A HIERARCHY ON NON-ARCHIMEDEAN POLISH GROUPS ADMITTING A COMPATIBLE COMPLETE LEFT-INVARIANT METRIC

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Abstract. In this article, we introduce a hierarchy on the class of non-archimedean Polish groups that admit a compatible complete left-invariant metric. We denote this hierarchy by α-CLI and L-α-CLI where α is a countable ordinal. We establish three results:

(1) G is 0-CLI iff $G = \{1_G\}$;
(2) G is 1-CLI iff G admits a compatible complete two-sided invariant metric; and
(3) G is L-α-CLI iff G is locally α-CLI, i.e., G contains an open subgroup that is α-CLI.

Subsequently, we show this hierarchy is proper by constructing non-archimedean CLI Polish groups G_α and H_α for $\alpha < \omega_1$, such that:

(1) H_α is α-CLI but not L-β-CLI for $\beta < \alpha$; and
(2) G_α is $(\alpha + 1)$-CLI but not L-α-CLI.

§1. Introduction. A Polish group is *non-archimedean* if it has a neighborhood basis of its identity element consisting of open subgroups. By a theorem of Becker and Kechris (cf. [1, Theorem 1.5.1]), a Polish group is non-archimedean iff it is homeomorphic to a closed subgroup of S_∞, the group of all permutations of $\mathbb{N}$ equipped with the pointwise convergence topology. A metric d on a group G is *left-invariant* if $d(gh, gk) = d(h, k)$ for all $g, h, k \in G$. A Polish group is *CLI* if it admits a compatible complete left-invariant metric.

Malicki [5] defined a notion of orbit tree T_G for each closed subgroup G of S_∞, and showed that G is CLI iff T_G is well-founded. Moreover, he proved that the heights of orbit trees of all CLI closed subgroups of S_∞ are cofinal in ω_1. Malicki proved that the family of all CLI groups is coanalytic non-Borel based on this cofinality. After that, Xuan defined a different kind of orbit trees and showed that, a closed subgroup of S_∞ is locally compact iff its orbit tree has finite height (cf. [6, Theorem 3.7]). It is worth noting that both kinds of orbit trees defined by Malicki and Xuan are all defined on closed subgroups of S_∞ rather than directly on non-archimedean Polish groups. As a result, two topologically isomorphic closed subgroups of S_∞ can have completely different orbit trees, and even the rank of their orbit trees can be different. This suggests that one cannot use ranks of orbit trees directly to define a hierarchy.

In this article, for a given non-archimedean CLI Polish group G, we use a neighborhood basis of the identity 1_G to define a new type of orbit trees. This differs from the approach which employs a closed subgroup of S_∞ topologically

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isomorphic to G. More specifically, let $\mathcal{G} = (G_n)$ be a decreasing sequence of open subgroups of G with $G_0 = G$, such that (G_n) forms a neighborhood basis of 1_G. We will define a well-founded tree $T_{\mathcal{G}}^{X(\mathcal{G})}$ and denote its rank by $\rho(\mathcal{G})$. We let $\mathrm{rank}(G)$ be the ordinal given by $\max\{\beta : \rho(\mathcal{G}) \geq \omega \cdot \beta\}$. Then we shall prove that the following are independent from the choice of $\mathcal{G}$: (a) the value of the ordinal $\mathrm{rank}(G)$; and (b) whether $\rho(\mathcal{G})$ is a limit ordinal or not. These facts allow us to form a well-defined hierarchy on the class of non-archimedean CLI Polish groups: given an ordinal $\alpha < \omega_1$,

(1) if $\rho(\mathcal{G}) \leq \omega \cdot \alpha$, we say G is α-CLI;
(2) if $\rho(\mathcal{G}) \leq \omega \cdot \alpha + m$ for some $m < \omega$, i.e., $\mathrm{rank}(G) \leq \alpha$, we say G is L-α-CLI.

It is clear that, if G is L-α-CLI, then it is also $(\alpha + 1)$-CLI.

The following theorem shows that the hierarchy classifies non-archimedean CLI Polish groups in a good manner:

THEOREM 1.1. *Let G be a non-archimedean CLI Polish group and α be a countable ordinal. Then*:

(1) *G is 0-CLI iff $G = \{1_G\}$*;
(2) *G is L-0-CLI iff G is discrete*;
(3) *G is 1-CLI iff G is TSI, i.e., G admits a compatible complete two-sided invariant metric*;
(4) *G is L-α-CLI iff G is locally α-CLI, i.e., G has an open subgroup which is α-CLI.*

It is well-known that all compact Polish groups are TSI (cf. [2, Theorem 2.1.5]), and all locally compact Polish groups are CLI (cf. [2, Theorem 2.2.5]). Now we know that all compact non-archimedean Polish groups are 1-CLI, and all locally compact non-archimedean Polish groups are L-1-CLI.

THEOREM 1.2. *Let G be a non-archimedean CLI Polish group and α be a countable ordinal. Assume H and N are closed subgroups of G, and that N is normal in G. If G is α-CLI (or L-α-CLI), so are H and G/N. In particular, we have* $\mathrm{rank}(H) \leq \mathrm{rank}(G)$ *and* $\mathrm{rank}(G/N) \leq \mathrm{rank}(G)$.

THEOREM 1.3. *Let (G^i) be a sequence of non-archimedean CLI Polish groups, $\alpha < \omega_1$, and let $G = \prod_i G^i$. Then we have*:

(1) *G is α-CLI iff all G^i are α-CLI*; *and*
(2) *G is L-α-CLI iff all G^i are L-α-CLI and for all but finitely many i, G^i is α-CLI.*

Finally, we prove the following theorem, which indicates that this hierarchy is proper:

THEOREM 1.4. *For any $\alpha < \omega_1$, there exist two non-archimedean CLI Polish groups G_α and H_α with* $\mathrm{rank}(G_\alpha) = \mathrm{rank}(H_\alpha) = \alpha$ *such that H_α is α-CLI and G_α is L-α-CLI but not α-CLI.*

§2. Preliminaries.

We denote the class of all ordinals by Ord. For any $\alpha \in \mathrm{Ord}$, we define

$$\omega(\alpha) = \max\{0, \lambda : \lambda \leq \alpha \text{ is a limit ordinal}\}.$$

Then $\alpha = \omega(\alpha) + m$ for some $m < \omega$.

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Let E be an equivalence relation on a set X, $x \in X$, and $A \subseteq X$. The E-equivalence class of x is $[x]_E = \{y \in X : xEy\}$. Similarly, the E-saturation of A is $[A]_E = \{y \in X : \exists z \in A\,(yEz)\}$.

The identity element of a group G is denoted by 1_G. Let H be a subgroup of G, we denote the set of all left-cosets of H by G/H.

A topological space is *Polish* if it is separable and completely metrizable. A topological group is *Polish* if its underlying topology is Polish. Let G be a Polish group and X a Polish space, an action of G on X, denoted by $G \curvearrowright X$, is a map $a : G \times X \to X$ that satisfies $a(1_G, x) = x$ and $a(gh, x) = a(g, a(h, x))$ for $g, h \in G$ and $x \in X$. The pair (X, a) is called a *Polish G-space* if a is continuous. For brevity, we write $g \cdot x$ in place of $a(g, x)$. The *orbit equivalence relation* E_G^X is defined as $xE_G^X y \iff \exists g \in G\,(g \cdot x = y)$. Note that the E_G^X-equivalence class of x is $G \cdot x = \{g \cdot x : g \in G\}$, which is also called the *G-orbit* of x. Similarly, for $A \subseteq X$, the E_G^X-saturation of A is $G \cdot A = \{g \cdot x : g \in G \wedge x \in A\}$.

Let $<$ be a binary relation on a set T. We say that $(T, <)$ is a *tree* if:

(1) $\forall s \in T\,(s \not< s)$,
(2) $\forall s, t, u \in T\,((s < t \wedge t < u) \Rightarrow s < u)$,
(3) $\forall s \in T\,(|\{t \in T : t < s\}| < \omega \wedge \forall t, u < s\,(t = u \vee t < u \vee u < t))$.

For $s \in T$, we define $\mathrm{lh}(s) = |\{t \in T : t < s\}|$, which is called the length of s. It is clear that $s < t$ implies $\mathrm{lh}(s) < \mathrm{lh}(t)$. For $n < \omega$, we denote the nth level of T by

$$L_n(T) = \{s \in T : \mathrm{lh}(s) = n\}.$$

Each element in $L_0(T)$ is called a *root* of T.

Let $(T, <)$ be a tree. We say that T is *well-founded* if any non-empty subset of T contains at least a maximal element, or equivalently (under AC), if T contains no infinite strictly increasing sequence. Let T be a well-founded tree. We define the rank function $\rho_T : T \to \mathrm{Ord}$ by transfinite induction as

$$\rho_T(s) = \sup\{\rho_T(t) + 1 : s < t \wedge t \in T\}.$$

If $\rho_T(s) = 0$, we say that s is a *terminal* of T. Then we define

$$\rho(T) = \sup\{\rho_T(s) + 1 : s \in T\}.$$

So $\rho(T) = 0$ iff $T = \emptyset$. It is clear that $\rho(T) = \sup\{\rho_T(s) + 1 : s \in L_0(T)\}$. If $L_0(T) = \{s_0\}$ is a singleton, then $\rho(T) = \rho_T(s_0) + 1$ is a successor ordinal.

For $s \in T$, we define

$$T_s = \{t \in T : s = t \vee s < t\}.$$

Since $L_0(T_s) = \{s\}$, we have $\rho(T_s) = \rho_T(s) + 1$. For the convenience of discussion, we let $T_s = \emptyset$ whenever $s \notin T$; in other words, $\rho(T_s) = 0$. This convention will be useful in some proofs (see Lemma 4.3). Note that $\rho(T_s)$ is always a non-limit ordinal regardless of $s \in T$ or not.

First, we note the following facts:

Proposition 2.1. *Let T be a well-founded tree, then*

$$\rho(T) = \sup\{\rho(T_s) : s \in T\} = \sup\{\rho(T_s) : s \in T \wedge s \in L_0(T)\},$$

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and for all $s \in T$,

$$\begin{aligned}\rho(T_s) &= \sup\{\rho(T_t) : s < t \wedge t \in T\} + 1\\ &= \sup\{\rho(T_t) : s < t \wedge t \in T \wedge \mathrm{lh}(t) = \mathrm{lh}(s) + 1\} + 1.\end{aligned}$$

PROPOSITION 2.2. *Let $(T, <)$ be a well-founded tree, $k < \omega$. Then we have:*

(1) $\sup\{\rho(T_s) : s \in L_k(T)\} \leq \rho(T) \leq \sup\{\rho(T_s) : s \in L_k(T)\} + k$;
(2) $\omega(\rho(T)) = \omega(\sup\{\rho(T_s) : s \in L_k(T)\})$;
(3) *if $\rho(T_s) \geq \alpha$ for some $s \in L_k(T)$, then $\rho(T) \geq \alpha + k$.*

PROOF. It is routine to prove clause (1) by induction on k based on Proposition 2.1. Clause (2) is an easy corollary of (1). And clause (3) is trivial. ⊣

Let $(S, <)$ and $(T, <)$ be two trees. A map $\phi : S \to T$ is said to be an *order-preserving map* if

$$\forall s, t \in S\, (s < t \Rightarrow \phi(s) < \phi(t)).$$

It is said to be an *order-preserving embedding (isomorphism)* if it is injective (bijective) and

$$\forall s, t \in S\, (s < t \iff \phi(s) < \phi(t)).$$

In particular, an order-preserving map ϕ is said to be *Lipschitz* if $\mathrm{lh}(\phi(s)) = \mathrm{lh}(s)$ for all $s \in S$.

PROPOSITION 2.3. *Let $(S, <)$ and $(T, <)$ be trees, and assume that $(T, <)$ is well-founded. If there exists an order-preserving map $\phi : S \to T$, then $(S, <)$ is also well-founded, and $\rho(S) \leq \rho(T)$ holds.*

PROOF. If S contains an infinite strictly increasing sequence (s_n), then $(\phi(s_n))$ is an infinite strictly increasing sequence in T, contradicting that $(T, <)$ is well-founded.

Now we prove that $\rho_S(s) \leq \rho_T(\phi(s))$ for $s \in S$ by induction on $\rho_S(s)$. For the basis of induction, note that $\rho_S(s) = 0 \leq \rho_T(\phi(s))$ clearly holds; and for the inductive step, we have the following inequality:

$$\begin{aligned}\rho_S(s) &= \sup\{\rho_S(u) + 1 : s < u \wedge u \in S\}\\ &\leq \sup\{\rho_T(\phi(u)) + 1 : s < u \wedge u \in S\}\\ &\leq \sup\{\rho_T(t) + 1 : \phi(s) < t \wedge t \in T\} = \rho_T(\phi(s)).\end{aligned}$$

This implies $\rho(S) \leq \rho(T)$. ⊣

§3. Definition of the hierarchy

DEFINITION 3.1. Let X be a set, and $\mathcal{E} = (E_n)$ be a decreasing sequence of equivalence relations on X, i.e., $E_n \supseteq E_{n+1}$ for each $n < \omega$. We define

$$T^X_{\mathcal{E}} = \{(n, C) : \exists x \in X\, (C = [x]_{E_n} \neq \{x\})\}.$$

For $(n, C), (m, D) \in T^X_{\mathcal{E}}$, we define

$$(n, C) < (m, D) \iff n < m \wedge C \supseteq D.$$

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It is straightforward to check that $(T^X_{\mathcal{E}}, <)$ is a tree. Note that $(n, C) \in T^X_{\mathcal{E}}$ iff C is a non-singleton equivalence class of E_n. Here we omit all singleton classes in our definition. This will be crucial in the proof of Lemma 3.3.

DEFINITION 3.2. Let G be a non-archimedean Polish group. We denote by $\mathrm{dgnb}(G)$ the set of all decreasing sequences $\mathcal{G} = (G_n)$ of open subgroups of G, such that $G_0 = G$ and (G_n) forms a neighborhood basis of 1_G. Here 'dgnb' stands for 'decreasing group neighborhood basis'.

Let X be a countable discrete Polish G-space, $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. We define $E_n = E^X_{G_n}$, i.e., $xE_ny \iff \exists g \in G_n\,(g \cdot x = y)$, and hence $[x]_{E_n} = G_n \cdot x$. Then we write $\mathcal{E} = (E_n)$ and

$$T^X_{\mathcal{G}} = T^X_{\mathcal{E}}.$$

Therefore, $(n, C) \in T^X_{\mathcal{G}}$ iff C is a non-singleton G_n-orbit. We point out that our definition is different from Malicki's and Xuan's, which are based on infinite orbits.

The following lemma builds a connection between non-archimedean CLI Polish groups and well-founded trees. Similar results also appear in [5, Theorem 3] and [6, Theorem 3.9].

LEMMA 3.3. *Let G be a non-archimedean CLI Polish group, X a countable discrete Polish G-space, and let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. Then $T^X_{\mathcal{G}}$ is well-founded.*

PROOF. Assume for contradiction that $T^X_{\mathcal{G}}$ is ill-founded, then there exists an infinite sequence (n, C_n), $n < \omega$ in $T^X_{\mathcal{G}}$ with $C_n \supseteq C_{n+1}$ for each $n < \omega$.

Let d be a compatible complete left-invariant metric on G.

Fix an $x_0 \in C_0$. Then we have $G_0 \cdot x_0 = C_0 \supseteq C_1$, so we can find a $g_0 \in G_0$ such that $g_0 \cdot x_0 \in C_1$. Inductively, we can find a $g_n \in G_n$ for each $n < \omega$ such that $g_ng_{n-1} \dots g_0 \cdot x_0 \in C_{n+1}$. Put $h_n = g_0^{-1} \dots g_n^{-1}$ for each $n < \omega$. Then, for any $n, p < \omega$, we have $d(h_{n+p}, h_n) = d(h_n^{-1}h_{n+p}, 1_G) = d(g_{n+1}^{-1} \dots g_{n+p}^{-1}, 1_G) \leq \mathrm{diam}(G_{n+1}) \to 0$ as $n \to \infty$. It follows that (h_n) is a d-Cauchy sequence in G, so it converges to some $h \in G$.

Let $x_\infty = h^{-1} \cdot x_0$. Since $h_n \to h$, we have $h_n^{-1} \to h^{-1}$, and hence $h_n^{-1} \cdot x_0 \to h^{-1} \cdot x_0 = x_\infty$. Note that X is discrete, so there exists an integer N such that $h_n^{-1} \cdot x_0 = x_\infty$ for any $n > N$, thus $x_\infty = g_ng_{n-1} \dots g_0 \cdot x_0 \in C_{n+1}$. This implies that $x_\infty \in \bigcap_n C_n$ and $G_n \cdot x_\infty = C_n$ for each $n < \omega$.

Finally, define $G_{x_\infty} = \{g \in G : g \cdot x_\infty = x_\infty\}$ and put $f : G \to X$ as $f(g) = g \cdot x_\infty$. Since f is continuous and $\{x_\infty\}$ is clopen in X, it follows that $G_{x_\infty} = f^{-1}(x_\infty)$ is a clopen subgroup of G. So there exists an $m < \omega$ such that $G_m \subseteq G_{x_\infty}$. We now have $C_m = G_m \cdot x_\infty = \{x_\infty\}$, which is a singleton G_m-orbit, contradicting that $(m, C_m) \in T^X_{\mathcal{G}}$. ⊣

Given two sets X and Y. Let $\mathcal{E} = (E_n)$ and $\mathcal{F} = (F_n)$ be two decreasing sequences of equivalence relations on X and Y, respectively. Let $\theta : X \to Y$ be an injection. For $(n, C) \in T^X_{\mathcal{E}}$, define $\phi(n, C) = (n, [\theta(C)]_{F_n})$.

PROPOSITION 3.4. (1) *If θ is an (E_n, F_n)-homomorphism for each $n < \omega$, i.e.,*

$$\forall n < \omega\, \forall x, x' \in X\, (xE_nx' \Rightarrow \theta(x)F_n\theta(x')),$$

then ϕ is an order-preserving map from $T^X_{\mathcal{E}}$ to $T^Y_{\mathcal{F}}$.

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(2) *If θ is an reduction of E_n to F_n for each $n < \omega$, i.e.,*

$$\forall n < \omega \, \forall x, x' \in X \, (x E_n x' \Leftrightarrow \theta(x) F_n \theta(x')),$$

then ϕ is an Lipschitz embedding from $T^X_{\mathcal{E}}$ to $T^Y_{\mathcal{F}}$. In particular, if θ is a bijection, then ϕ is an order-preserving isomorphism.

PROOF. Note that θ is injective. Let us prove that $\phi(n, C) \in T^Y_{\mathcal{F}}$ holds for an arbitrary $(n, C) \in T^X_{\mathcal{E}}$. Indeed, it suffices to note that since C is not a singleton, neither is $[\phi(C)]_{F_n}$. This proves $\phi(n, C) \in T^Y_{\mathcal{F}}$. The rest of the proof is trivial. ⊣

DEFINITION 3.5. Let $(T, <)$ be a tree, (n_i) a strictly increasing sequence of natural numbers. We define

$$T|(n_i) = \bigcup_i L_{n_i}(T).$$

Note that $(T|(n_i), <)$ is also a tree, and that $L_j(T|(n_i)) = L_{n_j}(T)$ holds for each $j < \omega$. We call $T|(n_i)$ a *level-subtree* of T.

LEMMA 3.6. *Let $(T, <)$ be a well-founded tree, (n_i) a strictly increasing sequence of natural numbers. Then we have*:

$$\omega(\rho(T)) \le \rho(T|(n_i)) \le \rho(T).$$

In particular, if $\rho(T)$ is a limit ordinal, then $\rho(T|(n_i)) = \rho(T)$.

PROOF. $\rho(T|(n_i)) \le \rho(T)$ follows from Proposition 2.3. We prove $\omega(\rho(T)) \le \rho(T|(n_i))$ by induction on $\rho(T)$.

First, if $\rho(T) < \omega$, then $\rho(T) = \min\{n : L_n(T) = \emptyset\}$, and hence $\rho(T|(n_i)) = \min\{i : L_{n_i}(T) = \emptyset\}$. So we have $\omega(\rho(T)) = 0 \le \rho(T|(n_i))$.

For $t \in T|(n_i)$, note that $(T|(n_i))_t = \{u \in T|(n_i) : t = u \vee t < u\}$ is a level-subtree of T_t as well.

Case 1: If $\rho(T)$ is a limit ordinal, then $\omega(\rho(T)) = \rho(T)$. Proposition 2.1 implies that $\rho(T) = \sup\{\rho(T_t) : t \in T\}$. Since $\rho(T)$ is a limit ordinal and $\rho(T_t)$ is a successor ordinal, we have $\rho(T_t) < \rho(T)$ for $t \in T$.

Subcase 1.1: If there is no maximum in $\{\omega(\rho(T_t)) : t \in T\}$, we have

$$\rho(T) = \sup\{\rho(T_t) : t \in T\} = \sup\{\omega(\rho(T_t)) : t \in T\}.$$

By the inductive hypothesis, we have $\omega(\rho(T_t)) \le \rho((T|(n_i))_t)$. Proposition 2.3 gives $\rho((T|(n_i))_t) \le \rho(T|(n_i))$ for each $t \in T$, so we have $\rho(T|(n_i)) = \rho(T)$.

Subcase 1.2: Otherwise, let $\alpha = \max\{\omega(\rho(T_t)) : t \in T\}$. Since $\rho(T_t) < \rho(T)$ for $t \in T$, we have $\rho(T) = \alpha + \omega$. We can find a sequence $t_m, m < \omega$ in L_0 such that $\rho(T_{t_m}) = \alpha + k_m$ with $\sup\{k_m : m < \omega\} = \omega$. By Proposition 2.1, for each $m < \omega$ and $n < k_m$ we can find $t^n_m \in L_n(T)$ such that $t_m = t^0_m < t^1_m < \cdots < t^{k_m - 1}_m$ and $\rho(T_{t^n_m}) = \alpha + (k_m - n)$. For $k_m > n_0$, let i_m be the largest i such that $n_i < k_m$, then $t^{n_j}_m \in L_j(T|(n_i)) = L_{n_j}(T)$ for $j \le i_m$. By the inductive hypothesis, $\alpha = \omega(\rho(T_{t^{n_j}_m})) \le \rho((T|(n_i))_{t^{n_j}_m})$. Since $\rho((T|(n_i))_{t^{n_j}_m}) \ge \rho((T|(n_i))_{t^{n_{j+1}}_m}) + 1$ for each $j < i_m$, we have $\rho((T|(n_i))_{t^{n_0}_m}) \ge \alpha + i_m$. By the definition of i_m, we have $\sup\{i_m : m < \omega\} = \omega$. This gives $\rho(T|(n_i)) = \alpha + \omega = \rho(T)$.

Case 2: If $\rho(T) = \omega(\rho(T)) + n$ with $1 \leq n < \omega$, then there exists some $t_0 \in L_0(T)$ such that $\rho(T) = \rho_T(t_0) + 1$. Since $\{u \in T|(n_i) : t_0 < u\}$ is a level-subtree of $\{u \in T : t_0 < u\}$ and $\rho(\{u \in T : t_0 < u\}) = \rho_T(t_0) < \rho(T)$, by the inductive hypothesis and Proposition 2.3, we have

$$\omega(\rho_T(t_0)) = \omega(\rho(\{u \in T : t_0 < u\})) \leq \rho(\{u \in T|(n_i) : t_0 < u\}) \leq \rho(T|(n_i)).$$

It follows that $\omega(\rho(T)) = \omega(\rho_T(t_0)) \leq \rho(T|(n_i))$. ⊣

In general, the tree $T_{\mathcal{G}}^X$ and the ordinal $\rho(T_{\mathcal{G}}^X)$ depend on $\mathcal{G}$, and not only on the action $G \curvearrowright X$. The following key lemma shows that $\omega(\rho(T_{\mathcal{G}}^X))$ is independent from the choice of $\mathcal{G}$.

Lemma 3.7. *Let G be a non-archimedean CLI Polish group, X be a countable discrete Polish G-space, and let $\mathcal{G} = (G_n), \mathcal{G}' = (G_n') \in \mathrm{dgnb}(G)$. Then*

$$\omega(\rho(T_{\mathcal{G}}^X)) = \omega(\rho(T_{\mathcal{G}'}^X)).$$

Proof. (1) First, we consider the case where (G_n') is a subsequence of (G_n), i.e., there is a strictly increasing sequence (n_i) of natural numbers such that $G_i' = G_{n_i}$ for each $i < \omega$.

We define $\psi : T_{\mathcal{G}'}^X \to T_{\mathcal{G}}^X$ as $\psi(i, C) = (n_i, C)$. It is clear that ψ is an order-preserving isomorphism from $T_{\mathcal{G}'}^X$ onto $T_{\mathcal{G}}^X|(n_i)$. It follows that

$$\omega(\rho(T_{\mathcal{G}}^X)) \leq \rho(T_{\mathcal{G}'}^X) = \rho(T_{\mathcal{G}}^X|(n_i)) \leq \rho(T_{\mathcal{G}}^X).$$

So we have $\omega(\rho(T_{\mathcal{G}}^X)) = \omega(\rho(T_{\mathcal{G}'}^X))$.

(2) Since $(G_n), (G_n') \in \mathrm{dgnb}(G)$, we can find two strictly increasing natural numbers (n_i) and (m_j) such that $n_0 = 0, m_0 = 0$, and

$$G_0 \supseteq G_{m_0}' \supseteq G_{n_1} \supseteq G_{m_1}' \supseteq G_{n_2} \supseteq \dots.$$

Define $H_{2i} = G_{n_i}$ and $H_{2i+1} = G_{m_i}'$ for each $i < \omega$. Then $(H_k) \in \mathrm{dgnb}(G)$. Put $\mathcal{H} = (H_k), \mathcal{K} = (G_{n_i})$, and $\mathcal{K}' = (G_{m_i}')$.

Note that (G_{n_i}) is a subsequence of (G_n) and also a subsequence of (H_k). From (1), we obtain

$$\omega(\rho(T_{\mathcal{G}}^X)) = \omega(\rho(T_{\mathcal{K}}^X)) = \omega(\rho(T_{\mathcal{H}}^X)).$$

Similarly, we obtain

$$\omega(\rho(T_{\mathcal{G}'}^X)) = \omega(\rho(T_{\mathcal{K}'}^X)) = \omega(\rho(T_{\mathcal{H}}^X)).$$

So we have $\omega(\rho(T_{\mathcal{G}}^X)) = \omega(\rho(T_{\mathcal{G}'}^X))$. ⊣

Now we are going to find a special G-space $X(\mathcal{G})$ such that $\omega(\rho(T_{\mathcal{G}}^{X(\mathcal{G})}))$ attains the maximum value among all $\omega(\rho(T_{\mathcal{G}}^X))$. This leads to the conclusion that the value of $\omega(\rho(T_{\mathcal{G}}^{X(\mathcal{G})}))$ is determined by G itself.

Lemma 3.8. *Given two sets X and Y. Let $\mathcal{E} = (E_n)$ and $\mathcal{F} = (F_n)$ be two decreasing sequences of equivalence relations on X and Y, respectively. Let $\theta : X \to Y$ be a surjection such that*

$$\forall n < \omega\, \forall x \in X\, (\theta([x]_{E_n}) = [\theta(x)]_{F_n}).$$

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Then there exists a Lipschitz embedding $\psi : T^Y_{\mathcal{F}} \to T^X_{\mathcal{E}}$. In particular, if $T^X_{\mathcal{E}}$ is well-founded, so is $T^Y_{\mathcal{F}}$, and $\rho(T^Y_{\mathcal{F}}) \leq \rho(T^X_{\mathcal{E}})$ holds.

PROOF. For any $(n, C) \in T^Y_{\mathcal{F}}$, we construct $\psi(n, C)$ by induction on n such that $\psi(n, C) = (n, [x]_{E_n})$ for some $x \in X$ with $[\theta(x)]_{F_n} = C$.

For $n = 0$, since θ is a surjection, we can find an $x \in X$ with $\theta(x) \in C$. Then we let $\psi(0, C) = (0, [x]_{E_0})$.

For $n > 0$, since C is an F_n-equivalence class, there exists a unique F_{n-1}-equivalence class $D \supseteq C$. By the inductive hypothesis, we can find an $x' \in X$ such that $\psi(n-1, D) = (n-1, [x']_{E_{n-1}})$ with $[\theta(x')]_{F_{n-1}} = D$. Since $\theta([x']_{E_{n-1}}) = [\theta(x')]_{F_{n-1}} = D \supseteq C$, we can find an $x \in [x']_{E_{n-1}}$ such that $\theta(x) \in C$. Then we put $\psi(n, C) = (n, [x]_{E_n})$.

Since C is not a singleton, the equality $\theta([x]_{E_n}) = [\theta(x)]_{F_n} = C$ implies that $[x]_{E_n}$ is not a singleton. So $\psi(n, C) \in T^X_{\mathcal{E}}$. From the construction, it is routine to check that $\psi : T^Y_{\mathcal{F}} \to T^X_{\mathcal{E}}$ is a Lipschitz embedding.

Finally, if $T^X_{\mathcal{E}}$ is well-founded, then by Proposition 2.3, $T^Y_{\mathcal{F}}$ is well-founded as well, and $\rho(T^Y_{\mathcal{F}}) \leq \rho(T^X_{\mathcal{E}})$. ⊣

DEFINITION 3.9. Let G be a non-archimedean CLI Polish group, and let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. For each $k < \omega$, we define an action $G \curvearrowright G/G_k$ as $g \cdot hG_k = ghG_k$ for $g, h \in G$, and let $\rho^k(\mathcal{G})$ denote $\rho(T^{G/G_k}_{\mathcal{G}})$. Afterwards, we let $X(\mathcal{G}) = \bigcup_k G/G_k$ and $\rho(\mathcal{G}) = \rho(T^{X(\mathcal{G})}_{\mathcal{G}})$.

Note that $G \cdot gG_k = \{hgG_k : h \in G\} = G/G_k$ for any $g \in G$.

It is clear that $\rho^0(\mathcal{G}) = 0$, since $G = G_0$.

LEMMA 3.10. (1) $\rho(\mathcal{G}) = \sup\{\rho^k(\mathcal{G}) : k < \omega\}$.

(2) *$(\rho^k(\mathcal{G}))$ is an increasing sequence of countable non-limit ordinals.*

PROOF. (1) Note that $L_0(T^{X(\mathcal{G})}_{\mathcal{G}}) = \{(0, G/G_k) : G \neq G_k\}$. We point out that $(T^{X(\mathcal{G})}_{\mathcal{G}})_{(0,G/G_k)} \cong T^{G/G_k}_{\mathcal{G}}$ for $G \neq G_k$, and $(T^{X(\mathcal{G})}_{\mathcal{G}})_{(0,G/G_k)} = T^{G/G_k}_{\mathcal{G}} = \emptyset$ for $G = G_k$. So

$$\begin{aligned}\rho(\mathcal{G}) &= \rho(T^{X(\mathcal{G})}_{\mathcal{G}}) = \sup\{\rho((T^{X(\mathcal{G})}_{\mathcal{G}})_{(0,G/G_k)}) : k < \omega\} \\ &= \sup\{\rho(T^{G/G_k}_{\mathcal{G}}) : k < \omega\} = \sup\{\rho^k(\mathcal{G}) : k < \omega\}.\end{aligned}$$

(2) Given $k < \omega$, we define $\theta : G/G_{k+1} \to G/G_k$ as $\theta(gG_{k+1}) = gG_k$ for $g \in G$. It is clear that θ is well-defined and is a surjection. Furthermore, for $n < \omega$ and $g \in G$, we have

$$\begin{aligned}\theta(G_n \cdot gG_{k+1}) &= \{\theta(hgG_{k+1}) : h \in G_n\} \\ &= \{hgG_k : h \in G_n\} = G_n \cdot gG_k = G_n \cdot \theta(gG_{k+1}).\end{aligned}$$

From Lemma 3.8, we have

$$\rho(T^{G/G_k}_{\mathcal{G}}) \leq \rho(T^{G/G_{k+1}}_{\mathcal{G}}),$$

i.e., $(\rho^k(\mathcal{G}))$ is increasing.

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For each $k < \omega$, since $T_{\mathcal{G}}^{G/G_k}$ is countable, $\rho^k(\mathcal{G})$ is countable as well. If $G = G_k$, then $T_{\mathcal{G}}^{G/G_k} = \emptyset$; otherwise if $G \neq G_k$, then $L_0(T_{\mathcal{G}}^{G/G_k}) = \{(0, G/G_k)\}$ is a singleton. So $\rho^k(\mathcal{G}) = \rho(T_{\mathcal{G}}^{G/G_k})$ is either 0 or a successor ordinal. ⊣

Recall that a G-space X is said to be *transitive* if X itself is an orbit.

LEMMA 3.11. *Let G be a non-archimedean CLI Polish group, X a countable discrete transitive Polish G-space, and let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. Then we can find some $k < \omega$ such that $\rho(T_{\mathcal{G}}^X) \leq \rho^k(\mathcal{G})$.*

PROOF. Fix an $x \in X$. Since $\{x\}$ is clopen in X, by the continuity of the group action of G on X, we know G_x is a clopen subgroup of G. So there is some $k < \omega$ such that $G_k \subseteq G_x$. Then we can define $\theta : G/G_k \to X$ as $\theta(gG_k) = g \cdot x$ for $g \in G$. Since X is a transitive G-space, θ is surjective and $\theta(G_n \cdot gG_k) = G_n \cdot \theta(gG_k)$ for each $n < \omega$ and $g \in G$. By Lemma 3.8, we have $\rho(T_{\mathcal{G}}^X) \leq \rho^k(\mathcal{G})$. ⊣

COROLLARY 3.12. *Let G be a non-archimedean CLI Polish group, X a countable discrete Polish G-space, and let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. Then we have*

$$\rho(T_{\mathcal{G}}^X) \leq \rho(\mathcal{G}).$$

PROOF. Note that $L_0(T_{\mathcal{G}}^X) = \{(0, G \cdot x) : x \in X \wedge G \cdot x \neq \{x\}\}$ and $T_{\mathcal{G}}^{G\cdot x} \cong (T_{\mathcal{G}}^X)_{(0,G\cdot x)}$ for $G \cdot x \neq \{x\}$, so we have

$$\rho(T_{\mathcal{G}}^X) = \sup\{\rho((T_{\mathcal{G}}^X)_{(0,G\cdot x)}) : x \in X\} = \sup\{\rho(T_{\mathcal{G}}^{G\cdot x}) : x \in X\}.$$

By Lemma 3.11, we have

$$\rho(T_{\mathcal{G}}^X) \leq \sup\{\rho^k(\mathcal{G}) : k < \omega\} = \rho(\mathcal{G}).$$ ⊣

COROLLARY 3.13. *Let G be a non-archimedean Polish group, $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. Then G is CLI iff $T_{\mathcal{G}}^{G/G_k}$ is well-founded for any $k < \omega$.*

PROOF. The $(\Rightarrow)$ part follows from Lemma 3.3.
$(\Leftarrow)$. Given a countable Polish G-space X. Following the arguments in the proof of Lemma 3.11, we can see that, for any $x \in X$, there is a $k < \omega$ and a Lipschitz embedding from $T_{\mathcal{G}}^{G\cdot x}$ to $T_{\mathcal{G}}^{G/G_k}$. Since $T_{\mathcal{G}}^{G/G_k}$ is well-founded, so is $T_{\mathcal{G}}^{G\cdot x}$. By the arbitrariness of $x \in X$, we conclude that $T_{\mathcal{G}}^X$ is also well-founded. Combining this with [5, Theorem 6], it follows that G is CLI. ⊣

THEOREM 3.14. *Let G be a non-archimedean CLI Polish group, and $\mathcal{G} = (G_n)$, $\mathcal{G}' = (G'_n) \in \mathrm{dgnb}(G)$. Then $\omega(\rho(\mathcal{G})) = \omega(\rho(\mathcal{G}'))$ holds.*

PROOF. By Corollary 3.12, we have $\rho(T_{\mathcal{G}'}^{X(\mathcal{G})}) \leq \rho(\mathcal{G}')$. Thus Lemma 3.7 implies

$$\omega(\rho(\mathcal{G})) = \omega(\rho(T_{\mathcal{G}}^{X(\mathcal{G})})) = \omega(\rho(T_{\mathcal{G}'}^{X(\mathcal{G})})) \leq \omega(\rho(\mathcal{G}')),$$

and vice versa. ⊣

By the preceding theorem, there is a unique ordinal $\beta < \omega_1$, which is independent from the choice of $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$, such that

$$\omega(\rho(\mathcal{G})) = \omega \cdot \beta.$$

Consequently, we can define a rank for a given non-archimedean CLI Polish group as follows:

DEFINITION 3.15. Suppose G is a non-archimedean CLI Polish group. We define

$$\mathrm{rank}(G) = \text{the unique } \beta \text{ such that } \omega(\rho(\mathcal{G})) = \omega \cdot \beta$$

for some $\mathcal{G} \in \mathrm{dgnb}(G)$.

LEMMA 3.16. (1) *If* $\rho(\mathcal{G}) = \omega \cdot \mathrm{rank}(G)$, *then either* $\mathrm{rank}(G) = 0$ *or* $\rho^k(\mathcal{G}) < \omega \cdot \mathrm{rank}(G)$ *for any* $k < \omega$.

(2) *If* $\rho(\mathcal{G}) > \omega \cdot \mathrm{rank}(G)$, *then there exists an* $m > 0$ *such that* $\rho^k(\mathcal{G}) = \omega \cdot \mathrm{rank}(G) + m$ *for large enough* $k < \omega$.

PROOF. (1) If $\mathrm{rank}(G) > 0$, then $\omega \cdot \mathrm{rank}(G)$ is a limit ordinal. By Lemma 3.10, $\rho^k(\mathcal{G})$ is either 0 or a successor ordinal, so $\rho^k(\mathcal{G}) < \omega \cdot \mathrm{rank}(G)$ for any $k < \omega$.

(2) Clearly, $\rho(\mathcal{G}) = \omega \cdot \mathrm{rank}(G) + m$ for some $0 < m < \omega$. Again by Lemma 3.10, $(\rho^k(\mathcal{G}))$ is increasing, so $\rho^k(\mathcal{G}) = \omega \cdot \mathrm{rank}(G) + m$ for large enough $k < \omega$. ⊣

THEOREM 3.17. *Let* G *be a non-archimedean CLI Polish group,* $\mathcal{G} = (G_n), \mathcal{G}' = (G_n') \in \mathrm{dgnb}(G)$. *Then* $\rho(\mathcal{G}) = \omega \cdot \mathrm{rank}(G)$ *iff* $\rho(\mathcal{G}') = \omega \cdot \mathrm{rank}(G)$.

PROOF. If $\mathrm{rank}(G) = 0$, then $\rho(\mathcal{G}) = \omega \cdot \mathrm{rank}(G)$ implies that $\rho^k(\mathcal{G}) = 0$ for any $k < \omega$. So $T_{\mathcal{G}}^{G/G_k} = \emptyset$, and hence $G_0 \cdot G_k \notin T_{\mathcal{G}}^{G/G_k}$. It follows that $G_0 \cdot G_k = \{G_k\}$, i.e., $G = G_0 = G_k$ for any $k < \omega$. This implies that $G = \{1_G\}$. Then we can easily see that $T_{\mathcal{G}'}^{G/G_k'} = \emptyset$ for $k < \omega$. So $\rho(\mathcal{G}') = \omega \cdot \mathrm{rank}(G)$ holds. And vice versa.

If $\mathrm{rank}(G) > 0$, assume for contradiction that $\rho(\mathcal{G}) = \omega \cdot \mathrm{rank}(G)$, but $\rho(\mathcal{G}') > \omega \cdot \mathrm{rank}(G)$. From Lemma 3.16, we have $\rho^k(\mathcal{G}) < \omega \cdot \mathrm{rank}(G)$ for any $k < \omega$, but $\rho^l(\mathcal{G}') = \omega \cdot \mathrm{rank}(G) + m$ for some $0 < m < \omega$ and large enough $l < \omega$. From Lemmas 3.7 and 3.11, we can conclude that for any $l < \omega$, there is some $k < \omega$ such that

$$\omega(\rho^l(\mathcal{G}')) = \omega(\rho(T_{\mathcal{G}'}^{G/G_l'})) = \omega(\rho(T_{\mathcal{G}}^{G/G_l'})) \leq \rho(T_{\mathcal{G}}^{G/G_l'}) \leq \rho^k(\mathcal{G}).$$

A contradiction! ⊣

Now we are ready to define a hierarchy on the class of non-archimedean CLI Polish groups.

DEFINITION 3.18. Let G be a non-archimedean CLI Polish group, $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$, and let $\alpha < \omega_1$ be an ordinal.

(1) If $\rho(\mathcal{G}) \leq \omega \cdot \alpha$, we say that G is *α-CLI*.
(2) If $\omega(\rho(\mathcal{G})) \leq \omega \cdot \alpha$, i.e., $\mathrm{rank}(G) \leq \alpha$, we say that G is *L-α-CLI*.

It is clear that, if G is L-α-CLI, then it is also $(\alpha+1)$-CLI.

From Theorems 3.14 and 3.17, we see that the definitions of α-CLI and L-α-CLI are independent from the choice of $\mathcal{G} \in \mathrm{dgnb}(G)$.

Recall that a metric d on a group G is *two-sided invariant* if $d(gh, gk) = d(h,k) = d(hg, kg)$ for all $g, h, k \in G$. A Polish group is *TSI* if it admits a compatible complete two-sided invariant metric.

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THEOREM 3.19. *Let G be a non-archimedean CLI Polish group. The following hold*:
(1) *G is 0-CLI iff $G = \{1_G\}$*;
(2) *G is L-0-CLI iff G is discrete*; *and*
(3) *G is 1-CLI iff G is TSI.*

PROOF. Fix a sequence $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$.

(1) It follows from the first paragraph of the proof of Theorem 3.17.

(2) If G is L-0-CLI, then $\mathrm{rank}(G) = 0$. So there is an $m < \omega$ such that $\rho(T_{\mathcal{G}}^{G/G_k}) = \rho^k(\mathcal{G}) = m$ for large enough $k < \omega$. Then we have $L_m(T_{\mathcal{G}}^{G/G_k}) = \emptyset$. This implies that $G_m \cdot G_k = \{G_k\}$. So $G_m \subseteq G_k$ for large enough $k < \omega$, and thus $G_m = \{1_G\}$. It follows that G is discrete.

On the other hand, if G is discrete, then there is an $m < \omega$ such that $G_m = \{1_G\}$. Therefore, for any $k < \omega$, we have $L_m(T_{\mathcal{G}}^{G/G_k}) = \emptyset$, and hence $\rho^k(\mathcal{G}) \le m$. This implies that $\mathrm{rank}(G) = 0$, i.e., G is L-0-CLI.

(3) If G is 1-CLI, then Lemma 3.16 implies that $\rho^k(\mathcal{G}) < \omega$ for any $k < \omega$. So there is an $m_k < \omega$ such that $L_{m_k}(T_{\mathcal{G}}^{G/G_k}) = \emptyset$. It follows that, for $g \in G$, $G_{m_k} \cdot gG_k = \{gG_k\}$, so $g^{-1}G_{m_k}g \subseteq G_k$. Put $U_k = \bigcup\{g^{-1}G_{m_k}g : g \in G\} \subseteq G_k$. We can see that (U_k) is a neighborhood basis of 1_G with $g^{-1}U_kg = U_k$ for all $g \in G$. By Klee's theorem (cf. [4] or [2, Exercise 2.1.4]), G is TSI.

On the other hand, if G is TSI, again by Klee's theorem, we can find a neighborhood basis (U_m) of 1_G with $g^{-1}U_mg = U_m$ for all $g \in G$. For any $n < \omega$, there is an $m_n < \omega$ such that $U_{m_n} \subseteq G_n$. Let $V_n = U_{m_n} \cap U_{m_n}^{-1}$ and $G_n' = \bigcup_i V_n^i$. Then G_n' is an open normal subgroup of G with $G_n' \subseteq G_n$. So $(G_n') \in \mathrm{dgnb}(G)$. Let $\mathcal{G}' = (G_n')$. Then $G_k' \cdot gG_k' = \{gG_k'\}$ for all $g \in G$ and $k < \omega$, thus $L_k(T_{\mathcal{G}'}^{G/G_k'}) = \emptyset$. So $\rho^k(\mathcal{G}') \le k < \omega$, and hence G is 1-CLI. ⊣

Clause (2) in the preceding theorem can be generalized to all $\alpha < \omega_1$.

DEFINITION 3.20. Let G be a non-archimedean CLI Polish group, and $\alpha < \omega_1$. We say that G is *locally α-CLI* if G has an open subgroup which is α-CLI.

THEOREM 3.21. *Let G be a non-archimedean CLI Polish group, and $\alpha < \omega_1$. Then G is L-α-CLI iff G is locally α-CLI.*

PROOF. ($\Rightarrow$). Suppose G is L-α-CLI. Without loss of generality, we may assume that G is not α-CLI. Fix a sequence $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. There exists an $m \ge 1$ such that $\rho(\mathcal{G}) = \omega \cdot \alpha + m$, and thus we can pick a $k_0 > m$ such that $\rho^k(\mathcal{G}) = \omega \cdot \alpha + m$ for any $k \ge k_0$. We will show that G_{k_0} is α-CLI.

Let $H = G_{k_0}$ and $H_n = G_{n+k_0}$ for $n < \omega$. Then $(H_n) \in \mathrm{dgnb}(H)$. Put $\mathcal{H} = (H_n)$. Given $k < \omega$, define $\phi : T_{\mathcal{H}}^{H/H_k} \to (T_{\mathcal{G}}^{G/G_{k+k_0}})_{(k_0, G_{k_0}/G_{k+k_0})}$ as $\phi(n, C) = (n + k_0, C)$. It is trivial to see that ϕ is an order-preserving isomorphism. From Proposition 2.2, since $k_0 > m$, we have

$$\rho^k(\mathcal{H}) = \rho(T_{\mathcal{H}}^{H/H_k}) = \rho((T_{\mathcal{G}}^{G/G_{k+k_0}})_{(k_0, G_{k_0}/G_{k+k_0})}) \le \omega \cdot \alpha.$$

So $\rho(\mathcal{H}) \le \omega \cdot \alpha$, and hence $H = G_{k_0}$ is α-CLI.

($\Leftarrow$). Suppose G is locally α-CLI. Let H be an open subgroup of G which is α-CLI, and let $\mathcal{H} = (H_n) \in \mathrm{dgnb}(H)$. Then $\rho^k(\mathcal{H}) \le \omega \cdot \alpha$ for $k < \omega$.

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Put $G_0 = G$ and $G_n = H_{n-1}$ for $n \geq 1$. Then $(G_n) \in \mathrm{dgnb}(G)$. Put $\mathcal{G} = (G_n)$. Given $g \in G$ and $k < \omega$, by similar arguments in the $(\Rightarrow)$ part, we have $T_{\mathcal{H}}^{H \cdot gH_k} \cong (T_{\mathcal{G}}^{G/G_{k+1}})_{(1, G_1 \cdot gG_{k+1})}$. By Lemma 3.11, there exists an $l < \omega$ such that $\rho(T_{\mathcal{H}}^{H \cdot gH_k}) \leq \rho^l(\mathcal{H})$. Therefore, by Proposition 2.1,

$$\begin{aligned}
\rho^{k+1}(\mathcal{G}) &= \rho(T_{\mathcal{G}}^{G/G_{k+1}}) \\
&\leq \sup\{\rho((T_{\mathcal{G}}^{G/G_{k+1}})_{(1, G_1 \cdot gG_{k+1})}) : g \in G\} + 1 \\
&= \sup\{\rho(T_{\mathcal{H}}^{H \cdot gH_k}) : g \in G\} + 1 \\
&\leq \sup\{\rho^l(\mathcal{H}) : l < \omega\} + 1 \\
&\leq \omega \cdot \alpha + 1.
\end{aligned}$$

So $\rho(\mathcal{G}) \leq \omega \cdot \alpha + 1$, and hence G is L-α-CLI. ⊣

§4. Properties of the hierarchy

THEOREM 4.1. *Let G be a non-archimedean CLI Polish group, H a closed subgroup of G, and $\alpha < \omega_1$. If G is α-CLI (or L-α-CLI), so is H. In particular, we have* $\mathrm{rank}(H) \leq \mathrm{rank}(G)$.

PROOF. Let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$, and put $H_n = H \cap G_n$ for $n < \omega$. It is clear that $(H_n) \in \mathrm{dgnb}(H)$. Put $\mathcal{H} = (H_n)$. We only need to show that $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$.

Given $k < \omega$, define $\theta : H/H_k \to G/G_k$ as $\theta(hH_k) = hG_k$ for $h \in H$. By Proposition 3.4, there is a Lipschitz order-preserving map from $T_{\mathcal{H}}^{H/H_k}$ to $T_{\mathcal{G}}^{G/G_k}$. So Proposition 2.3 implies

$$\rho^k(\mathcal{H}) = \rho(T_{\mathcal{H}}^{H/H_k}) \leq \rho(T_{\mathcal{G}}^{G/G_k}) = \rho^k(\mathcal{G}).$$

Then we have $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$ as desired. ⊣

THEOREM 4.2. *Let G be a non-archimedean CLI Polish group, N a closed normal subgroup of G, and $\alpha < \omega_1$. If G is α-CLI (or L-α-CLI), so is G/N. In particular, we have* $\mathrm{rank}(G/N) \leq \mathrm{rank}(G)$.

PROOF. Let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$, and put $H_n = G_n \cdot N = \{\hat{g}N : \hat{g} \in G_n\}$ for $n < \omega$. It is clear that $H_0 = G/N$ and $(H_n) \in \mathrm{dgnb}(G/N)$. Put $\mathcal{H} = (H_n)$. We only need to show that $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$.

Given $k < \omega$, define $\theta : G/G_k \to (G/N)/H_k$ as $\theta(gG_k) = (gN)H_k$ for $g \in G$. Note that for $n < \omega$,

$$\begin{aligned}
\theta(G_n \cdot gG_k) &= \theta(\{\hat{g}gG_k : \hat{g} \in G_n\}) \\
&= \{(\hat{g}gN)H_k : \hat{g} \in G_n\} \\
&= \{(\hat{g}N)(gN)H_k : \hat{g} \in G_n\} \\
&= \{(\hat{g}N)\theta(gG_k) : \hat{g} \in G_n\} \\
&= \{\hat{g}N : \hat{g} \in G_n\}\theta(gG_k) = H_n \cdot \theta(gG_k).
\end{aligned}$$

By Lemma 3.8, we have

$$\rho^k(\mathcal{H}) = \rho(T_{\mathcal{H}}^{(G/N)/H_k}) \leq \rho(T_{\mathcal{G}}^{G/G_k}) = \rho^k(\mathcal{G}).$$

So $\rho(\mathcal{H}) \leq \rho(\mathcal{G})$ holds as desired. ⊣

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The above two theorems involve closed subgroups and quotient groups. Now we turn to product groups, which are more complicated. We discuss finite product groups first.

LEMMA 4.3. *Let X, Y be two sets, $\mathcal{E} = (E_n)$ and $\mathcal{F} = (F_n)$ be two decreasing sequences of equivalence relations on X and Y, respectively. Let $\mathcal{E} \times \mathcal{F} = (E_n \times F_n)$.*

(1) *$T^{X\times Y}_{\mathcal{E}\times\mathcal{F}}$ is well-founded iff $T^X_{\mathcal{E}}$ and $T^Y_{\mathcal{F}}$ are well-founded.*

(2) *If $T^{X\times Y}_{\mathcal{E}\times\mathcal{F}}$ is well-founded, then we have*

$$\rho(T^{X\times Y}_{\mathcal{E}\times\mathcal{F}}) = \max\{\rho(T^X_{\mathcal{E}}), \rho(T^Y_{\mathcal{F}})\}.$$

PROOF. First, note that $[(x,y)]_{E_n\times F_n} = [x]_{E_n} \times [y]_{F_n}$ for all $(x,y) \in X \times Y$ and $n < \omega$.

(1) For any sequences $(x_n), (y_n)$ in X, Y, respectively, $((n, [(x_n, y_n)]_{E_n\times F_n}))$ is an infinite branch of $T^{X\times Y}_{\mathcal{E}\times\mathcal{F}}$ iff either $((n, [x_n]_{E_n}))$ or $((n, [y_n]_{F_n}))$ is an infinite branch of $T^X_{\mathcal{E}}$ or $T^Y_{\mathcal{F}}$, respectively.

(2) If $T^{X\times Y}_{\mathcal{E}\times\mathcal{F}}$ is well-founded, by (1), $T^X_{\mathcal{E}}$ and $T^Y_{\mathcal{F}}$ are also well-founded. For all $(x,y) \in X \times Y$ and $n < \omega$, note that

$$\begin{aligned} & (n, [(x,y)]_{E_n\times F_n}) \in T^{X\times Y}_{\mathcal{E}\times\mathcal{F}} \\ \iff & [(x,y)]_{E_n\times F_n} \neq \{(x,y)\} \\ \iff & [x]_{E_n} \neq \{x\} \vee [y]_{F_n} \neq \{y\} \\ \iff & (n, [x]_{E_n}) \in T^X_{\mathcal{E}} \vee (n, [y]_{F_n}) \in T^Y_{\mathcal{F}}. \end{aligned}$$

By Proposition 2.1, it is routine to prove

$$\rho((T^{X\times Y}_{\mathcal{E}\times\mathcal{F}})_{(n,[(x,y)]_{E_n\times F_n})}) = \max\{\rho((T^X_{\mathcal{E}})_{(n,[x]_{E_n})}), \rho((T^Y_{\mathcal{F}})_{(n,[y]_{F_n})})\}$$

by induction on $\rho((T^{X\times Y}_{\mathcal{E}\times\mathcal{F}})_{(n,[(x,y)]_{E_n\times F_n})})$. Taking supremum on both sides of the above formula, we get

$$\rho(T^{X\times Y}_{\mathcal{E}\times\mathcal{F}}) = \max\{\rho(T^X_{\mathcal{E}}), \rho(T^Y_{\mathcal{F}})\}. \qquad \dashv$$

COROLLARY 4.4. *Let G and H be two non-archimedean CLI Polish groups, $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$ and $\mathcal{H} = (H_n) \in \mathrm{dgnb}(H)$. Then we have $\mathcal{G} \times \mathcal{H} = (G_n \times H_n) \in \mathrm{dgnb}(G \times H)$ and*

$$\rho^k(\mathcal{G} \times \mathcal{H}) = \max\{\rho^k(\mathcal{G}), \rho^k(\mathcal{H})\} \quad (\forall k < \omega),$$

$$\rho(\mathcal{G} \times \mathcal{H}) = \max\{\rho(\mathcal{G}), \rho(\mathcal{H})\},$$

$$\mathrm{rank}(G \times H) = \max\{\mathrm{rank}(G), \mathrm{rank}(H)\}.$$

PROOF. For any $k < \omega$, put $X = G/G_k$, $Y = H/H_k$, and define E_n, F_n on X and Y for each $n < \omega$, respectively, by

$$(\hat{g}G_k, \tilde{g}G_k) \in E_n \iff \exists g \in G_n\,(g\hat{g}G_k = \tilde{g}G_k) \quad (\forall \hat{g}, \tilde{g} \in G),$$

$$(\hat{h}H_k, \tilde{h}H_k) \in F_n \iff \exists h \in H_n\,(h\hat{h}H_k = \tilde{h}H_k) \quad (\forall \hat{h}, \tilde{h} \in H).$$

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Then Lemma 4.3 gives $\rho^k(\mathcal{G}\times\mathcal{H}) = \max\{\rho^k(\mathcal{G}),\rho^k(\mathcal{H})\}$. The rest follows trivially. ⊣

Now we are ready to discuss countably infinite product groups.

LEMMA 4.5. *Let (G^i) be a sequence of non-archimedean CLI Polish groups and $\mathcal{G}^i = (G^i_n) \in \mathrm{dgnb}(G^i)$ for each $i<\omega$. We define $G = \prod_i G^i$ and*

$$G_n = \prod_{i<n} G^i_n \times \prod_{i\geq n} G^i \quad (\forall n<\omega).$$

Then $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$ and for $k<\omega$, we have

$$\rho^k(\mathcal{G}) \leq \max\{\rho^k(\mathcal{G}^i) : i<k\} + k.$$

PROOF. Given $k<\omega$, we put $Y = G^0/G^0_k \times\cdots\times G^{k-1}/G^{k-1}_k$ and define $\theta : G/G_k \to Y$ as

$$\theta((g_i)G_k) = (g_0G^0_k,\dots,g_{k-1}G^{k-1}_k)$$

for $(g_i)\in G=\prod_i G^i$. By the definition of G_k, it is easy to see that θ is a bijection. Moreover, we put

$$H_n = \begin{cases} \prod_{i<n} G^i_n \times \prod_{n\leq i<k} G^i, & n<k,\\ \prod_{i<k} G^i_n, & n\geq k,\end{cases}$$

then θ is a reduction of $E^{G/G_k}_{G_n}$ to $E^Y_{H_n}$ for each $n<\omega$. Put $\mathcal{H} = (H_n)$. By Proposition 3.4, $(n, G_n\cdot(g_i)G_k) \mapsto (n, H_n\cdot\theta((g_i)G_k))$ is an order-preserving isomorphism from $T^{G/G_k}_{\mathcal{G}}$ to $T^Y_{\mathcal{H}}$. So $\rho^k(\mathcal{G}) = \rho(T^{G/G_k}_{\mathcal{G}}) = \rho(T^Y_{\mathcal{H}})$.

For $n\geq k$ and $(g_i)\in G$, we have

$$H_n\cdot\theta((g_i)G_k) = (G^0_n\cdot g_0G^0_k)\times\cdots\times(G^{k-1}_n\cdot g_{k-1}G^{k-1}_k).$$

Lemma 4.3 gives

$$\rho((T^Y_{\mathcal{H}})_{(k,H_k\cdot\theta((g_i)G_k))}) = \max\{\rho((T^{G^i/G^i_k}_{\mathcal{G}^i})_{(k,G^i_k\cdot g_iG^i_k)}) : i<k\}.$$

Hence, by Proposition 2.2(1),

$$\begin{aligned}\rho^k(\mathcal{G}) &= \rho(T^{G/G_k}_{\mathcal{G}}) = \rho(T^Y_{\mathcal{H}})\\ &\leq \sup\{\rho((T^Y_{\mathcal{H}})_{(k,H_k\cdot\theta((g_i)G_k))}) : (g_i)\in G\}+k\\ &= \sup\{\max\{\rho((T^{G^i/G^i_k}_{\mathcal{G}^i})_{(k,G^i_k\cdot g_iG^i_k)}) : i<k\} : (g_i)\in G\}+k\\ &= \max\{\sup\{\rho((T^{G^i/G^i_k}_{\mathcal{G}^i})_{(k,G^i_k\cdot gG^i_k)}) : g\in G^i\} : i<k\}+k\\ &\leq \max\{\rho(T^{G^i/G^i_k}_{\mathcal{G}^i}) : i<k\}+k\\ &= \max\{\rho^k(\mathcal{G}^i) : i<k\}+k.\end{aligned}$$ ⊣

THEOREM 4.6. *Let (G^i) be a sequence of non-archimedean CLI Polish groups, $\alpha<\omega_1$, and let $G=\prod_i G^i$. Then we have:*

(1) *G is α-CLI iff all G^i are α-CLI; and*
(2) *G is L-α-CLI iff all G^i are L-α-CLI and for all but finitely many i, G^i is α-CLI.*

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Proof. Fix a $\mathcal{G}^i = (G^i_n) \in \mathrm{dgnb}(G^i)$ for each $i < \omega$. Put

$$G_n = \prod_{i<n} G^i_n \times \prod_{i \geq n} G^i \quad (\forall n < \omega).$$

(1) Assume G is α-CLI. Since each G^i is topologically isomorphic to a closed subgroup of G, by Theorem 4.1, G^i is also α-CLI.

On the other hand, assume all G^i are α-CLI. Now by Lemma 3.16(1), $\rho^k(\mathcal{G}^i) < \omega \cdot \alpha$ holds for all $i, k < \omega$. Then Lemma 4.5 implies

$$\rho^k(\mathcal{G}) \leq \max\{\rho^k(\mathcal{G}^i) : i < k\} + k < \omega \cdot \alpha$$

holds for each $k < \omega$. Consequently, G is α-CLI.

(2) Assume G is L-α-CLI. Since each G^i is topologically isomorphic to a closed subgroup of G, we can see G^i is L-α-CLI too. Moreover, by Theorem 3.21, there is an open subgroup H of G which is α-CLI. Hence, G_n is a clopen subgroup of H for some $n < \omega$, from which we can see this G_n is also α-CLI. Now by (1), we conclude that for all $i \geq n$, G^i is α-CLI.

On the other hand, assume all G^i are L-α-CLI, and there is an $m < \omega$ such that G^i is α-CLI for each $i \geq m$. Then (1) implies that $\prod_{i \geq m} G^i$ is α-CLI. Note that $G = G^0 \times \cdots \times G^{m-1} \times \prod_{i \geq m} G^i$. By Corollary 4.4, we have

$$\mathrm{rank}(G) = \max\{\mathrm{rank}(G^0), \ldots, \mathrm{rank}(G^{m-1}), \mathrm{rank}(\prod_{i \geq m} G^i)\} \leq \alpha,$$

i.e., G is L-α-CLI. ⊣

In the rest of this article, we will show that the notions of α-CLI, L-α-CLI, together with $\mathrm{rank}(G)$, form a proper hierarchy on the class of non-archimedean CLI Polish groups. For this purpose, we will construct groups which are α-CLI but not L-β-CLI for all $\beta < \alpha$, and groups which are L-α-CLI but not α-CLI, by induction on $\alpha < \omega_1$. We consider the case concerning successor ordinals first.

Corollary 4.7. *Let (G^i) be a sequence of non-archimedean CLI Polish groups, $\alpha < \omega_1$, and let $G = \prod_i G^i$. If all G^i are L-α-CLI but not α-CLI, then G is $(\alpha+1)$-CLI but not L-α-CLI.*

Proof. Note that all G^i are $(\alpha+1)$-CLI but not α-CLI. ⊣

From Theorem 3.19 and [3, Theorem 1.1], a non-archimedean CLI Polish group G is 1-CLI iff G is isomorphic to a closed subgroup of a product $\prod_i G^i$, where each G^i is L-0-CLI. However, we do not know whether the following generalization of Corollary 4.7 is true:

Question 4.8. *Let $0 < \alpha < \omega_1$, and let G be an $(\alpha+1)$-CLI group. Can we find a sequence of L-α-CLI groups G^i such that G is isomorphic to a closed subgroup of $\prod_i G^i$?*

Let G and Λ be two groups. Recall that the *wreath product* $\Lambda \wr G$ is the set $\Lambda \times G^\Lambda$ with the following group operation: given $(\hat{\lambda}, \hat{\chi}), (\tilde{\lambda}, \tilde{\chi}) \in \Lambda \times G^\Lambda$, we have

$$(\hat{\lambda}, \hat{\chi})(\tilde{\lambda}, \tilde{\chi}) = (\hat{\lambda}\tilde{\lambda}, \chi),$$

where $\chi(\lambda) = \hat{\chi}(\lambda)\tilde{\chi}(\hat{\lambda}^{-1}\lambda)$ for each $\lambda \in \Lambda$. If Λ is countable discrete and G is Polish, then $\Lambda \wr G$ equipped with the product topology of $\Lambda \times G^\Lambda$ is also a Polish group.

THEOREM 4.9. *Let G be a non-archimedean CLI Polish group, Λ an infinite countable discrete group, and $\alpha < \omega_1$. If G is $(\alpha+1)$-CLI but not α-CLI, then $\Lambda \wr G$ is L-$(\alpha+1)$-CLI but not $(\alpha+1)$-CLI.*

PROOF. For brevity, we denote $\Lambda \wr G$ by H. Let λ_i, $i < \omega$ be an enumeration of Λ without repetition. Note that the underlying space of $\Lambda \wr G$ is $\Lambda \times G^\Lambda$. For $(\lambda, \chi) \in \Lambda \wr G$, put $\pi_\Lambda(\lambda, \chi) = \lambda$ and $\pi_G^i(\lambda, \chi) = \chi(\lambda_i)$.

Let $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$. Since G is not α-CLI, $G \neq \{1_G\}$. Without loss of generality, we can assume that $G \neq G_1$. Put $H_0 = H$ and

$$H_{n+1} = \{(1_\Lambda, \chi) : \chi \in G^\Lambda \wedge \forall i < n\, (\chi(\lambda_i) \in G_n)\}$$

for $n < \omega$. Then $\mathcal{H} = (H_n) \in \mathrm{dgnb}(H)$. Note that the open subgroup $H_1 = \{1_\Lambda\} \times G^\Lambda$ is topologically isomorphic to G^ω. By Theorem 4.6, H_1 is $(\alpha+1)$-CLI. So we can see that H is L-$(\alpha+1)$-CLI from Theorem 3.21.

Since G is $(\alpha+1)$-CLI but not α-CLI, $\omega \cdot \alpha < \rho(\mathcal{G}) \leq \omega \cdot (\alpha+1)$. By Lemma 3.16, there exist $1 \leq m, k < \omega$ such that $\rho^k(\mathcal{G}) = \omega \cdot \alpha + m$. To see that H is not $(\alpha+1)$-CLI, we will show that $\rho^{k+1}(\mathcal{H}) > \omega \cdot (\alpha+1)$ as follows.

For any $(\lambda_l, \hat{\chi}) \in H$ and $(1_\Lambda, \tilde{\chi}) \in H_{k+1}$, note that $(\lambda_l, \hat{\chi})(1_\Lambda, \tilde{\chi}) = (\lambda_l, \hat{\chi}\tilde{\chi}_l)$, where $\tilde{\chi}_l(\lambda) = \tilde{\chi}(\lambda_l^{-1}\lambda)$ for $\lambda \in \Lambda$. It follows that

$$(\lambda_l, \hat{\chi})H_{k+1} = \{(\lambda_l, \chi) : \chi \in G^\Lambda \wedge \forall i < k\, (\chi(\lambda_l\lambda_i) \in \hat{\chi}(\lambda_l\lambda_i)G_k)\}.$$

There is a unique $l_i < \omega$ such that $\lambda_{l_i} = \lambda_l\lambda_i$ for $i < k$. It is clear that $\pi_\Lambda((\lambda_l, \hat{\chi})H_{k+1}) = \{\lambda_l\}$ and for $j < \omega$,

$$\pi_G^j((\lambda_l, \hat{\chi})H_{k+1}) = \begin{cases} \hat{\chi}(\lambda_{l_i})G_k, & j = l_i, i < k, \\ G, & \text{otherwise.} \end{cases}$$

Let $m_l = \max\{l_i : i < k\}$. Then it is clear that $m_l \geq k-1$ for $l < \omega$ and $\sup\{m_l : l < \omega\} = \omega$. Note that $\pi_G^{m_l}(H_{m_l+1} \cdot (\lambda_l, \hat{\chi})H_{k+1}) = G/G_k$ is not a singleton, and so $(m_l+1, H_{m_l+1} \cdot (\lambda_l, \hat{\chi})H_{k+1}) \in T_{\mathcal{H}}^{H/H_{k+1}}$. Also note that

$$(T_{\mathcal{H}}^{H/H_{k+1}})_{(m_l+2, H_{m_l+2}\cdot(\lambda_l,\hat{\chi})H_{k+1})} \cong T_{\mathcal{H}'}^{H_{m_l+2}\cdot(\lambda_l,\hat{\chi})H_{k+1}}, \text{ and}$$

$$(T_{\mathcal{G}}^{G/G_k})_{(m_l+1, G_{m_l+1}\cdot gG_k)} \cong T_{\mathcal{G}'}^{G_{m_l+1}\cdot gG_k},$$

where $\mathcal{H}' = (H_{n+m_l+2})$ and $\mathcal{G}' = (G_{n+m_l+1})$ for $n < \omega$. Define $\theta : H/H_{k+1} \to G/G_k$ as $\theta(C) = \pi_G^{m_l}(C)$. Now we restrict θ as a function that maps $H_{m_l+2} \cdot (\lambda_l, \hat{\chi})H_{k+1}$ onto $G_{m_l+1} \cdot \hat{\chi}(\lambda_{m_l})G_k$. Then apply Lemma 3.8 and Proposition 2.2 to obtain the following:

$$\begin{array}{rl} & \rho((T_{\mathcal{H}}^{H/H_{k+1}})_{(m_l+1,H_{m_l+1}\cdot(\lambda_l,\bar{\chi})H_{k+1})}) \\ \geq & \sup\{\rho((T_{\mathcal{H}}^{H/H_{k+1}})_{(m_l+2,H_{m_l+2}\cdot(\lambda_l,\hat{\chi})H_{k+1})}) : \forall i < m_l\, (\bar{\chi}(\lambda_i) = \hat{\chi}(\lambda_i))\} + 1 \\ \geq & \sup\{\rho((T_{\mathcal{G}}^{G/G_k})_{(m_l+1,G_{m_l+1}\cdot gG_k)}) : g \in G\} + 1 \\ \geq & \omega(\rho(T_{\mathcal{G}}^{G/G_k})) + 1 \\ = & \omega\cdot\alpha + 1. \end{array}$$

This implies $\rho(T_{\mathcal{H}}^{H/H_{k+1}}) \geq \omega\cdot\alpha + m_l + 2$ for all $l < \omega$, and hence

$$\rho^{k+1}(\mathcal{H}) = \rho(T_{\mathcal{H}}^{H/H_{k+1}}) \geq \omega\cdot\alpha + \omega = \omega\cdot(\alpha+1).$$

Since $\rho^{k+1}(\mathcal{H})$ is a successor ordinal, we have $\rho^{k+1}(\mathcal{H}) > \omega\cdot(\alpha+1)$. ⊣

Now we turn to the case concerning limit ordinals. To do this, we need two lemmas.

Lemma 4.10. *Let (G^i) be a sequence of Polish groups, and let H^i be an open subgroup of G^i for each $i < \omega$. Suppose $H = \prod_i H^i$ and*

$$G = \{(g_i) \in \prod_i G^i : \forall^\infty i\, (g_i \in H^i)\}$$

is equipped with the topology τ generated by the sets of the form $(g_i)U$ for $(g_i) \in G$ and U open in H. Then (G,τ) is a Polish group and τ is the unique group topology on G making H an open subgroup of G.

Proof. For each $(g_i) \in G$, the subspace $(g_i)H$ of (G,τ) is homeomorphic to H, so it is Polish. Let D^i be a subset of G^i which meets every coset of H^i at exactly one point. Note that D^i is countable for all $i < \omega$. We define

$$D = \{(g_i) \in \prod_i D^i : \forall^\infty i\, (g_i = 1_{G^i})\}.$$

It is clear that $G/H = \{(g_i)H : (g_i) \in D\}$ is countable. So (G,τ) is a sum of countably many Polish spaces, thus is a Polish space.

For $(g_i),(h_i) \in G$ and U open in H with $1_H \in H$, there exists an $m < \omega$ such that $g_i, h_i \in H^i$ for $i > m$ and $U^0 \times \cdots \times U^m \times \prod_{i>m} H^i \subseteq U$, where U^i is an open subset of H^i with $1_{H^i} \in U^i$ for each $i \leq m$. We can find open neighborhoods V^i and W^i of 1_{H^i} with $(g_i V^i)(h_i W^i)^{-1} \subseteq g_i h_i^{-1} U^i$ for $i \leq m$. Now let $V = V^0 \times \cdots \times V^m \times \prod_{i\geq m} H^i$ and $W = W^0 \times \cdots \times W^m \times \prod_{i\geq m} H^i$, then V and W are open neighborhoods of 1_H, and $((g_i)V)((h_i)W)^{-1} \subseteq (g_i h_i^{-1})U$. So (G,τ) is a Polish group.

Finally, suppose τ' is another group topology on G such that H is an open subgroup of G. Then for each $(g_i) \in G$, the subspace $(g_i)H$ of (G,τ') is homeomorphic to H, so the restrictions of τ and τ' on $(g_i)H$ are the same. Hence, $\tau = \tau'$. ⊣

Lemma 4.11. *Let (G^i) be a sequence of non-archimedean CLI Polish groups, $\mathcal{G}^i = (G_n^i) \in \mathrm{dgnb}(G^i)$ for each $i < \omega$, and let $0 < \alpha < \omega_1$. Suppose*

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$$\sup\{\rho^1(\mathcal{G}^i) : i < \omega\} = \omega \cdot \alpha, \text{ and}$$

$$G = \{(g_i) \in \prod_i G^i : \forall^\infty i\, (g_i \in G_1^i)\}$$

is equipped with the unique group topology making $\prod_i G_1^i$ an open subgroup of G. Then G is not α-CLI.

Proof. Put $G_0 = G$ and for $n \geq 1$, let

$$G_n = \prod_{i<n-1} G_n^i \times \prod_{i \geq n-1} G_1^i.$$

It is clear that $G_1 = \prod_i G_1^i$ and $\mathcal{G} = (G_n) \in \mathrm{dgnb}(G)$.

Given $j < \omega$, define $\theta : G/G_1 \to G^j/G_1^j$ as $\theta((g_i)G_1) = g_j G_1^j$ for $(g_i) \in G$. Applying Lemma 3.8 to the restriction of θ as in the proof of Theorem 4.9, we have

$$\begin{aligned} \rho(T_{\mathcal{G}}^{G/G_1}) &= \sup\{\rho((T_{\mathcal{G}}^{G/G_1})_{(1,G_1\cdot(g_i)G_1)}) : (g_i) \in G\} + 1 \\ &\geq \sup\{\rho((T_{\mathcal{G}^j}^{G^j/G_1^j})_{(1,G_1^j\cdot gG_1^j)}) : g \in G^j\} + 1 \\ &= \rho(T_{\mathcal{G}^j}^{G^j/G_1^j}) = \rho^1(\mathcal{G}^j). \end{aligned}$$

Therefore,

$$\rho^1(\mathcal{G}) = \rho(T_{\mathcal{G}}^{G/G_1}) \geq \sup\{\rho^1(\mathcal{G}^j) : j < \omega\} = \omega \cdot \alpha.$$

Since $\rho^1(\mathcal{G})$ is a non-limit ordinal and $\alpha > 0$, we have $\rho^1(\mathcal{G}) > \omega \cdot \alpha$. So G is not α-CLI. ⊣

Finally, we complete the construction in the following theorem.

Theorem 4.12. *For any $\alpha < \omega_1$, there exist two non-archimedean CLI Polish groups G_α and H_α with $\mathrm{rank}(G_\alpha) = \mathrm{rank}(H_\alpha) = \alpha$ such that H_α is α-CLI and G_α is L-α-CLI but not α-CLI.*

Proof. We construct G_α and H_α by induction on α. From Corollary 4.7 and Theorem 4.9, we only need to consider the case where α is a limit ordinal.

Let (α_i) be a sequence of ordinals less than α with $\sup\{\alpha_i : i < \omega\} = \alpha$. By the inductive hypothesis, we can find a non-archimedean CLI Polish group G^i for each $i < \omega$ such that G^i is L-α_i-CLI but not α_i-CLI. It is clear that $\mathrm{rank}(G^i) = \alpha_i < \alpha$.

Put $H_\alpha = \prod_i G^i$. Theorem 4.6(1) implies that H_α is α-CLI. By Theorem 4.1, $\mathrm{rank}(H_\alpha) \geq \mathrm{rank}(G^i)$ for each $i < \omega$. So $\mathrm{rank}(H_\alpha) = \alpha$.

For $i < \omega$, let $\mathcal{G}^i = (G_n^i) \in \mathrm{dgnb}(G^i)$. By Lemma 3.16, there exist $0 < k_i < \omega$ such that $\omega(\rho^{k_i}(\mathcal{G}^i)) = \omega \cdot \alpha_i$. Put $H_0^i = G^i$ and $H_{n+1}^i = G_{n+k_i}^i$ for $n < \omega$. Then $\mathcal{H}^i = (H_n^i) \in \mathrm{dgnb}(G^i)$ and $H_1^i = G_{k_i}^i$. By Lemma 3.7, we have

$$\omega(\rho^1(\mathcal{H}^i)) = \omega(\rho(T_{\mathcal{H}^i}^{G^i/H_1^i})) = \omega(\rho(T_{\mathcal{G}^i}^{G^i/G_{k_i}^i})) = \omega(\rho^{k_i}(\mathcal{G}^i)) = \omega \cdot \alpha_i.$$

So $\sup\{\rho^1(\mathcal{H}^i) : i < \omega\} = \omega \cdot \alpha$. Now we let

$$G_\alpha = \{(g_i) \in \prod_i G^i : \forall^\infty i \, (g_i \in G^i_{k_i})\}$$

be equipped with the unique group topology making $\prod_i G^i_{k_i}$ an open subgroup of G_α. By Lemma 4.11, G_α is not α-CLI. It is clear that the open subgroup $\prod_i G^i_{k_i}$ is α-CLI, so G_α is L-α-CLI, and hence $\operatorname{rank}(G_\alpha) = \alpha$. ⊣

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