so that g_1 and g_2 are respectively the accelerations at the outer surface of the outer shell due to the attraction of the inner shell and its own gravitation. Then to a first approximation the complete values for the strains and stresses in the outer shell are as follows:—

$$\left.\begin{aligned} u/r &= \delta a/a = -\tfrac{1}{4}(g_2+2g_1)\rho_2 a(1-\eta)/\mathrm{E}, \\ \frac{du}{dr} &= \delta h/h = \tfrac{1}{2}(g_2+2g_1)\rho_2 a\eta/\mathrm{E}, \\ \widehat{rr} &= -\tfrac{1}{2}g_2\rho_2 a\frac{\xi(h-\xi)}{ah}, \\ \overline{\mathrm{S}} &= -\widehat{\theta\theta} = \tfrac{1}{4}(g_2+2g_1)\rho_2 a \end{aligned}\right\} \quad \ldots \quad (95).$$

The intensity of the actual bodily force in the outer shell varies regularly from g_1 at the inner to g_1+g_2 at the outer surface. Thus the above results show that to a first approximation the strains and the transverse stress in the shell are the same as if the bodily forces had at every point of the thickness a constant value equal to the mean of the actual values. The value of the radial stress depends even to a first approximation on the law of distribution of the bodily forces, but this stress is negligible compared to the transverse stress. So far as concerns the results (95) the inner shell may be a solid core or a shell of any thickness. The only limitation is that the two shells must not be in contact.

The Elements of Quaternions (*Second Paper*).

Discussion of the Proofs of the Laws of the Quaternionic Algebra.

[*Abstract.*]

By Dr William Peddie.

Three main laws regulate the treatment of ordinary algebraic quantities. These are the Associative Law, the Distributive Law, and the Commutative Law. If a, b, c, ..., represent quantities dealt with in the algebra, the associative law of multiplication asserts that $a(bc)=(ab)c$, where the brackets have the usual meaning that the quantity within them is to be regarded as a single quantity: the distributive law of multiplication asserts that $(a+b)(c+d)=ac+bc+ad+bd$: and the commutative law gives $ab=ba$. With regard to addition, the associative law asserts that $(a+b)+c=a+(b+c)$: and the commutative law gives $a+b=b+a$.

https://doi.org/10.1017/S0013091500031291 Published online by Cambridge University Press

In ordinary algebra, all the quantities are scalars. In a vector algebra, the further idea of direction is ntroduced, and so we cannot assert *a priori* that the quantities dealt with in that algebra will satisfy the laws of ordinary algebra. The matter is one for investigation. With certain fundamental assumptions, some of the ordinary laws may hold and others may not; and the particular set which holds, and that which does not, will depend on the nature of these assumptions.

In framing a new algebra, the first care should be to make its laws agree as far as possible with the laws of the old; former assumptions are only to be discarded when they stand in the way of farther development. The adoption of certain assumptions may make the algebra more readily applicable in some directions than in others; in which case the maximum of general applicability, consistent with ease of application in the most important directions, is to be aimed at. And, in dealing solely with the new quantities introduced in the new algebra, we may assume characteristics totally different from those which typify the old quantities if such assumptions enable us to follow out the above rules, while all others prevent us from doing so. Indeed, such a choice of characteristics might be most advantageous even in circumstances in which the old characteristics would also enable us to observe these rules.

When one vector α is changed into another β, the change may be represented as due to the addition of a third vector γ to the former: so that $\alpha + \gamma = \beta$. And, since the notion of a vector quantity involves only the ideas of magnitude and direction—*not* the idea of position—we see that a geometrical interpretation of this equation is that the relative position of two points in a plane is fully given either by means of the straight line joining the two or by means of the two sides of any triangle described with that line as base: and, similarly, it may be given by the remaining sides of any polygon of which the line joining the two points forms the other side. From this we at once see that the associative and commutative laws must apply to the addition (and subtraction) of vectors.

Again when α is changed into β, we may represent the result as due, not to some addition to α but, to some operation performed upon α. This is represented by the equation $q\alpha = \beta$, where q is the required operator. Such a method is as natural, as important, and, in many cases, more appropriate than the former. The Calculus of

Quaternions, regarded as a vector algebra, recognises and employs both methods.

The operator q turns α, in a plane parallel to the directions of α and β, through an angle equal to that contained between two lines drawn in these directions respectively through a fixed point; and it also changes its length if necessary until it becomes equal to that of β. Now, to determine the plane, two numbers (such as (1) the azimuth, in a fixed plane, of the line of intersection of the fixed plane with the required one, and (2) the obliquity of the planes) are needed. Then another number is needed to determine the amount of rotation in the plane; after which yet another is needed to determine the amount of lengthening (or shortening). In all *four* numbers are required; and hence q is conveniently called a *quaternion*.

An algebra which deals with such operators is, *ipso facto*, an algebra of vectors *plus* quaternions, and so *may be* more complex than another in which the subject is not regarded from this operational point of view. On the other hand, since we have $q\alpha = \alpha + \gamma$, it is evident that we can, by means of suitable definitions, express a quaternion in terms of vectors; and it may be possible to do this so simply that special symbols for quaternions need never be introduced, while, on the one hand, the greater complexity spoken of becomes vanishingly small, and, on the other, greater freedom of treatment is attained. In accordance with the usages of ordinary algebra, we may regard $q\alpha$ as the *product* of q into α. That is to say, $q\alpha$ is the product of two vector quantities; or, more strictly, of a vector and a function of vectors. Now, in physical enquiries, we have constantly to deal with products of vector quantities—which products may be either scalar or vector. Hence a vector algebra, which recognises the quaternion, may be made to deal *naturally* (and, it may be, very simply) with such physical investigations. On the contrary, the algebra which does not recognise the quaternion must have introduced into it new fundamental definitions, totally unconnected with anything else, if it is to deal with scalar or vector products of directed quantities. And the introduction of these new definitions into the algebra will make possible the quaternionic treatment of vectors by its means; so that it would be quite correct to call it a calculus of quaternions whether developed or not. Indeed the cry that vectors should be treated vectorially is merely a play upon words. A vector calculus deals with vectors and functions of vectors; and, as we have seen, in any quaternion calculus, a quaternion can be represented as a function of vectors; so that the quaternion calculus is, in a sense, *purely* a calculus of vectors. This is preeminently the case with Hamilton's system.

We know of only two fundamental classes of vectors—vectors having reference to translation along a line, and vectors having reference to rotation around an axis. Hamilton's system takes account of both ideas without introducing separate symbols: the same vector acts translationally, or rotationally, according as it is added to another, or is multiplied into another: and there is no possible confusion of meaning. And, further, provision is made, simply, for the treatment of scalar products of vectors. But, before considering the assumptions by means of which these advantages are attained, it is necessary to consider the laws of multiplication of quaternions: and, in doing so, it is not necessary to consider the stretching part (or Tensor) of the quaternion—for that part is a mere number and so obeys all the laws of ordinary algebra.

We may represent quaternions by plane angles or by arcs of great circles on a unit sphere. Thus, if PQR be a spherical triangle whose sides p, q, r are portions of great circles on the unit sphere, the quantities p, q, r may represent the corresponding quaternions. Let α be the vector from the origin to the point Q. Then $p\alpha$ is the vector to the point R, and $q \cdot p\alpha$ is the vector to P. But this is also $r\alpha$, if r is measured from Q to P while p and q are measured from Q to R, and from R to P, respectively. And we are at liberty to define $r = qp$, so that $q \cdot p\alpha = qp\alpha$. This makes the associative law hold when α, $p\alpha$, and $qp\alpha$ are vectors—a fact which is pointed out by Hamilton, *Lectures*, § 310, and by Tait, *Elements*, § 54. It defines quaternion multiplication.

Various proofs that the associative law holds in the multiplication of quaternions have been given. Of these, Hamilton's proof (*Lectures*, § 296; *Elements*, § 270; and Tait's *Elements*, §§ 57-60) by spherical arcs and elementary properties of spherical conics involves, by definition, the particular assumption of association just alluded to. His alternative proof, by more elementary geometry (*Lectures*, §§ 298-301), makes use of the same definition; and the same remark applies to the proof given in §§ 358, 359 of the *Lectures*. On the other hand, the geometrical proof given in Hamilton's *Elements*, §§ 266, 267, 272, is based upon the definition of the reciprocal of a quaternion, which makes the product of a quaternion and its reciprocal unity, and leads to the result that the versor of a product of quaternions is equal to the product of their versors. It involves the definition, above alluded to, of a quaternion in terms of vectors: which, in turn, partially assumes the associative law for vectors ($\beta a^{-1}.a = \beta.a^{-1}a$).

The complete proof of the law, by this method, is given in § 272 of the *Elements*. Other possible proofs are indicated in the *Elements*. In the proof, by spherical conics, given in the *Lectures*, § 302, and the *Elements*, §§ 265, 271, a quaternion is represented by a spherical angle.

Hamilton also gives proofs (*Lectures*, § 489; *Elements*, § 223) of the associative law for quaternions when the distributive law for vector multiplication is granted or proved. This proof is also given by Tait, *Elements*, § 85. It involves the representation of a quaternion as the sum of a scalar and a vector. The proof that this representation is possible and definite (*Lectures*, § 406; *Elements*, §§ 201, 202; Tait's *Quaternions*, § 77) necessitates the association of vectors, as above, to the extent $\beta a^{-1}.a = \beta.a^{-1}a$, (and the distributive law to the extent $(a+\gamma a)a^{-1} = aa^{-1} + \gamma aa^{-1}$). Indeed all the laws of combination of rectangular vectors are taken for granted in this proof of the associative law.

The addition of quaternions is defined by the equation $(q+r)a = qa + ra$ where a is a vector. From this (*Lectures*, § 449) it at once follows that the associative law holds in such addition. This definition of course is virtually an assumption of the distributive law in the particular case when a, qa, ra, are vectors.

Hamilton's proof (*Lectures*, §§ 451–455; *Elements*, §§ 210–212; Tait's *Quaternions*, § 81) of the distributive law in the multiplication of quaternions employs this definition of the addition of quaternions together with the partial assumption of the associative law of vectors involved in the definition of a quaternion in terms of vectors. It also assumes the possibility of representing a quaternion as the sum of a scalar and a vector. The law may be proved, as Tait indicates (§ 62), by means of this definition and the assumption of the laws of combination of vectors. Tait's other proof (§ 62) by means of the properties of spherical conics involves, in its complete generality, the proof of the commutativeness of quaternion addition.

When the partial assumption of association of vectors, used in Hamilton's fundamental expression for a quaternion in terms of vectors, is made along with the partial assumption of distribution used in the definition of quaternion addition, the commutativeness of quaternion addition follows at once (*Lectures*, §§ 448, 449; *Elements*, §§ 195–207; Tait's *Elements*, § 61) from the obvious commutativeness of vector addition.

The results obtained up to this point are the following: (1) The addition of vectors is commutative and associative. (2) A quaternion may be represented as a function of vectors. In Hamilton's system the quaternion q in the equation $q\alpha=\beta$ is defined to be $\beta\alpha^{-1}$; and so $q\alpha=\beta\alpha^{-1}.\alpha=\beta.\alpha^{-1}\alpha=\beta$, for $\alpha^{-1}\alpha$ is defined to be unity; and the steps of the process are consistent with association in vector multiplication. (3) A definition of quaternion addition, which does not conflict with the distributive law of multiplication, and which subjects the process to the associative law, is given. (4) With no further definitions, it is found that the associative and distributive laws hold in the multiplication of quaternions. Thus all the results yet obtained are consistent with the rules which must be observed in the formation of a new calculus.

The graphical representation of quaternion (or versor) multiplication shows at once (*Elements*, § 168, Tait's *Quaternions*, § 54) that quaternion multiplication is not in general commutative. And another peculiarity is that, if q be a versor which turns any vector in a given plane through a right angle, the double application of the operation q reverses any vector in that plane. If α be such a vector, we get $q.q\alpha=q^2\alpha=-\alpha$; so that we may put $q^2=-1$ in the case of any quadrantal versor. And if p, q, r be rectangular quadrantal versors we get

$$p^2=q^2=r^2=-1;\ pq=-qp=r,\ qr=-rq=p,\ rp=-pr=q.$$

Now consider three rectangular unit vectors i, j, k; and let them be perpendicular respectively to the planes of rotation of p, q, r, so that we may say that i is parallel to the axis of p, etc. We get at once $pj=k$, $pq=r$; $qi=-k$, $qp=-r$; $r^2=-1$. Whence if we write $p=i$, $q=j$, $r=k$, we shall have the immense simplification that no special symbols are needed for versors—a vector acting translationally in addition (or subtraction), rotationally in multiplication (or division).

With this assumption, vector multiplication is associative, and distributive; but is not commutative; and the square of a unit vector is negative unity; the laws for unit rectangular vectors being $ij=-ji$, and $i^2=-i$, etc.

Now the idea of a vector is one entirely foreign to ordinary algebra, in which the square of any unit is positive unity. Hence the fact that the square of a unit vector is negative unity has no disadvantage. It makes the scalar part of the product of β into α equal to the product of the lengths of these vectors into the cosine of the supplement of the angle between their positive directions; and it makes the reciprocal of a vector have a direction opposite to that of the vector itself; all of which conditions are as natural and simple as their opposites.

Finally, it is shown by Hamilton, by strict reasoning (*Lectures*, §§ 49–56), that these laws for the multiplication of unit rectangular vectors *must* hold if no one direction in space is to be regarded as eminent above another and if the ordinary rules of algebra are to apply in so far that, (1) to multiply either factor by any number positive or negative, multiplies the product by the same, (2) the product of two determined factors is itself determined, (3) the distributive and associative principles hold. We see then that Hamilton's system is one which preeminently satisfies the conditions of correspondence to ordinary algebra as far as possible.

Note on a Problem in Analytical Geometry.

By A. J. PRESSLAND, M.A.

[*Abstract.*]

The theorem, "If upon the sides of a triangle as diagonals parallelograms be described, whose sides are parallel to two given lines, then the other three diagonals will intersect in the same point," occurs in Hutton's *Course of Mathematics*, 12th ed., vol. II., p. 191.

For a proof, see Smith's *Conic Sections*, p. 40.

If we are given the point of intersection of the diagonals, and wish to find the directions of the sides of the parallelograms, the discussion resolves itself into describing a conic through three points to have its centre at a given point. The asymptotes of this conic are the directions required. For a solution, see Eagles' *Constructive Geometry of Plane Curves*, pp. 124, 173, and notice Taylor, *Ancient and Modern Geometry of Conics*, p. 164, Ex. 454.

If A, B, C be the three points, D, E, F the mid points of BC, CA, AB then if the centre lies inside DEF the asymptotes are imaginary, but they are real if the centre lies inside AEF, etc.

https://doi.org/10.1017/S0013091500031291 Published online by Cambridge University Press