DENSENESS OF OPERATORS
WHICH ATTAIN THEIR NUMERICAL RADIUS

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Abstract

We show that a bounded linear operator on a uniformly convex space may be perturbed by a compact operator of arbitrarily small norm to yield an operator which attains its numerical radius.

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The second author, in his dissertation [Sims, 1972], raised the question of the norm-denseness of those operators which attained their numerical radii, paralleling the norm-attaining operator investigations of Lindenstrauss [1963] and the related results of Bishop and Phelps [1961]. He observed there that self-adjoint operators on a Hilbert space could be approximated by self-adjoint operators which attained their numerical radii.

In this note we show that any bounded operator on a Hilbert space can be approximated in norm by operators which attain their numerical radii and, incidentally, differ from the original operator by a compact operator. Indeed, our argument establishes this result for any uniformly convex Banach space.

It is not clear how far the argument or result extend. The numerical radius is a natural object of contemplation for Hilbert space but is less accessible for general Banach spaces. At present we do not know a counterexample to the theorem on any Banach space.

We acknowledge valuable correspondence with J. Williams who, in particular, pointed out directions of investigation which are not pursued in this paper but indicate promising lines for future investigation.

We let X be a Banach space, X^* its dual space and $B(X)$ the algebra of bounded linear operators on X.

Let

$$\Pi := \{(x, f) \in X \times X^* : f(x) = \|f\| = \|x\| = 1\}.$$

Recall: The *numerical radius* of $T \in B(X)$ is

$$\nu(T) := \mathrm{Sup}\{|f(Tx)| : (x, f) \in \Pi\}$$

and we say T *attains* its numerical radius if there exists $(x_0, f_0) \in \Pi$ with

$$|f_0(Tx_0)| = \nu(T).$$

(TECHNICAL) LEMMA. *For X a normed linear space and $T \in B(X)$, given constants $a_1, b, a_2 > 0$ and $(x_1, f_1) \in \Pi$ with $|f_1(Tx_1)| > \nu(T) - a_1$, there exists $T' \in B(X)$ with*

(1) $T' - T$ *a rank-one operator of norm b,*

and there exists $(x_2, f_2) \in \Pi$ with

(2) $|f_2(T'x_2)| > \nu(T') - a_2$

and

(3) $f_1(x_2) > 1 - \dfrac{a_1 + a_2}{b}.$

Moreover,

(4) $\left\|\dfrac{x_1 + x_2}{2}\right\| > 1 - \dfrac{a_1 + a_2}{2b}.$

PROOF. (1) is satisfied by taking

$$T'(x) = T(x) + be^{i\theta}f_1(x)x_1$$

for all $x \in X$, where $\theta = \arg f_1(Tx_1)$.

Now choose $(x_2, f_2) \in \Pi$ such that (2) is satisfied and simultaneously $f_1(x_2) \geqslant 0$. This is possible since $(x, f) \in \Pi$ implies $(\lambda x, \bar{\lambda} f) \in \Pi$ for any λ with $|\lambda| = 1$. To establish (3) note that

$$
\begin{aligned}
\nu(T') - a_2 &< |f_2(T'x_2)| = |f_2(Tx_2) + be^{i\theta}f_1(x_2)f_2(x_1)| \\
&\leqslant |f_2(Tx_2)| + b|f_1(x_2)||f_2(x_1)| \\
&\leqslant \nu(T) + bf_1(x_2) \quad \text{as } |f_1(x_2)| = f_1(x_2) \text{ and } |f_2(x_1)| \leqslant 1.
\end{aligned}
$$

Further,

$$
\begin{aligned}
\nu(T') &\geqslant |f_1(T'x_1)| = |f_1(Tx_1) + be^{i\theta}| \\
&= |f_1(Tx_1)| + b \quad \text{(by choice of } \theta) \\
&\geqslant \nu(T) - a_1 + b.
\end{aligned}
$$

Combining these yields

$$\nu(T) - a_1 + b - a_2 < \nu(T) + bf_1(x_2)$$

or

$$f_1(x_2) > 1 - \frac{a_1 + a_2}{b} \quad \text{as required.}$$

We now establish (4) by noting

$$\left\| \frac{x_1 + x_2}{2} \right\| \geq f_1\left(\frac{x_1 + x_2}{2} \right) \geq \frac{1}{2}\left(1 + 1 - \frac{a_1 + a_2}{b} \right) = 1 - \frac{a_1 + a_2}{2b}.$$

Now let X be uniformly convex with modulus of convexity

$$\delta(\varepsilon) := \inf\left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \right\}.$$

Then $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$ and $\|x - y\| < \varepsilon$ whenever $\|(x + y)/2\| > 1 - \delta(\varepsilon)$. Let $\varepsilon > 0$ be given. Choose positive sequences (ε_n) and (ε_n') such that: ε_n is monotonically decreasing to 0; $\sum \varepsilon_n' < \varepsilon$ and

$$\frac{\varepsilon_n}{\varepsilon_n'} < \delta\left(\frac{1}{2^n} \right).$$

[For example, take $\varepsilon_n' = \varepsilon/2^n$ and $\varepsilon_n = (\varepsilon\delta(1/2^n))/2^{n+1}$.]

Using the above lemma we may inductively construct sequences (T_n) and $(x_n, f_n) \in \Pi$ as follows: $T_1 = T$ and (x_1, f_1) is such that $|f_1(T_1 x_1)| > \nu(T_1) - \varepsilon_1$; T_2 is such that $T_2 - T_1$ is a rank-one operator of norm ε_1' and (x_2, f_2) is such that $|f_2(T_2 x_2)| > \nu(T_2) - \varepsilon_2$ and

$$\left\| \frac{x_1 + x_2}{2} \right\| > 1 - \frac{\varepsilon_1 + \varepsilon_2}{2\varepsilon_1'} > 1 - \frac{\varepsilon_2}{\varepsilon_1'} > 1 - \delta\left(\frac{1}{2} \right).$$

So $\|x_2 - x_1\| < \frac{1}{2}$ [take $a_1 = \varepsilon_1$, $b = \varepsilon_1'$ and $a_2 = \varepsilon_2$]; and in general: T_n is such that $T_n - T_{n-1}$ is a rank-one operator of norm ε_n' and (x_n, f_n) is such that $|f_n(T_n x_n)| > \nu(T_n) - \varepsilon_n$ and $\|x_n - x_{n-1}\| < 1/2^n$. It now follows that (T_n) is a Cauchy sequence [$\|T_m - T_n\| \leq \sum_{k=n+1}^{m} \|T_k - T_{k-1}\| \leq \sum_{k=n+1}^{m} \varepsilon_k' \to 0$ as $m > n \to \infty$] as is (x_n). Thus T_n converges to $T_\infty \in B(X)$ and $x_n \to x_\infty$ where $\|x_\infty\| = 1$ and $T_\infty - T$ is a compact operator of norm less than or equal to $\sum \varepsilon_n' < \varepsilon$. By passing to a subsequence if necessary we may assume that f_n converges weakly to f_∞ with $\|f_\infty\| \leq 1$. (X is reflexive, so the unit ball of X^* is weakly compact.)

We show $(x_\infty, f_\infty) \in \Pi$ and $|f_\infty(T_\infty x_\infty)| = \nu(T_\infty)$. Now,

$$f_\infty(x_\infty) = \lim f_n(x_\infty) = \lim(f_n(x_n) + f_n(x_\infty - x_n))$$
$$= 1, \quad \text{as } f_n(x_n) = 1 \text{ and } \|x_\infty - x_n\| \to 0.$$

So $(x_\infty, f_\infty) \in \Pi$. Further,

$$\begin{aligned}
\nu(T_\infty) &= \lim \nu(T_n) && \text{as } \|T_n - T_\infty\| \to 0 \\
&= \lim |f_n(T_n x_n)| && \text{as } \nu(T_n) - |f_n(T_n x_n)| \to 0 \\
&= \lim |f_n(T_\infty x_\infty)| && \text{as } \|T_n x_n - T_\infty x_\infty\| \to 0 \\
&= |f_\infty(T_\infty x_\infty)| && \text{as } f_n \xrightarrow{w} f_\infty,
\end{aligned}$$

and we have established:

THEOREM. *For X a uniformly convex Banach space, given $T \in B(X)$ and $\varepsilon > 0$, there exists a compact operator C with $\|C\| < \varepsilon$ and $(x, f) \in \Pi$ such that $\nu(T + C) = |f((T + C)x)|$.*

That is; T may be perturbed by a compact operator of arbitrarily small norm to obtain an operator which attains its numerical radius. In particular, the operators which attain their numerical radius are norm dense in $B(X)$. We remark that, as a consequence of a theorem of Weyl [Halmos, 1967, problem 43], in the case of a self-adjoint operator on a Hilbert space the necessary perturbation can be realized by a rank-one self-adjoint operator.

References

Errett Bishop and R. R. Phelps, 'A proof that every Banach space is subreflexive', *Bull. Amer. Math. Soc.* **67** (1961), 97–98.

P. Halmos, *A Hilbert space problem book* (Van Nostrand, New York, 1967).

Joram Lindenstrauss, 'On operators which attain their norm', *Israel J. Math.* **1** (1963), 139–148.

Brailey Sims, *On numerical range and its application to Banach algebra* (Ph.D. dissertation, University of Newcastle, Australia, 1972).

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