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ON A CONJECTURE FOR A REFINEMENT OF THE SUM OF MINIMAL EXCLUDANTS

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Abstract

In 2019, Andrews and Newman [‘Partitions and the minimal excludant’, *Ann. Comb.* **23**(2) (2019), 249–254] introduced the arithmetic function $\sigma\text{mex}(n)$, which denotes the sum of minimal excludants over all the partitions of n. Baruah *et al.* [‘A refinement of a result of Andrews and Newman on the sum of minimal excludants’, *Ramanujan J.*, to appear] showed that the sum of minimal excludants over all the partitions of n is the same as the number of partition pairs of n into distinct parts. They proved three congruences modulo 4 and 8 for two functions appearing in this refinement and conjectured two further congruences modulo 8 and 16. We confirm these two conjectures by using q-series manipulations and modular forms.

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1. Introduction

A partition π of a positive integer n is a finite weakly decreasing sequence of positive integers $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r$ such that $\sum_{i=1}^{r} \pi_i = n$. The π_i are called the parts of the partition π. Fraenkel and Peled [9] originally defined the minimal excludant for any set S of positive integers as the least positive integer not in S. In 2019, Andrews and Newman [3] defined the minimal excludant of an integer partition π as the least positive integer missing from the partition, denoted by $\text{mex}(\pi)$. For example, there are five partitions of 4: 4 with $\text{mex}(\pi) = 1$; $3 + 1$ with $\text{mex}(\pi) = 2$; $2 + 2$ with $\text{mex}(\pi) = 1$; $2 + 1 + 1$ with $\text{mex}(\pi) = 3$; $1 + 1 + 1 + 1$ with $\text{mex}(\pi) = 2$. Andrews and Newman [3, Theorem 1.1] established an elegant identity involving the quantity $\sigma\text{mex}(n)$, which

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denotes the sum of minimal excludants over all the partitions of n. More precisely, they proved that

$$\sum_{n=0}^{\infty} \sigma\mathrm{mex}(n)q^n = (-q;q)_\infty^2 = \sum_{n=0}^{\infty} Q_2(n)q^n, \tag{1.1}$$

where $Q_2(n)$ denotes the number of partition pairs of n into distinct parts. Throughout the rest of this paper, we always assume that q is a complex number and adopt the standard notation:

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j),$$

$$(a_1,a_2,\ldots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

Interestingly, (1.1) was derived earlier by Grabner and Knopfmacher [11, (4.2)] under a different terminology. Recently, Ballantine and Merca [4] also proved (1.1) by employing purely combinatorial arguments.

Quite recently, Baruah *et al.* [5] investigated a refinement of the arithmetic function $\sigma\mathrm{mex}(n)$ by considering the parity of the minimal excludant. More specifically, in [5, (1.2) and (1.3)] they defined the two functions

$$\sigma_o\mathrm{mex}(n) = \sum_{\substack{\pi\vdash n\\ \mathrm{mex}(n)\ \mathrm{odd}}} \mathrm{mex}(\pi) \quad \text{and} \quad \sigma_e\mathrm{mex}(n) = \sum_{\substack{\pi\vdash n\\ \mathrm{mex}(n)\ \mathrm{even}}} \mathrm{mex}(\pi). \tag{1.2}$$

For instance, with $n=4$, $\sigma_o\mathrm{mex}(4) = 1+1+3 = 5$ and $\sigma_e\mathrm{mex}(4) = 2+2 = 4$. By some q-series manipulations, Baruah *et al.* [5, Theorem 2.1] proved the following two partition identities which can be viewed as a refinement of (1.1): for any $n \geq 0$,

$$\sigma_o\mathrm{mex}(n) = Q_2^e(n) \quad \text{and} \quad \sigma_e\mathrm{mex}(n) = Q_2^o(n), \tag{1.3}$$

where $Q_2^e(n)$ and $Q_2^o(n)$ denote the number of partition pairs of n into distinct parts with an even number of parts and an odd number of parts, respectively. As a consequence of (1.3), [5, Theorem 2.2] gives the following three congruences modulo 4 and 8 for $\sigma_o\mathrm{mex}(n)$ and $\sigma_e\mathrm{mex}(n)$:

$$\begin{aligned} \sigma_o\mathrm{mex}(2n+1) &\equiv 0 \pmod{4},\\ \sigma_o\mathrm{mex}(4n+1) &\equiv 0 \pmod{8}, \end{aligned} \tag{1.4}$$

$$\sigma_e\mathrm{mex}(4n) \equiv 0 \pmod{4}. \tag{1.5}$$

Based on numerical evidence, Baruah *et al.* proposed the following conjecture.

CONJECTURE 1.1 [5, Conjecture 6.1]. For any $n \geq 0$,

$$\sigma_o\mathrm{mex}(8n+1) \equiv 0 \pmod{16}, \tag{1.6}$$

$$\sigma_e\mathrm{mex}(8n) \equiv 0 \pmod{8}. \tag{1.7}$$

The main purpose of this paper is to confirm (1.6) and (1.7).

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THEOREM 1.2. *The congruences (1.6) and (1.7) are valid for any $n \geq 0$.*

The rest of this paper is constructed as follows. In Section 2, we first collect some necessary identities, and next introduce some notation, terminology and theorems in the theory of modular forms. The proof of Theorem 1.2 is presented in Section 3. We conclude this paper with two remarks.

2. Preliminaries

To prove (1.6) and (1.7), we first need the following identities.

LEMMA 2.1 (Jacobi's triple product identity, [1, Lemma 1.2.2]). *We have*

$$\sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty, \quad \text{where } |ab| < 1. \tag{2.1}$$

For notational convenience, we denote

$$J_{a,b} := (q^a, q^{b-a}, q^b; q^b)_\infty, \quad \overline{J}_{a,b} := (-q^a, -q^{b-a}, q^b; q^b)_\infty, \quad J_a := J_{a,3a} = (q^a; q^a)_\infty.$$

LEMMA 2.2. *We have*

$$\frac{1}{J_1} = \frac{1}{J_2^2}(\overline{J}_{6,16} + q\overline{J}_{2,16}), \tag{2.2}$$

$$J_1^2 = \frac{J_2 J_8^5}{J_4^2 J_{16}^2} - 2q\frac{J_2 J_{16}^2}{J_8}, \tag{2.3}$$

$$\frac{1}{J_1^2} = \frac{J_8^5}{J_2^5 J_{16}^2} + 2q\frac{J_4^2 J_{16}^2}{J_2^5 J_8}, \tag{2.4}$$

$$J_1^4 = \frac{J_4^{10}}{J_2^2 J_8^4} - 4q\frac{J_2^2 J_8^4}{J_4^2}, \tag{2.5}$$

$$\frac{1}{J_1^4} = \frac{J_4^{14}}{J_2^{14} J_8^4} + 4q\frac{J_4^2 J_8^4}{J_2^{10}}, \tag{2.6}$$

$$\frac{J_1^2}{J_2^2} = \frac{J_2^{22}}{J_1^{14} J_4^8} - 16q\frac{J_4^8}{J_1^6 J_2^2}. \tag{2.7}$$

PROOF. The identity (2.2) appears in [2, Lemma 4.1]. The identities (2.3)–(2.6) follow from [6, page 40, Entries 25(i), (ii), (v) and (vi)] (see also [16, Lemmas 2.2 and 2.3]). It follows immediately from [6, page 40, Entry 25(vii)] that

$$\frac{J_2^{20}}{J_1^8 J_4^8} - 16q\frac{J_4^8}{J_2^4} = \frac{J_1^8}{J_2^4}. \tag{2.8}$$

Multiplying by the factor J_2^2/J_1^6 on both sides of (2.8) yields (2.7). □

LEMMA 2.3 [15, (2.10)]. *We have*

$$J_1^3 \equiv \overline{J}_{28,64} - 3q\overline{J}_{20,64} + 5q^3\overline{J}_{12,64} - 7q^6\overline{J}_{4,64} \pmod{16}. \tag{2.9}$$

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Next, we collect some notation and terminology on the theory of modular forms. The full modular group is given by

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{and } ad - bc = 1 \right\},$$

and for a positive integer N, the congruence subgroup $\Gamma_1(N)$ is defined by

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod{N}, c \equiv 0 \pmod{N} \right\}.$$

We denote by γ the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, if not specified otherwise. Let γ act on $\tau \in \mathbb{C}$ by the linear fractional transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \gamma\infty = \lim_{\tau\to\infty} \gamma\tau.$$

Let k be a positive integer and $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form with weight k for $\Gamma_1(N)$ if it satisfies the following two conditions:

(1) $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for all $\gamma \in \Gamma_1(N)$;
(2) for any $\gamma \in \Gamma$, $(c\tau + d)^{-k} f(\gamma\tau)$ has a Fourier expansion of the form

$$(c\tau + d)^{-k} f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} a(n) q_{w_\gamma}^n,$$

where $a(n_\gamma) \neq 0$, $n_\gamma \geq 0$, $q_{w_\gamma} = e^{2\pi i\tau/w_\gamma}$ and w_γ is the minimal positive integer h such that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Gamma_1(N)\gamma.$$

For a modular form $f(\tau)$ of weight k with respect to $\Gamma_1(N)$, the order of $f(\tau)$ at the cusp $a/c \in \mathbb{Q} \cup \{\infty\}$ is defined by

$$\mathrm{ord}_{a/c}(f) = n_\gamma$$

for some $\gamma \in \Gamma$ such that $\gamma\infty = a/c$; $\mathrm{ord}_{a/c}(f)$ is well defined (see [8, page 72]). If the orders of f at all cusps are strictly greater than 0, then f is called a cusp form for $\Gamma_1(N)$.

Let $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$. The Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

The function $\eta^{24}(\tau)$ is a cusp form with weight 12 for Γ and also for $\Gamma_1(N)$ for any positive integer N. For a positive integer δ and a residue class $g \pmod{\delta}$, the generalised Dedekind eta-function $\eta_{\delta,g}(\tau)$ is defined by

$$\eta_{\delta,g}(\tau) = q^{\delta P_2(g/\delta)/2} \prod_{\substack{n>0 \\ n\equiv g \pmod{\delta}}} (1 - q^n) \prod_{\substack{n>0 \\ n\equiv -g \pmod{\delta}}} (1 - q^n),$$

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where

$$P_2(t) = \{t\}^2 - \{t\} + \tfrac{1}{6}$$

is the second Bernoulli function and $\{t\}$ is the fractional part of t (see, for example, [12, 13]). Notice that

$$\eta_{\delta,0}(\tau) = \eta^2(\delta\tau) \quad \text{and} \quad \eta_{\delta,\delta/2}(\tau) = \frac{\eta^2(\delta\tau/2)}{\eta^2(\delta\tau)}.$$

A generalised eta-quotient is a function of the form

$$\prod_{\substack{\delta \mid N \\ 0 \le g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \tag{2.10}$$

where $N \ge 1$ and

$$r_{\delta,g} \in \begin{cases} \frac{1}{2}\mathbb{Z} & \text{if } g = 0 \text{ or } g = \dfrac{\delta}{2}; \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Although the work of Robins [12, Theorem 3] which gives a criterion for a generalised eta-quotient to be modular is for the zero weight case, the following theorem is true for nonzero weight as well (see [7, Theorem 2.5]).

THEOREM 2.4. *If $k = \frac{1}{2}\sum_{\delta \mid N} r_{\delta,0} \in \mathbb{Z}$ and $f(\tau) = \prod_{\delta \mid N,\, 0 \le g < \delta} \eta_{\delta,g}^{r_{\delta,g}}(\tau)$ is a generalised eta-quotient such that*

$$\sum_{\substack{\delta \mid N \\ 0 \le g < \delta}} \delta P_2\Big(\frac{g}{\delta}\Big) r_{\delta,g} \equiv 0 \pmod 2$$

and

$$\sum_{\substack{\delta \mid N \\ 0 \le g < \delta}} \frac{N}{\delta} P_2(0) r_{\delta,g} \equiv 0 \pmod 2,$$

then

$$f(\gamma\tau) = (c\tau + d)^{2k} f(\tau)$$

for all $\gamma \in \Gamma_1(N)$.

We can obtain a formula for the order of a generalised eta-quotient at the cusp of $\Gamma_1(N)$ by [10, Theorem 2.3].

THEOREM 2.5. *The order of the function*

$$f(\tau) = \prod_{\substack{\delta \mid N \\ 0 < g \le \lfloor \frac{\delta}{2} \rfloor}} \eta_{\delta,g}^{a_{\delta,g}}(\tau)$$

at the cusp a/c is given by

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$$\sum_{\substack{\delta|N\\0<g\le\lfloor\frac{\delta}{2}\rfloor}} w_\gamma a_{\delta,g}\Big(\frac{e^2}{2\delta}\Big(\frac{ag}{e}-\Big\lfloor\frac{ag}{e}\Big\rfloor-\frac{1}{2}\Big)^2-\frac{h^2}{6\delta}\Big(\frac{a\delta}{h}-\Big\lfloor\frac{a\delta}{h}\Big\rfloor-\frac{1}{2}\Big)^2\Big), \tag{2.11}$$

where γ satisfies $\gamma\infty = a/c$, $e = \gcd(\delta, c)$ and $h = \gcd(3\delta, c)$.

The following theorem of Sturm [14, Theorem 1] plays an important role in proving congruences using the theory of modular forms.

THEOREM 2.6. *Let Γ' be a congruence subgroup of Γ, and let k be an integer and $g(\tau) = \sum_{n=0}^{\infty} c(n)q^n$ a modular form of weight k for Γ'. For any given positive integer u, if $c(n) \equiv 0 \pmod u$ holds for all $n \le (1/12)k[\Gamma : \Gamma']$, then $c(n) \equiv 0 \pmod u$ holds for any $n \ge 0$.*

There is an explicit formula for the index [8, page 13]:

$$[\Gamma : \Gamma_1(N)] = N^2 \cdot \prod_{\substack{p|N\\ p \text{ is prime}}} \Big(1-\frac{1}{p^2}\Big).$$

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. In [5], Baruah *et al.* stated without proof the following identity:

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma_o\mathrm{mex}(8n+1)q^n &= (\overline{J}_{3,8}\overline{J}_{7,16}+q^2\overline{J}_{1,8}\overline{J}_{1,16})\\ &\quad\times\Big(-\frac{J_2^2J_4^5}{J_1^7J_8^2}-\frac{J_4^5}{J_1^3J_8^2}+2\frac{J_2^7J_4^8}{J_1^{13}J_8^4}+8q\frac{J_2^{11}J_8^4}{J_1^{13}J_4^4}\Big)\\ &\quad+2q(\overline{J}_{1,8}\overline{J}_{7,16}+q\overline{J}_{3,8}\overline{J}_{1,16})\Big(-\frac{J_2^4J_8^2}{J_1^7J_4}+\frac{J_2^2J_8^2}{J_1^3J_4}+4\frac{J_2^9J_4^2}{J_1^{13}}\Big).\end{aligned} \tag{3.1}$$

For the sake of completeness, we present a proof of (3.1) here.

According to [5, (4.25)],

$$\sum_{n=0}^{\infty}\sigma_o\mathrm{mex}(4n+1)q^n = \overline{J}_{3,8}\cdot J_4^2\cdot\frac{1}{J_1}\cdot\frac{1}{J_1^2}\Big(2\frac{J_4^5}{J_2J_8^2}\cdot\frac{1}{J_1^2}-\frac{1}{J_2^2}\cdot J_1^4-1\Big). \tag{3.2}$$

From (2.1),

$$\begin{aligned}\overline{J}_{3,8} &= (-q^3,-q^5,q^8;q^8)_\infty = \sum_{n=-\infty}^{\infty} q^{4n^2+n}\\ &= \sum_{n=-\infty}^{\infty} q^{4(2n)^2+2n}+\sum_{n=-\infty}^{\infty} q^{4(2n-1)^2+(2n-1)}\\ &= \sum_{n=0}^{\infty} q^{16n^2+2n}+\sum_{n=-\infty}^{\infty} q^{16n^2-14n+3} = \overline{J}_{14,32}+q^3\overline{J}_{2,32}.\end{aligned} \tag{3.3}$$

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Substituting (2.2), (2.4), (2.5) and (3.3) into (3.2), after simplification,

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma_o\mathrm{mex}(8n+1)q^n &= (\overline{J}_{3,8}\overline{J}_{7,16}+q^2\overline{J}_{1,8}\overline{J}_{1,16})\\ &\quad\times\Big(-\frac{J_2^{12}J_4}{J_1^{11}J_8^2}+8q\frac{J_2^2J_4^3J_8^2}{J_1^7}-\frac{J_2^2J_4^5}{J_1^7J_8^2}+2\frac{J_2^7J_4^8}{J_1^{13}J_8^4}+8q\frac{J_2^{11}J_8^4}{J_1^{13}J_4^4}\Big)\\ &\quad+2q(\overline{J}_{1,8}\overline{J}_{7,16}+q\overline{J}_{3,8}\overline{J}_{1,16})\\ &\quad\times\Big(2\frac{J_4^9}{J_1^7J_8^2}-\frac{J_2^{14}J_8^2}{J_1^{11}J_4^5}-\frac{J_2^4J_8^2}{J_1^7J_4}+4\frac{J_2^9J_4^2}{J_1^{13}}\Big).\end{aligned}\tag{3.4}$$

Thanks to (2.5) and (2.6),

$$\begin{aligned}-\frac{J_2^{12}J_4}{J_1^{11}J_8^2}+8q\frac{J_2^2J_4^3J_8^2}{J_1^7} &= -\frac{J_2^{12}J_4}{J_1^7J_8^2}\Big(\frac{J_4^{14}}{J_2^{14}J_8^4}+4q\frac{J_4^2J_8^4}{J_2^{10}}\Big)+8q\frac{J_2^2J_4^3J_8^2}{J_1^7}\\ &= -\frac{J_4^{15}}{J_1^7J_2^2J_8^6}+4q\frac{J_2^2J_4^3J_8^2}{J_1^7}\\ &= -\frac{J_4^5}{J_1^7J_8^2}\Big(\frac{J_4^{10}}{J_2^2J_8^4}-4q\frac{J_2^2J_8^4}{J_4^2}\Big)=-\frac{J_4^5}{J_1^3J_8^2},\end{aligned}\tag{3.5}$$

$$\begin{aligned}2\frac{J_4^9}{J_1^7J_8^2}-\frac{J_2^{14}J_8^2}{J_1^{11}J_4^5} &= 2\frac{J_4^9}{J_1^7J_8^2}-\frac{J_2^{14}J_8^2}{J_1^7J_4^5}\Big(\frac{J_4^{14}}{J_2^{14}J_8^4}+4q\frac{J_4^2J_8^4}{J_2^{10}}\Big)\\ &= \frac{J_4^9}{J_1^7J_8^2}-4q\frac{J_2^4J_8^6}{J_1^7J_4^3}\\ &= \frac{J_2^2J_8^2}{J_1^7J_4}\Big(\frac{J_4^{10}}{J_2^2J_8^4}-4q\frac{J_2^2J_8^4}{J_4^2}\Big)=\frac{J_2^2J_8^2}{J_1^3J_4}.\end{aligned}\tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain (3.2).

Moreover, Baruah *et al.* [5, (4.34)] proved that

$$\begin{aligned}2\sum_{n=0}^{\infty}\sigma_e\mathrm{mex}(4n)q^n &= \frac{J_2^6}{J_1^2J_4^4}\sum_{n=0}^{\infty}\sigma_o\mathrm{mex}(4n+1)q^n\\ &= \overline{J}_{3,8}\cdot\frac{J_2^6}{J_4^2}\cdot\frac{1}{J_1}\cdot\frac{1}{J_1^4}\Big(2\frac{J_4^5}{J_2J_8^2}\cdot\frac{1}{J_1^2}-\frac{1}{J_2^2}\cdot J_1^4-1\Big).\end{aligned}$$

Substituting (2.2), (2.3), (2.5), (2.6) and (3.3), upon simplification, we deduce that

$$\begin{aligned}&2\sum_{n=0}^{\infty}\sigma_e\mathrm{mex}(8n)q^n\\ &\quad=(\overline{J}_{3,8}\overline{J}_{7,16}+q^2\overline{J}_{1,8}\overline{J}_{1,16})\end{aligned}$$

$$\times\Big(-\frac{J_2^{22}}{J_1^{14}J_4^8}+16q\frac{J_4^8}{J_1^6J_2^2}-\frac{J_2^{12}}{J_1^{10}J_4^4}+2\frac{J_2^{17}}{J_1^{16}J_4J_8^2}+16q\frac{J_2^7J_4J_8^2}{J_1^{12}}\Big)$$
$$+4q(\overline{J}_{1,8}\overline{J}_{7,16}+q\overline{J}_{3,8}\overline{J}_{1,16})\Big(-\frac{J_4^4}{J_1^6}+2\frac{J_2^5J_4^7}{J_1^{12}J_8^2}+\frac{J_2^{19}J_8^2}{J_1^{16}J_4^7}\Big). \tag{3.7}$$

Substituting (2.7) into (3.7) yields

$$2\sum_{n=0}^{\infty}\sigma_e\mathrm{mex}(8n)q^n=(\overline{J}_{3,8}\overline{J}_{7,16}+q^2\overline{J}_{1,8}\overline{J}_{1,16})$$
$$\times\Big(-\frac{J_1^2}{J_2^2}-\frac{J_2^{12}}{J_1^{10}J_4^4}+2\frac{J_2^{17}}{J_1^{16}J_4J_8^2}+16q\frac{J_2^7J_4J_8^2}{J_1^{12}}\Big)$$
$$+4q(\overline{J}_{1,8}\overline{J}_{7,16}+q\overline{J}_{3,8}\overline{J}_{1,16})\Big(-\frac{J_4^4}{J_1^6}+2\frac{J_2^5J_4^7}{J_1^{12}J_8^2}+\frac{J_2^{19}J_8^2}{J_1^{16}J_4^7}\Big).$$

Replacing q by $-q$ in (3.1) and using the identity

$$(-q;-q)_\infty=\frac{J_2^3}{J_1J_4}, \tag{3.8}$$

after simplification,

$$\sum_{n=0}^{\infty}\sigma_o\mathrm{mex}(8n+1)(-q)^n=(J_{3,8}J_{7,16}+q^2J_{1,8}J_{1,16})$$
$$\times\Big(-\frac{J_1^7J_4^{12}}{J_2^{19}J_8^2}-J_1^3\cdot\frac{J_4^8}{J_2^9J_8^2}+2\frac{J_1^{13}J_4^{21}}{J_2^{32}J_8^4}-8q\frac{J_1^{13}J_4^9J_8^4}{J_2^{28}}\Big)$$
$$-2q(J_{1,8}J_{7,16}-qJ_{3,8}J_{1,16})$$
$$\times\Big(-\frac{J_1^7J_4^6J_8^2}{J_2^{17}}+J_1^3\cdot\frac{J_4^2J_8^2}{J_2^7}+4\frac{J_1^{13}J_4^{15}}{J_2^{30}}\Big). \tag{3.9}$$

Also, we note that

$$2\sum_{n=0}^{\infty}\sigma_e\mathrm{mex}(8n)q^n=(\overline{J}_{3,8}\overline{J}_{7,16}+q^2\overline{J}_{1,8}\overline{J}_{1,16})$$
$$\times\Big(-J_1^3\cdot\frac{1}{J_1J_2^2}-\frac{J_2^{12}}{J_1^{10}J_4^4}+2\frac{J_2^{17}}{J_1^{16}J_4J_8^2}+16q\frac{J_2^7J_4J_8^2}{J_1^{12}}\Big)$$
$$+4q(\overline{J}_{1,8}\overline{J}_{7,16}+q\overline{J}_{3,8}\overline{J}_{1,16})\Big(-\frac{J_4^4}{J_1^6}+2\frac{J_2^5J_4^7}{J_1^{12}J_8^2}+\frac{J_2^{19}J_8^2}{J_1^{16}J_4^7}\Big). \tag{3.10}$$

Substituting (2.9) into (3.9) and (3.10) and using the identity

$$\overline{J}_{a,b}=\frac{J_{2a,2b}J_b^2}{J_{a,b}J_{2b}},$$

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we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sigma_o \mathrm{mex}(8n+1)(-q)^n \\
&\quad \equiv (J_{3,8}J_{7,16} + q^2 J_{1,8}J_{1,16})\Big(-\frac{J_1^7 J_4^{12}}{J_2^{19} J_8^2} + 2\frac{J_1^{13} J_4^{21}}{J_2^{32} J_8^4} - 8q\frac{J_1^{13} J_4^9 J_8^4}{J_2^{28}}\Big) \\
&\qquad - \frac{J_4^8}{J_2^9 J_8^2}\Big(\frac{J_{56,128}J_{64}^2}{J_{28,64}J_{128}} - 3q\frac{J_{40,128}J_{64}^2}{J_{20,64}J_{128}} + 5q^3\frac{J_{24,128}J_{64}^2}{J_{12,64}J_{128}} - 7q^6\frac{J_{8,128}J_{64}^2}{J_{4,64}J_{128}}\Big) \\
&\qquad\quad \times (J_{3,8}J_{7,16} + q^2 J_{1,8}J_{1,16}) \\
&\qquad - 2q(J_{1,8}J_{7,16} - qJ_{3,8}J_{1,16})\Big(-\frac{J_1^7 J_4^6 J_8^2}{J_2^{17}} + 4\frac{J_1^{13} J_4^{15}}{J_2^{30}}\Big) \\
&\qquad - 2q\frac{J_4^2 J_8^2}{J_2^7}\Big(\frac{J_{56,128}J_{64}^2}{J_{28,64}J_{128}} - 3q\frac{J_{40,128}J_{64}^2}{J_{20,64}J_{128}} + 5q^3\frac{J_{24,128}J_{64}^2}{J_{12,64}J_{128}} - 7q^6\frac{J_{8,128}J_{64}^2}{J_{4,64}J_{128}}\Big) \\
&\qquad\quad \times (J_{1,8}J_{7,16} - qJ_{3,8}J_{1,16}) \pmod{16}
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
&2\sum_{n=0}^{\infty} \sigma_e \mathrm{mex}(8n) q^n \\
&\quad \equiv \Big(\frac{J_{6,16}J_{14,32}J_8^2 J_{16}}{J_{3,8}J_{7,16}J_{32}} + q^2\frac{J_{2,16}J_{2,32}J_8^2 J_{16}}{J_{1,8}J_{1,16}J_{32}}\Big)\Big(-\frac{J_2^{12}}{J_1^{10}J_4^4} + 2\frac{J_2^{17}}{J_1^{16}J_4 J_8^2} + 16q\frac{J_2^7 J_4 J_8^2}{J_1^{12}}\Big) \\
&\qquad - \frac{1}{J_1 J_2^2}\Big(\frac{J_{56,128}J_{64}^2}{J_{28,64}J_{128}} - 3q\frac{J_{40,128}J_{64}^2}{J_{20,64}J_{128}} + 5q^3\frac{J_{24,128}J_{64}^2}{J_{12,64}J_{128}} - 7q^6\frac{J_{8,128}J_{64}^2}{J_{4,64}J_{128}}\Big) \\
&\qquad\quad \times \Big(\frac{J_{6,16}J_{14,32}J_8^2 J_{16}}{J_{3,8}J_{7,16}J_{32}} + q^2\frac{J_{2,16}J_{2,32}J_8^2 J_{16}}{J_{1,8}J_{1,16}J_{32}}\Big) \\
&\qquad + 4q\Big(\frac{J_{2,16}J_{14,32}J_8^2 J_{16}}{J_{1,8}J_{7,16}J_{32}} + q\frac{J_{6,16}J_{2,32}J_8^2 J_{16}}{J_{3,8}J_{1,16}J_{32}}\Big) \\
&\qquad\quad \times \Big(-\frac{J_4^4}{J_1^6} + 2\frac{J_2^5 J_4^7}{J_1^{12} J_8^2} + \frac{J_2^{19} J_8^2}{J_1^{16} J_4^7}\Big) \pmod{16}.
\end{aligned} \tag{3.12}$$

Therefore, to prove (1.6) and (1.7), we need to prove that the coefficients on the right-hand sides of (3.11) and (3.12) vanish modulo 16.

Let f and g denote the right-hand sides of (3.11) and (3.12), respectively. By Theorem 2.4,

$$F(\tau) = q^{13/96}\frac{\eta^{122}(4\tau)\eta^2(8\tau)\eta_{16,8}^7(\tau)}{\eta^{48}(2\tau)\eta_{16,7}^{15}(\tau)} f \tag{3.13}$$

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and

$$G(\tau) = q^{1/96} \frac{\eta^{168}(4\tau)\eta_{16,7}^{21}(\tau)\eta_{16,8}^{10}(\tau)}{\eta^{72}(2\tau)} g \tag{3.14}$$

satisfy the transformation formulae

$$F(\gamma\tau) = (c\tau + d)^{38} F(\tau) \quad \text{and} \quad G(\gamma\tau) = (c\tau + d)^{48} G(\tau)$$

for any $\gamma \in \Gamma_1(128)$. From (2.11), the orders of $F(\tau)$ and $G(\tau)$ at every cusp of $\Gamma_1(128)$ are nonnegative, and so they are modular forms for $\Gamma_1(128)$ of weight 38 and 48, respectively. One can check the coefficients of the first 38912 terms of (3.13) are congruent to 0 modulo 16, and the coefficients of the first 49152 terms of (3.14) are congruent to 0 modulo 16. Therefore, by Theorem 2.6, $f \equiv 0 \pmod{16}$ and $g \equiv 0 \pmod{16}$. This completes the proof of Theorem 1.2.

4. Concluding remarks

We conclude this paper with two remarks.

First, Baruah *et al.* [5] proved (1.4) and (1.5) by using several identities involving $\varphi(q)$ and $\psi(q)$ and the Lambert series representations of $\varphi^2(q)$ and $\varphi(q)\varphi(q^2)$, where $\varphi(q)$ and $\psi(q)$ are two of Ramanujan's three classical theta functions. We provide a simplified proof of (1.4) based on (2.2), (2.9) and (3.3).

Baruah *et al.* [5, (4.17)] derived

$$\sum_{n=0}^{\infty} \sigma_o \mathrm{mex}(2n+1) q^n = \frac{J_2^2 J_8^2}{J_1^3 J_4} - \frac{J_1 J_8^2}{J_4}. \tag{4.1}$$

Replacing q by $-q$ in (4.1) and using (3.8),

$$\sum_{n=0}^{\infty} \sigma_o \mathrm{mex}(2n+1)(-q)^n = \frac{J_4^2 J_8^2}{J_2^7} \cdot J_1^3 - \frac{J_2^3 J_8^2}{J_4^2} \cdot \frac{1}{J_1}. \tag{4.2}$$

Substituting (2.2) and (2.9) into (4.2), taking all the terms of the form q^{2n}, after simplification,

$$\begin{aligned}
\sum_{n=0}^{\infty} \sigma_o \mathrm{mex}(4n+1) q^n &\equiv \frac{J_2^2 J_4^2}{J_1^7} (\overline{J}_{14,32} - 7q^3 \overline{J}_{2,32}) - \frac{J_1 J_4^2}{J_2^2} \overline{J}_{3,8} \pmod{16} \\
&= \frac{J_2^2 J_4^2}{J_1^7} (\overline{J}_{14,32} - 7q^3 \overline{J}_{2,32}) - \frac{J_1 J_4^2}{J_2^2} (\overline{J}_{14,32} + q^3 \overline{J}_{2,32}) \\
&\equiv \frac{J_1 J_4^2}{J_2^2} (\overline{J}_{14,32} - 7q^3 \overline{J}_{2,32} - \overline{J}_{14,32} - q^3 \overline{J}_{2,32}) \pmod{8} \\
&= -8q^3 \frac{J_1 J_4^2}{J_2^2} \overline{J}_{2,32} \equiv 0 \pmod{8},
\end{aligned}$$

where the second identity follows from (3.3). The congruence (1.4) thus follows.

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Second, the numerical evidence suggests the following conjecture.

CONJECTURE 4.1. We have

$$\lim_{X\to\infty} \frac{\#\{0 \le n < X : \sigma_o\mathrm{mex}(31n+18) \equiv 0 \pmod{16}\}}{X} = 1,$$

$$\lim_{X\to\infty} \frac{\#\{0 \le n < X : \sigma_e\mathrm{mex}(31n+18) \equiv 0 \pmod{16}\}}{X} = 1.$$

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