerical invariants of the four possible cases.

The rest of this section is devoted to proving Theorem 1.1. We start with some general remarks.

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Table 1. Invariants of (X, Δ).

Case	$\chi(X)$	$q(X)$	$p_a(\Delta)$	$h^0(K_X + \Delta)$
(P)	1	0	3	3
(dP)	1	0	2	2
(E_-)	1	0	2	2
(E_+)	0	1	2	1

Lemma 2.3. *Let X be a normal surface and let L be an ample line bundle of X such that $L^2 = 1$. Then:*

(i) *every curve $C \in |L|$ is irreducible and $h^0(L) \leqslant 3$;*
(ii) *$h^0(L) = 3$ if and only if $X = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$;*
(iii) *if $h^0(L) = 2$, then the system $|L|$ has one simple base point P that is smooth for X.*

Proof. For (i) and (ii), we have $LC = 1$, and hence C is irreducible since L is ample. Denote by $\nu : \widetilde{C} \to C$ the normalization: since $\deg L|_C = 1$, one has $h^0(\nu^* L) \leqslant 2$, with equality holding if and only if $\widetilde{C}$ is a smooth rational curve. Since $h^0(L|_C) \leqslant h^0(\nu^* L)$, the usual restriction sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) = L \to L|_C \to 0$$

gives $h^0(L) \leqslant 3$. Moreover, if $h^0(L) = 3$, then $h^0(L|_C) = h^0(\nu^* L) = 2$, C is a smooth rational curve, and the system $|L|$ is base-point-free. The morphism $X \to \mathbb{P}^2$ defined by $|L|$ has degree 1 and is finite, since L is ample. Because X and $\mathbb{P}^2$ are normal, it is an isomorphism.

Assertion (iii) follows from (i) and the fact that $L^2 = 1$. □

Lemma 2.4. *Let Y be a smooth surface, let $D > 0$ be a nef and big divisor of Y, and let D_{red} be the underlying reduced divisor. Then:*

(i) *$p_a(D_{\text{red}}) \leqslant p_a(D)$;*
(ii) *the natural map $\operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(D_{\text{red}})$ is injective.*

Proof. (i) One has $h^1(K_Y + D) = 0$ by Kawamata–Viehweg's vanishing; thus, upon taking cohomology in the usual restriction sequence $0 \to K_Y \to K_Y + D \to K_D \to 0$, one obtains

$$p_a(D) = \chi(K_D) + 1 = \chi(K_Y + D) - \chi(K_Y) + 1 = h^0(K_Y + D) - \chi(K_Y) + 1.$$

Applying the same argument to D_{red}, one obtains instead the inequality

$$p_a(D_{\text{red}}) \leqslant h^0(K_Y + D_{\text{red}}) - \chi(K_Y) + 1,$$

since $h^2(K_Y + D_{\text{red}}) = h^0(-D_{\text{red}}) = 0$. Then the claim follows since $h^0(K_Y + D_{\text{red}}) \leqslant h^0(K_Y + D)$.

(ii) This is a slight generalization of [CFML97, Proposition 1.6] and can be proved by exactly the same argument. □

We now fix the notation and assumptions that will be used in the rest of the section: (X, Δ) is a lc pair satisfying the assumptions of Theorem 1.1 and $\varepsilon : \widetilde{X} \to X$ is the minimal desingularization. We set $L := K_X + \Delta$ and $\widetilde{L} := \varepsilon^* L$; $\widetilde{L}$ is a nef and big divisor with $\widetilde{L}^2 = 1$ and $h^0(L) = h^0(\widetilde{L})$. We define the divisor $\widetilde{\Delta}$ by the equality $\widetilde{L} = K_{\widetilde{X}} + \widetilde{\Delta}$ and the requirement that $\varepsilon_* \widetilde{\Delta} = \Delta$.

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Lemma 2.5. *In the above set-up:*

(i) $K_{\widetilde{X}}\widetilde{L} < 0$ *and* $h^2(\widetilde{L}) = 0$;

(ii) $\widetilde{X}$ *is ruled.*

Proof. (i) Using the projection formula, we compute

$$\widetilde{L}\widetilde{\Delta} = \varepsilon^* L(\varepsilon^{-1})_*\Delta = L\Delta = (K_X + \Delta)\Delta,$$

so $\widetilde{L}\widetilde{\Delta}$ is a positive number and is even by adjunction. Thus

$$\widetilde{L}K_{\widetilde{X}} = \widetilde{L}^2 - \widetilde{L}\widetilde{\Delta} = 1 - \widetilde{L}\widetilde{\Delta} < 0.$$

By Serre duality, we have $h^2(\widetilde{L}) = h^0(-\widetilde{\Delta}) = 0$, since $\widetilde{L}\widetilde{\Delta} = L\Delta > 0$ and $\widetilde{L}$ is nef.

(ii) Since $\widetilde{L}$ is nef, the condition $K_{\widetilde{X}}\widetilde{L} < 0$ implies that $\kappa(\widetilde{X}) = -\infty$. □

Next we look at the adjoint divisor $K_{\widetilde{X}} + \widetilde{L}$.

Lemma 2.6. *Assume that* $h^0(\widetilde{L}) \leqslant 2$*; then* $K_{\widetilde{X}}\widetilde{L} = -1$*, and there are the following two possibilities:*

(R) $h^0(K_{\widetilde{X}} + \widetilde{L}) = \chi(\widetilde{X}) = 1$ *and* $h^0(\widetilde{L}) = 2$*;*

(E) $h^0(K_{\widetilde{X}} + \widetilde{L}) = \chi(\widetilde{X}) = 0$ *and* $h^0(\widetilde{L}) = 2$ *or* 1.

Proof. Since $\widetilde{L}$ is nef and big, Riemann–Roch and Kawamata–Viehweg vanishing give

$$h^0(K_{\widetilde{X}} + \widetilde{L}) = \chi(\widetilde{X}) + \frac{\widetilde{L}^2 + K_{\widetilde{X}}\widetilde{L}}{2} = \chi(\widetilde{X}) + \frac{1 + K_{\widetilde{X}}\widetilde{L}}{2} \leqslant \chi(\widetilde{X}), \tag{2.1}$$

where the last inequality follows from Lemma 2.5. Since $\widetilde{X}$ is ruled by Lemma 2.5, we have $\chi(\widetilde{X}) \leqslant 1$, so $h^0(K_{\widetilde{X}} + \widetilde{L}) \leqslant 1$; and if equality holds, then $\chi(\widetilde{X}) = 1$ and $\widetilde{L}K_{\widetilde{X}} = -1$.

Assume $h^0(K_{\widetilde{X}} + \widetilde{L}) = 0$. Then equation (2.1) implies that either $\chi(\widetilde{X}) = 1$ and $K_{\widetilde{X}}\widetilde{L} = -3$, or $\chi(\widetilde{X}) = 0$ and $K_{\widetilde{X}}\widetilde{L} = -1$. In the first case, by using Lemma 2.5 and Riemann–Roch we obtain $h^0(\widetilde{L}) \geqslant \chi(\widetilde{L}) = 3$, which violates the assumptions. In the second case, since $K_{\widetilde{X}}\widetilde{L} = -1$, the same argument gives $h^0(\widetilde{L}) \geqslant \chi(\widetilde{L}) = \chi(\widetilde{X}) + 1$, which yields the listed cases. □

Case (R) of the above lemma gives case (dP) in our classification.

Lemma 2.7. *If* (X, Δ) *is as in case* (R) *of Lemma 2.6, then it is of type* (dP).

Proof. By Lemma 2.3, the base locus of the pencil $|\widetilde{L}| = \varepsilon^*|L|$ is a simple point $\widetilde{P}$ which is the preimage of a smooth point $P \in X$; by adjunction, the general $C \in |\widetilde{L}|$ is a smooth elliptic curve. Upon blowing up the point P, we get an elliptic fibration $p : \widehat{X} \to \mathbb{P}^1$ with a section Γ.

Denote by Z the only effective divisor in $|K_{\widetilde{X}} + \widetilde{L}|$. Since $\widetilde{L}Z = 0$, Z does not contain the point $\widetilde{P}$ and is contained in a finite union of curves of $|\widetilde{L}|$; hence it can be identified with a divisor Z' of $\widehat{X}$ that is contained in a union of fibers of p and does not intersect the section Γ. By the Kodaira classification of elliptic fibers, Z' is either 0 or supported on a set $R_1, \ldots, R_k$ of (-2)-curves; the same is true of Z, since Z' does not meet Γ. In particular, we have $K_{\widetilde{X}}Z = 0$, hence

$$Z^2 = ZK_{\widetilde{X}} + Z\widetilde{L} = 0,$$

and therefore $Z = 0$ by the index theorem. Thus $\widetilde{L} = -K_{\widetilde{X}}$, X is the anti-canonical model of $\widetilde{X}$, and $\widetilde{\Delta} \in |-2K_{\widetilde{X}}|$. □

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We now turn to studying case (E) of Lemma 2.6. This gives rise to the cases (E_-) and (E_+) in our classification, depending on the value of $h^0(\widetilde{L})$.

LEMMA 2.8. *If (X,Δ) is as in case (E) of Lemma 2.6, then there exists an elliptic curve E and a vector bundle $\mathcal{E}$ on E of rank 2 and degree 1 such that $\widetilde{X} = \mathbb{P}(\mathcal{E})$ and $\widetilde{L} = \mathcal{O}_{\widetilde{X}}(1)$.*

Proof. By Lemmas 2.5 and 2.6, the surface $\widetilde{X}$ is ruled and $q(\widetilde{X}) = 1$; we denote by $a : \widetilde{X} \to E$ the Albanese map and by F a fiber of a.

Step 1: one has $\widetilde{L}F = 1$. The linear system $|\widetilde{L}|$ is non-empty by Lemma 2.6. Fix $C \in |\widetilde{L}|$ and denote by C_{red} the underlying reduced divisor. One has $p_a(C) = 1$ by adjunction and $p_a(C_{\mathrm{red}}) \leqslant 1$ by Lemma 2.4. The natural map $\mathrm{Pic}^0(E) = \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(C_{\mathrm{red}})$ is an inclusion by Lemma 2.4. Thus $p_a(C_{\mathrm{red}}) = 1$ and $\mathrm{Pic}^0(E) \to \mathrm{Pic}^0(C_{\mathrm{red}})$ is an isomorphism. By [BLR90, § 9.2, Corollary 12], $C_{\mathrm{red}} = C_0 + Z$, where C_0 is an elliptic curve that is mapped isomorphically onto E by a, Z is a sum of smooth rational curves and the dual graph of C_{red} is a tree. We write $C = bC_0 + Z'$, where $b > 0$ is an integer and Z' has the same support as Z. If $b = 1$, then $\widetilde{L}F = 1$ as claimed.

So assume to the contrary that $b > 1$. In this case, $1 = \widetilde{L}^2 \geqslant b\widetilde{L}C_0$ gives $\widetilde{L}C_0 = 0$. Then $C_0^2 < 0$, C_0 is contracted by $\widetilde{L}$ to an elliptic singularity and it does not intersect any other ε-exceptional curve. Since $\widetilde{L}$ is nef and $\widetilde{L}C = \widetilde{L}^2 = 1$, there is exactly one component Γ of C that has non-zero intersection with $\widetilde{L}$, and Γ appears in C with multiplicity 1. In particular, $Z' - \Gamma$ is contracted by ε and therefore $C_0(Z' - \Gamma) = 0$. We have $C_0\Gamma \leqslant 1$, since Γ is contained in a fiber of a. Hence

$$0 = C_0\widetilde{L} = C_0(bC_0 + \Gamma + (Z' - \Gamma)) = bC_0^2 + C_0\Gamma \leqslant 1 - b < 0,$$

which is a contradiction.

Step 2: conclusion of the proof. We claim that $a : \widetilde{X} \to E$ is a $\mathbb{P}^1$-bundle. Indeed, assume to the contrary that $\widetilde{X}$ contains an irreducible (-1)-curve Γ; then $\widetilde{L}\Gamma > 0$, because $\widetilde{X} \to X$ is the minimal resolution and $\widetilde{L}$ is the pull-back of an ample line bundle on X. On the other hand, $\widetilde{L}\Gamma \leqslant \widetilde{L}F = 1$, since Γ is contained in a fiber F of a. Hence $\widetilde{L}\Gamma = 1$. But then $\widetilde{L}(F - \Gamma) = 0$ and $K_{\widetilde{X}}(F - \Gamma) = -1$; that is, $F - \Gamma$ contains a (-1)-curve Γ_1 with $\widetilde{L}\Gamma_1 = 0$, which is a contradiction.

Finally, we set $\mathcal{E} = a_*\widetilde{L}$. □

LEMMA 2.9. *Assume that we are in case (E) of Lemma 2.6.*

(i) *If $h^0(\widetilde{L}) = 2$, then (X,Δ) is of type (E_-).*
(ii) *If $h^0(\widetilde{L}) = 1$, then (X,Δ) is of type (E_+).*

Proof. By Lemma 2.8, there exists an elliptic curve E and a vector bundle $\mathcal{E}$ on E of rank 2 and degree 1 such that $\widetilde{X} = \mathbb{P}(\mathcal{E})$ and $\widetilde{L} = \mathcal{O}_{\widetilde{X}}(1)$. Denote by $x \in E$ the point such that $\det \mathcal{E} = \mathcal{O}_E(x)$. We will freely use the general theory of $\mathbb{P}^1$-bundles, especially the classification of such bundles over an elliptic curve; see [Har77, ch. V.2].

Assume that $\mathcal{E}$ is decomposable, i.e. there are line bundles A and B on E such that $\mathcal{E} = A \oplus B$. Then we have $\deg A + \deg B = \deg \mathcal{E} = 1$ and $1 \leqslant h^0(A) + h^0(B) = h^0(\widetilde{L}) \leqslant 2$. So there are three possibilities:

(a) $\deg A = -1$ and $\deg B = 2$;
(b) $\deg A = 0$, $A \neq \mathcal{O}_E$ and $\deg B = 1$;
(c) $A = \mathcal{O}_E$ and $B = \mathcal{O}_E(x)$.

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We denote by C_0 the section of $\widetilde{X}$ corresponding to the surjection $\mathcal{E} \twoheadrightarrow A$. In case (a), the system $|\widetilde{L}| = |\mathcal{O}_{\widetilde{X}}(1)|$ has dimension 1 and has C_0 as its fixed part, contradicting Lemma 2.3; so this case does not occur. In case (b), we have $\widetilde{L}C_0 = 0$, but $\widetilde{L}|_{C_0}$ is non-trivial; this contradicts the assumption that $\widetilde{L}$ is the pull-back of an ample line bundle via the birational map $\varepsilon : \widetilde{X} \to X$. So (c) is the only possibility. In this case C_0 is contracted to an elliptic singularity of degree 1 by ε, and C_0 is the only curve contracted by ε since $\mathrm{NS}(\widetilde{X})$ has rank 2. We have $\widetilde{\Delta} = \widetilde{L} - K_{\widetilde{X}} = 3C_0 + 2F$. Since $K_{\widetilde{X}} = \varepsilon^* K_X - C_0$ and Δ does not go through the elliptic singularity of X because the pair (X, Δ) is lc, we obtain that $\varepsilon^*\Delta = \widetilde{\Delta} - C_0 = 2C_0 + 2F$ and (X, Δ) is a log surface of type (E_-).

If $\mathcal{E}$ is indecomposable, then $\mathcal{E}$ is the only non-trivial extension $0 \to \mathcal{O}_E \to \mathcal{E} \to \mathcal{O}_E(x) \to 0$ and $h^0(\widetilde{L}) = h^0(\mathcal{E}) = 1$. Up to a translation in E, we may assume that x is the origin $0 \in E$. Hence $\widetilde{X} = S^2E$ and $C = C_0 = \widetilde{L}$ is the image of the curve $\{0\} \times E + E \times \{0\}$ via the quotient map $E \times E \to S^2E$ (cf. the description of case (E_+) at the beginning of this section). Since $\widetilde{L}$ is ample, we have $\widetilde{X} = X$, $\widetilde{L} = L$ and that $\widetilde{\Delta} = \Delta = L - K_X$ is numerically equivalent to $3C_0 - F$. So the pair (X, Δ) is of type (E_+). □

Finally, we summarize all the above results.

Proof of Theorem 1.1. If $h^0(\widetilde{L}) \geqslant 3$, then by Lemma 2.3 we have $X = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$, and thus (X, Δ) is of type (P).

So we may assume $h^0(\widetilde{L}) \leqslant 2$, which by Lemma 2.6 leaves us with the cases (R) and (E), according to the value of $\chi(\widetilde{X})$. The first case gives type (dP) by Lemma 2.7, whereas the second splits into the cases (E_+) and (E_-) by Lemma 2.9. This concludes the proof of the theorem. □

3. Applications to stable surfaces

In this section we explore some consequences of the classification of pairs in Theorem 1.1 for the study of stable surfaces with $K^2 = 1$.

3.1 Definitions and Kollár's gluing construction

Our main reference for this section is [Kol13, §§ 5.1–5.3].

3.1.1 *Stable surfaces.* Let X be a demi-normal surface; that is, X satisfies S_2, and at each point of codimension one X either is regular or has an ordinary double point. We denote by $\pi : \bar{X} \to X$ the normalization of X. Contrary to our previous assumptions, X is not assumed to be irreducible; in particular, $\bar{X}$ is possibly disconnected. The conductor ideal $\mathscr{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X)$ is an ideal sheaf in both $\mathcal{O}_X$ and $\mathcal{O}_{\bar{X}}$ and as such defines subschemes $D \subset X$ and $\bar{D} \subset \bar{X}$, both of which are reduced and pure of codimension one; we will often refer to D as the non-normal locus of X.

Definition 3.1. The demi-normal surface X is said to have *semi-log-canonical (slc)* singularities if it satisfies the following conditions.

(i) The canonical divisor K_X is $\mathbb{Q}$-Cartier.

(ii) The pair $(\bar{X}, \bar{D})$ has log-canonical (lc) singularities.

It is called a stable surface if, in addition, K_X is ample. In that case, we define the geometric genus of X to be $p_g(X) = h^0(X, K_X) = h^2(X, \mathcal{O}_X)$ and the irregularity to be $q(X) = h^1(X, K_X) = h^1(X, \mathcal{O}_X)$. A *Gorenstein stable surface* is a stable surface such that K_X is a Cartier divisor.

The importance of these surfaces lies in the fact that they generalize stable curves: there is a projective moduli space of stable surfaces which compactifies the Gieseker moduli space of canonical models of surfaces of general type [Kol14].

3.1.2 *Kollár's gluing principle.* Let X be a demi-normal surface as above. Since X has at most double points in codimension one, the map $\pi : \bar{D} \to D$ on the conductor divisors is generically a double cover and thus induces a rational involution on $\bar{D}$. Upon normalizing the conductor loci, we get an honest involution $\tau : \bar{D}^\nu \to \bar{D}^\nu$ such that $D^\nu = \bar{D}^\nu/\tau$ and $\mathrm{Diff}_{\bar{D}^\nu}(0)$ is τ-invariant (for the definition of Diff, the different, see [Kol13, 5.11], for example).

THEOREM 3.2 [Kol13, Theorem 5.13]. *Associating to a stable surface X the triple $(\bar{X}, \bar{D}, \tau : \bar{D}^\nu \to \bar{D}^\nu)$ induces a one-to-one correspondence*

$$\left\{ \begin{matrix} \text{stable} \\ \text{surfaces} \end{matrix} \right\} \leftrightarrow \left\{ (\bar{X}, \bar{D}, \tau) \;\middle|\; \begin{matrix} (\bar{X}, \bar{D}) \text{ log-canonical pair with } K_{\bar{X}} + \bar{D} \text{ ample,} \\ \tau \colon \bar{D}^\nu \to \bar{D}^\nu \text{ involution such that } \mathrm{Diff}_{\bar{D}^\nu}(0) \text{ is } \tau\text{-invariant.} \end{matrix} \right\}$$

ADDENDUM: *In the above correspondence the surface X is Gorenstein if and only if $K_{\bar{X}} + \bar{D}$ is Cartier and τ induces a fixed-point-free involution on the preimages of the nodes of $\bar{D}$.*

An important consequence, which allows us to understand the geometry of stable surfaces from the normalization, is that

$$\begin{array}{ccccc} \bar{X} & \xleftarrow{\bar{\iota}} & \bar{D} & \xleftarrow{\bar{\nu}} & \bar{D}^\nu \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle /\tau} \\ X & \xleftarrow{\iota} & D & \xleftarrow{\nu} & D^\nu \end{array} \tag{3.1}$$

is a pushout diagram.

Proof of the addendum in Theorem 3.2. By [KSB88, 4.20, 4.21, 4.27], the non-normal Gorenstein slc singularities are normal-crossing points, pinch points or degenerate cusps; the normalization $\bar{X}$ is smooth at the smooth points of the conductor $\bar{D}$ and has at most cyclic quotient singularities at the nodes of $\bar{D}$.

If X is Gorenstein, then $K_{\bar{X}} + \bar{D} = \pi^* K_X$ is an ample Cartier divisor. The different $\mathrm{Diff}_{\bar{D}^\nu}(0)$ is the sum of preimages of the nodes of $\bar{D}$, each with coefficient 1, and thus τ induces an action on the preimages of the nodes. The preimage of a degenerate cusp in the normalization consists of a set of points which are nodes of $\bar{D}$ and are glued in a cycle in such a way that gluing the log-resolutions of each component gives a cycle of rational curves or a nodal rational curve (see [FPR14, Figure 1] for an illustration). Therefore, τ does not fix the preimage of a node.

Conversely, assume that we are given a triple $(\bar{X}, \bar{D}, \tau)$ satisfying the conditions in the addendum. The assumption that $K_{\bar{X}} + \bar{D}$ is Cartier implies that the only singular points of $\bar{X}$ along $\bar{D}$ are contained in nodes of $\bar{D}$, again by classification,[1] so that the different $\mathrm{Diff}_{\bar{D}^\nu}(0)$ is the reduced sum of the preimages of nodes. By assumption, τ acts freely on these points and in particular preserves the different. Thus, by Theorem 3.2, there is a corresponding slc surface X. Outside the non-normal locus D, the surface X is clearly Gorenstein since $\bar{X}$ is. Let $P \in D \subset X$ be a non-normal point of X. If the preimage $\pi^{-1}(P)$ contains a smooth point of $\bar{D}$, then it

[1] In the classification of lc pairs [Kol13, § 3.3], there are several instances where $p \in \bar{D} \subset \bar{X}$ is a smooth point of D and a quotient singularity of X. However, in those cases $K_{\bar{X}} + \bar{D}$ is never Cartier; this can be seen, for example, by computing the (log-)discrepancies for the minimal resolution, which turn out to be non-integral for some components of the exceptional set.

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contains only smooth points of $\bar{D}$, and $P \in X$ is semi-smooth and hence Gorenstein. Otherwise, $\pi^{-1}(P)$ contains at least one node of $\bar{D}$ and thus only nodes because of the invariance of the different. Since the action of τ on $\pi^{-1}(P)$ has no fixed point, we get a degenerate cusp as described above. □

Computing the main invariants of a stable surface from its normalization is not difficult; see, for example, [LR13, Proposition 2.5].

Proposition 3.3. *Let X be a stable surface with normalization $(\bar{X}, \bar{D})$. Then $K_X^2 = (K_{\bar{X}} + \bar{D})^2$ and $\chi(X) = \chi(\bar{X}) + \chi(D) - \chi(\bar{D})$.*

Note in particular that, by Nakai–Moishezon, a Gorenstein stable surface with $K_X^2 = 1$ is irreducible. Summing up, we now state explicitly our main motivation for the classification in Theorem 1.1.

Corollary 3.4. *Let X be a Gorenstein stable surface with $K_X^2 = 1$, and let $(\bar{X}, \bar{D}, \tau)$ be the corresponding triple as above. Then $(\bar{X}, \bar{D})$ is of one of the types classified in Theorem 1.1.*

3.2 Numerology

In this section we feed the classification from § 2 into Kollár's gluing construction. The result is a precise list of the possible normalizations of a non-normal Gorenstein stable surface with $K_X^2 = 1$. We also give the possible values of $\chi(X)$ for each type, showing in particular that there are no Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(X) < 0$.

We start with a preliminary lemma. In order to state it, we keep the notation from § 3.1.2 and introduce some additional numerical invariants of a stable surface X:

- μ_1, the number of degenerate cusps;
- μ_2, the number of $\mathbb{Z}/2\mathbb{Z}$-quotients of degenerate cusps of X;
- ρ, the number of ramification points of the map $\bar{D}^\nu \to D^\nu$;
- $\bar{\mu}$, the number of nodes of $\bar{D}$.

Lemma 3.5. *Let X be a non-normal stable surface. With the above notation:*

(i) $\chi(D) = \frac{1}{2}(\chi(\bar{D}) - \bar{\mu}) + \rho/4 + \mu_1$;
(ii) *if $K_{\bar{X}} + \bar{D}$ is Cartier, then* $\chi(D) \geqslant 2\chi(\bar{D}) + \rho/4 + \mu_1$;
(iii) *if X is Gorenstein, then* $\chi(D) \geqslant 2\chi(\bar{D}) + 1$.

In addition, if equality holds in (ii) or (iii), then $\bar{D}$ is a union of rational curves and has $-3\chi(\bar{D})$ nodes.

We remark that there exist examples of non-Gorenstein stable surfaces for which the inequalities (ii) and (iii) of Lemma 3.5 fail.

Proof. The curve $\bar{D}$ has nodes by the classification of lc pairs. Recall that diagram (3.1) is a pushout diagram in the category of schemes. In particular, the points of D correspond to equivalence classes of points on $\bar{D}^\nu$ with respect to the relation generated by $x \sim y$ if $\bar{\nu}(x) = \bar{\nu}(y)$ or $\tau(x) = y$. Note that if an equivalence class contains the preimage of a node of $\bar{D}$, then either it contains no fixed point of τ and the image point is a degenerate cusp, or it contains exactly two fixed points of τ and the image is a $\mathbb{Z}/2\mathbb{Z}$-quotient of a degenerate cusp. (Compare with the discussions in [LR14, § 4.2] and [KSB88, § 4].)

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Thus, of the $2\bar{\mu}$ preimages of nodes of $\bar{D}$ in $\bar{D}^\nu$, exactly $2\mu_2$ are fixed by τ, and there are exactly $\bar{\mu} + \mu_2$ points in D^ν that map to images of nodes in D. By the normalization sequences, we have

$$\chi(\bar{D}^\nu) = \chi(\bar{D}) + \bar{\mu},$$
$$\chi(D) = \chi(D^\nu) - ((\bar{\mu} + \mu_2) - (\mu_1 + \mu_2)) = \chi(D^\nu) + \mu_1 - \bar{\mu}.$$

Combining this with the Hurwitz formula for $\bar{D}^\nu \to D^\nu$, which gives

$$\chi(D^\nu) = \frac{1}{2}\chi(\bar{D}^\nu) + \frac{\rho}{4},$$

we get

$$\chi(D) = \frac{1}{2}\chi(\bar{D}^\nu) + \frac{\rho}{4} + \mu_1 - \bar{\mu} = \frac{1}{2}\left(\chi(\bar{D}) - \bar{\mu}\right) + \frac{\rho}{4} + \mu_1$$

as claimed in (i).

Now assume, in addition, that $K_{\bar{X}} + \bar{D}$ is Cartier. Then, by adjunction (see e.g. [Kol13, § 4.1]), $K_{\bar{D}} = (K_{\bar{X}} + \bar{D})|_{\bar{D}}$ is ample, so $\bar{D}$ is a stable curve. Therefore, every rational component of the normalization has at least three marked points mapping to nodes in $\bar{D}$, and thus $\chi(\bar{D}^\nu) \leqslant 2\bar{\mu}/3$, which implies $-\bar{\mu} \geqslant 3\chi(\bar{D})$. This gives (ii) and proves the last sentence in the statement.

Equality in (ii) is attained if and only if $\bar{D}^\nu$ consists of $-2\chi(\bar{D})$ rational curves, each with three marked points; then the curve $\bar{D}$ has $-3\chi(\bar{D})$ nodes.

In order to prove (iii), we only need to show that if equality occurs in (ii) and X is Gorenstein, then there is at least one degenerate cusp. But if equality holds in (ii), then $\bar{D}$ has $-3\chi(\bar{D}) > 0$ nodes and, since X is Gorenstein, each node of $\bar{D}$ maps to a degenerate cusp, i.e. $\mu_1 > 0$. □

THEOREM 3.6. *There exists a non-normal Gorenstein stable surface with normalization of given type (as defined and classified in § 2) exactly in the following cases:*

Normalization	$\chi(X) = 0$	$\chi(X) = 1$	$\chi(X) = 2$	$\chi(X) = 3$
(P)	✓	✓	✓	✓
(dP)		✓	✓	✓
(E_-)			✓	✓
(E_+)		✓	✓	

One could extend the above numerical analysis to all stable surfaces with $K_X^2 = 1$ and Gorenstein normalization $(\bar{X}, \bar{D})$. From a moduli perspective, such surfaces do not form a good class: they would include some but not all 2-Gorenstein surfaces.

Proof. The restrictions follow from Proposition 3.3, the invariants given in Table 1 and Lemma 3.5, where in the cases $(E_\pm)$ we use that not all components of $\bar{D}$ can be rational.

The existence of examples is settled below in § 3.3. □

The above results allow us to refine in the $K^2 = 1$ case the P_2-inequality $\chi \geqslant -K^2$, proved in [LR13] for Gorenstein stable surfaces.

COROLLARY 3.7. *If X is a Gorenstein stable surface with $K_X^2 = 1$, then $\chi(X) \geqslant 0$.*

Proof. Let X be a Gorenstein stable surface with $K_X^2 = 1$. If X is normal, then $\chi(X) \geqslant 1$ by [Bla94, Theorem 2]. If X is not normal, then $\chi(X) \geqslant 0$ by Theorem 3.6. □

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3.3 Examples

For completeness, we now provide some explicit examples to show that each case given in Theorem 3.6 actually occurs. We will analyse such surfaces more systematically in another paper.

By Theorem 3.2 and Corollary 3.4, for each type we need to specify a (nodal) boundary $\bar{D}$ and an involution τ on the normalization of $\bar{D}$ which induces a fixed-point-free action on the preimages of the nodes. The holomorphic Euler characteristic is then computed by Proposition 3.3.

Case (P). Examples with $0 \leqslant \chi(X) \leqslant 3$ are given in [LR13, § 5.1].

Case (E_-).

- Take $\bar{D}$ to be a general section in $|-2K_{\bar{X}}|$ which is smooth, and take τ to be the hyperelliptic involution. This gives $\chi(X) = 3$.
- Let $E_1, E_2 \in |-K_{\bar{X}}|$ be two distinct smooth isomorphic curves, and fix the intersection point as a base point on both. Let $\bar{D} = E_1 + E_2$ and let τ be the involution that exchanges the two curves, preserving the base point. Then $\chi(X) = 2$.
- Take as $\bar{X}$ the blow-up of $\mathbb{P}^2$ at eight of the nine base points of a pencil spanned by two nodal cubics C_1 and C_2 meeting transversely, and take as $\bar{D}$ the union of the strict transforms of C_1 and C_2. The normalization $\bar{D}^\nu$ consists of two copies of $\mathbb{P}^1$, each with three marked points which are the preimages of the nodes of $\bar{D}$.

  An involution on $\bar{D}^\nu$ interchanging the components is uniquely determined by its action on the marked points, and we can choose it in such a way that the preimage of the base point of the pencil is not preserved by the involution (see Figure 1). One can easily see that this gives a rational curve of genus 2 (not nodal) as the non-normal locus, and thus $\chi(X) = 1$. Such surfaces are studied in detail in [Rol14].

Case (E_-). The divisor $\bar{D}$ is a curve of arithmetic genus 2, which after pull-back to the minimal resolution becomes a degree-2 cover of the base curve of the projective bundle. If $\bar{D}$ is smooth, choosing as τ either the hyperelliptic involution or the involution corresponding to the double cover of the elliptic base curve gives the two possible values for $\chi(X)$.

Case (E_+).

- A general $\bar{D}$ is a smooth curve of genus 2, and by choosing τ to be the hyperelliptic involution we get $\chi(X) = 2$.
- For the numerical Godeaux case, let $E \cong \mathbb{C}/\mathbb{Z}[i]$. Then multiplication by $\xi := 1 + i$ induces an endomorphism of degree 2 on E, that is, an isomorphism $E \cong E/\xi$. For this choice of E, the surface $\bar{X} = S^2E$ contains a bisection $E_1 \cong E$ of the Albanese map $a : \bar{X} \to E$ such that $E_1 C_0 = 1$, where C_0 is a section of a with $C_0^2 = 1$ (cf. [CC93, § 2, (5)]). We choose $\bar{D} = E_1 \cup C_0 \cong E \cup E$. Thus there is an involution τ on $\bar{D}$ with quotient E which exchanges the two components and fixes their intersection point. With this choice, $\chi(X) = 1$.

4. Normal Gorenstein stable surfaces with $K^2 = 1$

In this section we complement the results of § 2 by omitting the condition that the boundary should be non-empty; that is, we study Gorenstein log-canonical surfaces X with K_X ample and $K_X^2 = 1$. In the terminology of § 3.1, these are normal Gorenstein stable surfaces and they occur in the compactified Gieseker moduli space.

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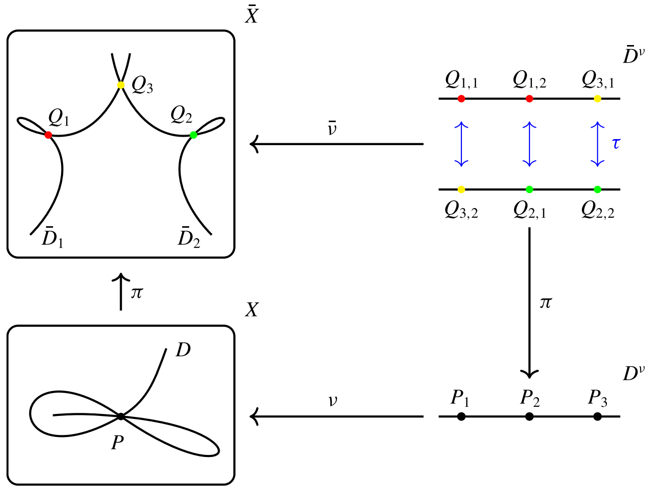

FIGURE 1. Construction of a numerical Godeaux surface with normalization of type (dP).

Of course, in this case we cannot hope for a complete picture; for instance, surfaces of general type with $K^2 = \chi = 1$, known as Godeaux surfaces, have been an object of study for decades and a full classification has not been achieved yet.

Still, we are able to give a rough description according to the Kodaira dimension of $\widetilde{X}$.

THEOREM 4.1. *Let X be a normal Gorenstein stable surface with $K_X^2 = 1$, and let $\varepsilon : \widetilde{X} \to X$ be its minimal desingularization. Then the following hold.*

(i) *If $\kappa(\widetilde{X}) = 2$, then X has canonical singularities.*

(ii) *If $\kappa(\widetilde{X}) = 1$, then $\widetilde{X}$ is a minimal properly elliptic surface and X has precisely one elliptic singularity of degree* 1.

(iii) *If $\kappa(\widetilde{X}) = 0$, denote by $X_{\min}$ the minimal model of $\widetilde{X}$. Then there exists a nef effective divisor $D_{\min}$ on $X_{\min}$ and a point P such that:*

- *$D_{\min}^2 = 2$ and $P \in D_{\min}$ has multiplicity* 2*;*
- *$\widetilde{X}$ is the blow-up of $X_{\min}$ at P;*
- *X is obtained from $\widetilde{X}$ by blowing down the strict transform of $D_{\min}$, and it has either one elliptic singularity of degree* 2 *or two elliptic singularities of degree* 1.

(iv) *If $\kappa(\widetilde{X}) = -\infty$, then there are two possibilities:*

(a) *$\chi(\widetilde{X}) = 1$ and $\widetilde{X}$ has one or two elliptic singularities;*

(b) *$\chi(\widetilde{X}) = 0$ and $\widetilde{X}$ has one, two or three elliptic singularities; in this case, the exceptional divisors arising from the elliptic singularities are smooth elliptic curves.*

One can show that all the cases actually occur (see [FPR15b, FPR15a]). The proof of Theorem 4.1 occupies the rest of this section. First we fix the set-up and notation to be used throughout: X is a normal Gorenstein stable surface with $K_X^2 = 1$, $\varepsilon : \widetilde{X} \to X$ is the minimal resolution and $\widetilde{L} := \varepsilon^* K_X$, so $\widetilde{L}$ is a nef and big line bundle with $\widetilde{L}^2 = 1$. One has $\widetilde{L} = K_{\widetilde{X}} + \widetilde{D}$, where $\widetilde{D}$ is effective and $\widetilde{L}\widetilde{D} = 0$. It follows, in particular, that $\widetilde{L}K_{\widetilde{X}} = 1$.

By the classification of normal Gorenstein lc singularities (cf. [KSB88, Theorem 4.21]), the singularities of X are either canonical or elliptic. The elliptic Gorenstein singularities are described in [Rei97, 4.21]: denoting by $x_1, \ldots, x_k \in X$ the elliptic singular points, we can

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write $\widetilde{D} = \widetilde{D}_1 + \cdots + \widetilde{D}_k$, where $\widetilde{D}_i$ is a divisor supported on $\varepsilon^{-1}(x_i)$ such that $p_a(Z) < p_a(\widetilde{D}_i) = 1$ for every $0 < Z < \widetilde{D}_i$. The divisors $\widetilde{D}_i$ are called the *elliptic cycles* of $\widetilde{X}$. The degree of the elliptic singularity x_i is the positive integer $-\widetilde{D}_i^2$.

The invariants of X and $\widetilde{X}$ are related as follows.

Lemma 4.2. *In the above set-up:*

$$p_g(X) = h^0(\widetilde{L}) \geqslant p_g(\widetilde{X}), \quad q(X) \leqslant q(\widetilde{X}), \quad \chi(X) = \chi(\widetilde{X}) + k.$$

Proof. By the projection formula we have $h^0(\widetilde{L}) = h^0(\varepsilon_*\widetilde{L}) = h^0(K_X) = p_g(X)$; in addition, there is an inclusion $H^0(K_{\widetilde{X}}) \hookrightarrow H^0(\widetilde{L})$ since $\widetilde{D}$ is effective.

The remaining inequalities follow by the five-term exact sequence associated with the Leray spectral sequence for $\mathcal{O}_{\widetilde{X}}$,

$$0 \to H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{\widetilde{X}}) \to H^0(R^1\varepsilon_*\mathcal{O}_{\widetilde{X}}) \to H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_{\widetilde{X}}) \to 0,$$

since $R^1\varepsilon_*\mathcal{O}_{\widetilde{X}}$ has length one at each of the points $x_1, \ldots, x_k$ and is zero elsewhere. □

We start by dealing with the case $\kappa(\widetilde{X}) > 0$.

Lemma 4.3. *If $\kappa(\widetilde{X}) > 0$, then there are the following possibilities:*

(i) *X has canonical singularities;*
(ii) *$\widetilde{X}$ is a minimal properly elliptic surface and X has precisely one elliptic singularity of degree* 1.

Proof. Let $\eta : \widetilde{X} \to X_{\min}$ be the morphism to the minimal model. Let $M = \eta^* K_{X_{\min}}$ so that $K_{\widetilde{X}} = M + E$, where E is exceptional for η. We have $\widetilde{L}(M + E) = \widetilde{L}K_{\widetilde{X}} = \widetilde{L}^2 = 1$. Since $\widetilde{L}$ is nef and big and some multiple of M moves, we have $\widetilde{L}M = 1$ and $\widetilde{L}E = 0$. Thus, since $\widetilde{L}$ is the pull-back of an ample divisor, E is also contracted by ε. Since ε is assumed to be minimal, there is no ε-exceptional (-1)-curve, while on the other hand η is a composition of blow-ups of a smooth surface. Hence $E = 0$, i.e. $\widetilde{X}$ is minimal.

If $\kappa(\widetilde{X}) = 2$, then the index theorem applied to $\widetilde{L}$ and $K_{\widetilde{X}}$ gives $K_{\widetilde{X}}^2 = 1$ and that $K_{\widetilde{X}}$ and $\widetilde{L}$ are numerically equivalent (otherwise they would span a two-dimensional subspace on which the intersection form is positive). It follows that $\widetilde{D} \geqslant 0$ is numerically trivial; hence $\widetilde{D} = 0$ and $K_{\widetilde{X}} = \varepsilon^* K_X$, i.e. X has canonical singularities.

If $\kappa(\widetilde{X}) = 1$, then $\widetilde{X}$ is minimal properly elliptic and $K_{\widetilde{X}}^2 = 0$. It follows that $(\widetilde{D}_1 + \cdots + \widetilde{D}_k)K_{\widetilde{X}} = \widetilde{D}K_{\widetilde{X}} = \widetilde{L}K_{\widetilde{X}} = 1$. Since $\widetilde{D}_iK_{\widetilde{X}} > 0$ for every i, we have $k = 1$; that is, $\widetilde{D}$ is connected and $\widetilde{D}^2 = -1$. □

Next we consider the case where $\kappa(\widetilde{X}) = 0$.

Lemma 4.4. *If $\kappa(\widetilde{X}) = 0$, then X is as in Theorem 4.1(iii).*

Proof. Let $\eta : \widetilde{X} \to X_{\min}$ be the morphism to the minimal model, so η is a composition of m blow-ups in smooth points $P_1, \ldots, P_m$, possibly infinitely near. Denote by E_i the total transform on $\widetilde{X}$ of the exceptional curve that appears at the ith blow-up; then $E_i^2 = E_iK_{\widetilde{X}} = -1$, $E_iE_j = 0$ if $i \neq j$, and $K_{\widetilde{X}}$ is numerically equivalent to $\sum_{i=1}^m E_i$. Observe that each E_i contains at least one irreducible (-1)-curve. Since ε is relatively minimal, $\widetilde{L}$ is positive on irreducible (-1)-curves.

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Hence we have $1 = \widetilde{L}K_{\widetilde{X}} = \sum_{i=1}^m \widetilde{L}E_i \geqslant m$, and we conclude that $m = 1$, i.e. that ε is a single blow-up. We set $E = E_1$.

Write $\widetilde{D} = \widetilde{D}_1 + \cdots + \widetilde{D}_k$, with the $\widetilde{D}_i$ being disjoint elliptic cycles. We have $2 = (\widetilde{L} - K_{\widetilde{X}})K_{\widetilde{X}} = \widetilde{D}K_{\widetilde{X}} = \widetilde{D}_1 K_{\widetilde{X}} + \cdots + \widetilde{D}_k K_{\widetilde{X}}$, so either $k = 1$ and $2 = \widetilde{D}_1 K_{\widetilde{X}} = \widetilde{D}_1 E$, or $k = 2$ and $1 = \widetilde{D}_i K_{\widetilde{X}} = \widetilde{D}_i E$ for $i = 1, 2$. In the former case we have $\widetilde{D}_1^2 = \widetilde{D}^2 = -2$, and in the latter case we have $\widetilde{D}_1^2 = \widetilde{D}_2^2 = -1$ since $p_a(\widetilde{D}_i) = 1$.

We set $D_{\min} = \eta_* \widetilde{D}$. The divisor $D_{\min}$ has $D_{\min}^2 = 2$ and contains P with multiplicity 2.

In order to complete the proof, we need to show that $D_{\min}$ is nef. Let Γ be an irreducible curve of $X_{\min}$. We have $\Gamma D_{\min} = (\eta^* \Gamma)(\eta^* D_{\min}) = \eta^* \Gamma(\widetilde{L} + E) = \eta^* \Gamma \widetilde{L} \geqslant 0$, since $\widetilde{L}$ is nef. □

Finally, we consider the case where $\kappa(\widetilde{X}) = -\infty$.

Lemma 4.5. *If $\kappa(\widetilde{X}) = -\infty$, then there are the following possibilities:*

(a) *$\chi(\widetilde{X}) = 1$ and $\widetilde{X}$ has one or two elliptic singularities;*

(b) *$\chi(\widetilde{X}) = 0$, $\widetilde{X}$ has one, two or three elliptic singularities, and $\widetilde{D}$ is a union of disjoint smooth elliptic curves.*

Proof. Since $\widetilde{X}$ is ruled, we have $\chi(\widetilde{X}) \leqslant 1$, with equality if and only if $\widetilde{X}$ is rational.

Assume $\chi(\widetilde{X}) \leqslant 0$ and let $a : X \to B$ be the Albanese map, where B is a smooth curve of genus $b > 0$. Write $\widetilde{D} = \widetilde{D}_1 + \cdots + \widetilde{D}_k$; since the general fiber of a is a smooth rational curve and $p_a(\widetilde{D}_i) = 1$ for all i, no $\widetilde{D}_i$ can be contracted to a point by a, and hence $\widetilde{D}_i$ dominates B. It follows that $b = 1$ and $\widetilde{D}_i$ contains a smooth elliptic curve D_i'. Since $\widetilde{D}_i$ is minimal among the divisors $Z > 0$ supported on $\varepsilon^{-1}(x_i)$ such that $p_a(Z) = 1$, it follows that $\widetilde{D}_i = D_i'$.

One has $\chi(X) \geqslant 1$ by [Bla94, Theorem 2] and $\chi(X) \leqslant 3$ by the stable Noether inequality for normal Gorenstein stable surfaces [Sak80, LR13]. Since $k > 0$, Lemma 4.2 gives $1 \leqslant k \leqslant 3$ if $\chi(\widetilde{X}) = 0$ and $1 \leqslant k \leqslant 2$ if $\chi(\widetilde{X}) = 1$. □

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Marco Franciosi franciosi@dm.unipi.it
Dipartimento di Matematica, Università di Pisa,
Largo B. Pontecorvo 5, I-56127 Pisa, Italy

Rita Pardini pardini@dm.unipi.it
Dipartimento di Matematica, Università di Pisa,
Largo B. Pontecorvo 5, I-56127 Pisa, Italy

Sönke Rollenske rollenske@math.uni-bielefeld.de
Fakultät für Mathematik, Universität Bielefeld,
Universitätsstr. 25, 33615 Bielefeld, Germany