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On the deformation chirality of real cubic fourfolds

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Abstract

According to our previous results, the conjugacy class of the involution induced by the complex conjugation in the homology of a real non-singular cubic fourfold determines the fourfold up to projective equivalence and deformation. Here, we show how to eliminate the projective equivalence and obtain a pure deformation classification, that is, how to respond to the chirality problem: which cubics are not deformation equivalent to their image under a mirror reflection. We provide an arithmetical criterion of chirality, in terms of the eigen-sublattices of the complex conjugation involution in homology, and show how this criterion can be effectively applied taking as examples M-cubics (that is, those for which the real locus has the richest topology) and $(M-1)$-cubics (the next case with respect to complexity of the real locus). It happens that there is one chiral class of M-cubics and three chiral classes of $(M-1)$-cubics, in contrast to two achiral classes of M-cubics and three achiral classes of $(M-1)$-cubics.

L'univers est un ensemble dissymétrique, et je suis persuadé que la vie, telle qu'elle manifeste à nous, est fonction de la dissymétrie de l'univers ou des conséquences qu'elle entraîne.

Louis Pasteur

Observations sur les forces dissymétriques,
C. R. Acad. Sci. Paris **78** (1874), 1515–1518

1. Introduction

Recall that the projective non-singular cubic fourfolds form the complement in a projective space $P_{4,3} = P(Sym^3(\mathbb{C}^6))$ of dimension $\binom{5+3}{3} - 1 = 55$ to the so-called *discriminant hypersurface*. The discriminant hypersurface, which we denote by $\Delta_{4,3}$, is defined over reals and its real part $\Delta_{4,3}(\mathbb{R})$ is represented by real singular cubics, so that the space under study is nothing but $P_{4,3}(\mathbb{R}) \setminus \Delta_{4,3}(\mathbb{R})$. (Such a notation specifies the dimension, $n = 4$, and the degree, $d = 3$, of the hypersurfaces under consideration; we make use of it in § 8 for arbitrary n and d).

The space $\mathcal{C} = P_{4,3}(\mathbb{C}) \setminus \Delta_{4,3}(\mathbb{C})$ is connected, while $\mathcal{C}_{\mathbb{R}} = P_{4,3}(\mathbb{R}) \setminus \Delta_{4,3}(\mathbb{R})$ is not. Understanding the nature of the connected components of the latter space is a natural and classical task, it can be rephrased as a *deformation classification* of real projective non-singular

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cubic fourfolds. In our previous paper [FK08] we performed a classification with respect to a weaker *coarse deformation equivalence*: we call two real projective non-singular hypersurfaces *coarse deformation equivalent* if one hypersurface is deformation equivalent to a projective transformation of the other.

The difference between these two equivalence relations shows up in the case of subvarieties of real projective spaces of odd dimension. It is due to the orientability of real projective spaces of odd dimension, which implies that the group $\mathrm{PGL}(n+2,\mathbb{R})$ of real projective transformations of P^{n+1} has two connected components if n is even. In our case, $n=4$, so some of the coarse deformation classes of real projective non-singular cubic fourfolds may *a priori* consist of two deformation classes.

This leads us to a study of the following chirality phenomenon. We say that a real non-singular cubic $X \subset P^5$ and its coarse deformation class are *chiral* if X and its mirror image X' (that is, the image of X under a reflection in a hyperplane) belong to different connected components of $\mathcal{C}_{\mathbb{R}}$, and *achiral* if they belong to the same component (that is, if X and X' can be connected by a continuous family of real non-singular cubics). Clearly, a coarse deformation class consists of two deformation classes if and only if it is chiral.

In the present paper we reduce the chirality problem to a specific problem of the arithmetics of lattices and use this reduction to show that certain real cubic fourfolds are chiral, while certain other real cubic fourfolds are achiral. We pay a special attention to real cubic fourfolds with extremal values of the sum of the Betti numbers. Namely, we consider in details the cases of M-*cubics*, in which $\dim H_*(X(\mathbb{R});\mathbb{Z}/2)=\dim H_*(X(\mathbb{C});\mathbb{Z}/2)$ (the maximal value), and the cases of $(M-1)$-*cubics*, in which $\dim H_*(X(\mathbb{R});\mathbb{Z}/2)=\dim H_*(X(\mathbb{C});\mathbb{Z}/2)-2$ (the next value). As shown in [FK08], the M-cubics form three and the $(M-1)$-cubics form six coarse projective classes. In the present paper we prove that one coarse class of M-cubics and three coarse classes of $(M-1)$-cubics are achiral, while the other coarse classes of M- and $(M-1)$-cubics are chiral.

As a by-product, we give a new proof (in a sense, more natural and more direct) of the *homological quasi-simplicity* of cubic fourfolds, where the latter means that two real non-singular cubic hypersurfaces X_1, X_2 in P^5 are coarse deformation equivalent if and only if the involutions induced by the complex conjugation on $H_4(X_i(\mathbb{C}))$, $i=1,2$, regarded as a lattice via the intersection index form, are isomorphic (cf. [FK08, Theorem 1.1] and Theorem 4.1.2 below).

In our previous paper [FK08], we were using a relation between the nodal cubics in P^5 and the complete intersections of bi-degree $(2,3)$ in P^4. Since these complete intersections are the 6-polarized $K3$-surfaces, it had allowed us to apply Nikulin's coarse deformation classification of real 6-polarized $K3$-surfaces in terms of involutions on the $K3$-lattice and his results on the arithmetics of such involutions, see [Nik79, Nik08].

Such a roundabout approach was imposed by a lack of sufficiently complete understanding of the moduli of cubic hypersurfaces, in contrast to that of $K3$-surfaces. In particular, in the case of $K3$-surfaces one had in one's hands the surjectivity of the period map, while for cubic fourfolds the characterization of the image of the period map remained unknown. The situation has changed recently, after Laza [Laz07] and Looijenga [Loo09] established a suitable surjectivity statement for cubic fourfolds.

In our opinion, the two approaches are complementary and both deserve to be developed further. Combined together they should give us a better understanding of the topology of the moduli space of real cubic fourfolds on the one hand, and of the topology of the discriminant of cubic fourfolds on the other hand. Note that we had already found not only the coarse deformation classes but also their adjacencies in [FK08]. Now, via the period map, the deformation classes become endowed with a certain polyhedral structure. This opens

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a way for a full understanding of some natural stratifications of the moduli and the coefficient spaces of real cubic fourfolds.

The topological study of non-singular real cubic hypersurfaces has a long history, see [FK08] for a brief account. In addition, we would like to add a reference to the recent investigation of the moduli space of real cubic surfaces performed by Allcock *et al.* [ACT03].

Let us recall that according to Klein's classification of real cubic surfaces, see [Kle73] (the classification statement is reproduced in [FK08]), all of the real non-singular cubic surfaces are achiral. It may be worth mentioning that Klein's achirality argument in [Kle73] contained a mistake, which was corrected by Klein in his collected papers, see [Kle21].

This paper is organized as follows. In § 2, we review some properties of the period map for complex cubic fourfolds. In § 3, we introduce the real period spaces with the real period map and derive the properties of the latter from the corresponding properties of the complex period map. The results of § 3 are then applied in § 4 to reduce the chirality problem to some arithmetics of hyperbolic integer lattices and their reflection groups. Section 5 collects necessary information about Vinberg's algorithm for finding the fundamental domains of the arithmetical reflection groups. The technique developed in §§ 3–5 is applied in §§ 6 and 7 to treat the chirality of M- and $(M-1)$-cubic fourfolds. Section 8 is devoted to concluding remarks. We mention some other cases which were studied using similar methods, and mention some other related results and possible directions of their development. In particular, we discuss a notion of reversibility, which is closely related to chirality.

Index of notation

$\mathbb{M}(X)$, $\mathbb{M}^0(X)$, $\mathbb{M}$, $\mathbb{M}^0$, $\mathcal{D}$, $\widehat{\mathcal{D}}$, $\mathrm{Aut}^{\pm}(\mathbb{M}^0)$, $\mathcal{C}$, $\mathcal{C}^{\#}$	§ 2.1
R_v, V_2, V_6, H_v, $\mathcal{H}_{\Delta}$, $\mathcal{H}_{\infty}$, ω_p, $\mathcal{E}$	§ 2.2
$\mathbb{M}_{\pm}(c)$, $\mathbb{M}^0_{\pm}(c)$, $\mathbb{M}_{\pm}(X)$, $\mathbb{M}^0_{\pm}(X)$	§ 3.1
$\mathcal{C}^c_{\mathbb{R}}$, $\mathcal{C}^{c\#}_{\mathbb{R}}$	§ 3.2
$\overline{\mathcal{D}}$, $\mathcal{D}^c_{\mathbb{R}}$, $\widehat{\mathcal{D}}^c_{\mathbb{R}}$	§ 3.3
$\Upsilon_{\pm}(c)$, $\Lambda_{\pm}(c)$, $\Lambda^{\#}_{\pm}(c)$, $\mathrm{per}^{c\#}_{\mathbb{R}}$, $\mathcal{D}^{c\#}_{\mathbb{R}}$	§ 3.4
$\mathcal{H}_{\pm}(c)$, $\mathcal{H}^{\#}_{\pm}(c)$, $\mathrm{Per}^c_{\mathbb{R}}$	§ 3.5
$\pm P^{\#}$	§ 4.2
discr , discr_2, discr_3, $\mathfrak{q}_{\mathbb{L}}$, δ, δ_3, $P_{\pm}$, $P^{\#}_{\pm}$, $\mathrm{Aut}(P_{\pm})$, $\mathrm{Aut}^{\pm}(P_{\pm})$	§ 4.4
Φ, $\Phi^{\pm}$, Φ^b,[v], $[v]^{\#}$, $H^{\pm}_v$, $[H_v]$, $[H_v]^{\#}$	§ 5.1
Γ_J, G_J	§ 5.2
e_i, e^*_i	§ 6.1

2. Period map for complex cubic fourfolds

2.1 The period domain for marked cubic fourfolds

Consider a non-singular cubic fourfold $X \subset P^5$. It is well known that its non-zero Hodge numbers in dimension four are $h^{3,1} = h^{1,3} = 1$ and $h^{2,2} = 21$. The lattice $\mathbb{M}(X) = H^4(X)$ is odd with signature $(21, 2)$. The *polarization class* $h(X) \in \mathbb{M}(X)$, that is, the square of the hyperplane section, is a characteristic element of $\mathbb{M}(X)$ with $h^2 = 3$, and so the primitive sublattice $\mathbb{M}^0(X) = \{x \in \mathbb{M}(X) \mid xh = 0\}$ is even and has discriminant group $\mathbb{Z}_3$. This implies that there is

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a lattice isomorphism between $\mathbb{M}(X)$ and $\mathbb{M} = 3I + 2U + 2E_8$, which sends $h(X)$ to $h = (1,1,1) \in 3I$, so that $\mathbb{M}^0(X)$ is identified with $\mathbb{M}^0 = A_2 + 2U + 2E_8$. (The above transitivity property of the lattice automorphism group action holds because both indices of the inertia, 21 and 2, are at least two, see [Wal62].) A particular choice of such an isomorphism $\phi\colon (\mathbb{M}(X), h(X)) \to (\mathbb{M}, h)$ will be called a *marking of* X. We restrict the choice of markings as specified below.

The complex line $\phi(H^{3,1}(X)) \subset \mathbb{M}^0 \otimes \mathbb{C}$ is isotropic and has negative pairing with the conjugate (and, thus, also isotropic) line $\phi(H^{1,3}(X)) = \overline{\phi(H^{3,1}(X))}$, that is to say, $w^2 = 0$ and $w\overline{w} < 0$ (and, thus, $\overline{w}^2 = 0$) for all $w \in \phi(H^{3,1}(X))$. Writing $w = u + iv$, $u, v \in \mathbb{M}^0 \otimes \mathbb{R}$, we can reformulate it as $u^2 = v^2 < 0$ and $uv = 0$, which implies that the real plane $\langle u, v\rangle \subset \mathbb{M}^0 \otimes \mathbb{R}$ spanned by u and v is negative definite and bears a natural orientation given by $u = \operatorname{Re} w$, $v = \operatorname{Im} w$. Note that the orientation determined similarly by the complex line $\phi(H^{1,3}(X)) \subset \mathbb{M}^0 \otimes \mathbb{C}$ is the opposite orientation.

The line $\phi(H^{3,1}(X)) \subset \mathbb{M}^0 \otimes \mathbb{C}$ specifies a point $\Omega(X) \in P(\mathbb{M}^0 \otimes \mathbb{C})$ (as usual, P stands for the projectivization) called the *period point of* X. This period point belongs to the quadric $Q = \{w^2 = 0\} \subset P(\mathbb{M}^0 \otimes \mathbb{C})$, and more precisely, to its open subset, $\widehat{\mathcal{D}} = \{w \in Q \mid w\overline{w} < 0\}$. This subset has two connected components, which are exchanged by the complex conjugation (this reflects also switching from the given complex structure on X to the complex conjugate structure).

The orthogonal projection of a negative-definite real plane in $\mathbb{M}^0 \otimes \mathbb{R}$ to another is non-degenerate. Thus, to select one of the two connected components of $\widehat{\mathcal{D}}$ we fix an orientation of negative definite real planes in $\mathbb{M}^0 \otimes \mathbb{R}$ so that the orthogonal projection preserves it. We call it the *prescribed orientation* and restrict the choice of markings to those for which the orientation of $\phi(H^{3,1}(X))$ defined by the pairs $u = \operatorname{Re} w$, $v = \operatorname{Im} w$ for $w \in \phi(H^{3,1}(X))$ is the prescribed orientation. We denote this component by $\mathcal{D}$ and call it the *period domain*. By $\operatorname{Aut}^+(\mathbb{M}^0)$ we denote the group of those automorphisms of $\mathbb{M}^0$ which preserve the prescribed orientation (and, thus, preserve $\mathcal{D}$). We put $\operatorname{Aut}^-(\mathbb{M}^0) = \operatorname{Aut}(\mathbb{M}^0) \setminus \operatorname{Aut}^+(\mathbb{M}^0)$. This complementary coset consists of automorphisms exchanging the connected components of $\widehat{\mathcal{D}}$.

The projective space $P_{4,3}$ formed by all cubic fourfolds splits into the *discriminant hypersurface* $\Delta_{4,3}$ formed by singular cubics and its complement, $\mathcal{C}$. Let $\mathcal{C}^\sharp$ denote the space of marked non-singular cubics. The natural projection $\mathcal{C}^\sharp \to \mathcal{C}$ is obviously a covering with the deck transformation group $\operatorname{Aut}^+(\mathbb{M}^0)$. The above conventions define the *period map* $\operatorname{per}\colon \mathcal{C}^\sharp \to \mathcal{D}$, $(X, \phi) \mapsto \phi(H^{3,1}(X))$.

2.2 Principal properties of the period map

The global Torelli theorem for cubic fourfolds proved in [Voi86] claims injectivity of the period map. We need the following version of this theorem.

Theorem 2.2.1 (Global Torelli theorem [Voi86]). *Assume that (X, ϕ) and (X', ϕ') are non-singular marked cubic fourfolds such that $\operatorname{per}(X, \phi) = \operatorname{per}(X', \phi')$. Then there exists one and only one isomorphism $f\colon X' \to X$ such that $\phi' \circ f^* = \phi$.* □

The existence statement is explicit in [Voi86]. The uniqueness statement is implicit there. It follows easily from two well-known observations: first, each automorphism of a non-singular cubic fourfold is induced by a projective transformation and, second, if a projective transformation acts trivially in the cohomology then it is trivial.

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Theorem 2.2.2 (Construction of (anti-)isomorphisms). *Let X and X' be non-singular cubic fourfolds and $F : H^4(X;\mathbb{Z}) \to H^4(X';\mathbb{Z})$ an isometry such that $F(h(X)) = h(X')$.*

(1) *If $F(H^{3,1}(X)) = H^{3,1}(X')$, then there exists one and only one isomorphism $f : X' \to X$ such that $f^* = F$.*

(2) *If $F(H^{1,3}(X)) = H^{3,1}(X')$, then there exists one and only one isomorphism $f : X' \to \overline{X}$ such that $f^* = F$.*

Here and in what follows we denote by $\overline{X}$ the variety complex conjugate to X. If $X \subset P^5$ is given by a polynomial, then $\overline{X} \subset P^5$ can be seen as the variety given by the polynomial with the complex conjugate coefficients.

Proof of Theorem 2.2.2. The first statement is nothing but an equivalent version of Theorem 2.2.1. The second statement follows from the first or directly from Theorem 2.2.1 applied to (X', ϕ') and $(\overline{X}, \phi' \circ F)$, where ϕ' is any marking of X'. □

Consider the reflection R_v in $\mathbb{M}^0 \otimes \mathbb{C}$ across the mirror-hyperplane $H_v = \{x \in \mathbb{M}^0 \otimes \mathbb{C} \mid xv = 0\}$ defined as $x \mapsto x - 2(xv/v^2)v$, and note that it preserves the lattice $\mathbb{M}^0$ invariant if $v \in \mathbb{M}^0$ is such that $v^2 = 2$, or such that $v^2 = 6$ and xv is divisible by 3 for all $x \in \mathbb{M}^0$. We call these two types of lattice elements 2-*roots* and 6-*roots*, respectively, and denote their sets by V_2 and V_6. Note that $R_v \in \mathrm{Aut}^+(\mathbb{M}^0)$ for any $v \in V_2 \cup V_6$. If $v \in V_2$, then the reflection R_v extends (as a reflection) to $\mathbb{M}$ and h is preserved by this extension. In contrast, if $v \in V_6$, then the reflection R_v does not extend to a reflection in $\mathbb{M}$ and, moreover, the unique extension of R_v to $\mathbb{M}$ maps h to $-h$ (cf. Lemma 4.3.2 below). On the other hand, if $v \in V_6$, then the *anti-reflection* $-R_v$ extends to an isometry of $\mathbb{M}$ preserving h. This extension is the anti-reflection with respect to the 2-plane generated by h and v. In particular, it also represents an element of $\mathrm{Aut}^+(\mathbb{M}^0)$.

The union of the mirrors H_v for all $v \in V_2$ gives after projectivization a union of hyperplanes $\mathcal{H}_\Delta \subset P(\mathbb{M}^0 \otimes \mathbb{C})$, and a similar union for all $v \in V_6$ gives another union of hyperplanes, $\mathcal{H}_\infty \subset P(\mathbb{M}^0 \otimes \mathbb{C})$.

Theorem 2.2.3 (Surjectivity of the period map [Laz07, Loo09]). *The image of the period map $\mathrm{per} : \mathcal{C}^\sharp \to \mathcal{D}$ is the complement of $\mathcal{H}_\Delta \cup \mathcal{H}_\infty$.* □

According to the Griffiths theory, for any non-singular cubic $X \subset P^5$ the line $H^{3,1}(X)$ is spanned by the class $[\omega_p] \in H^4(X;\mathbb{C})$ of the 4-form $\omega_p = \mathrm{Res}(\mathcal{E}/p^2)$. Here $\mathcal{E}$ stands for the Euler 5-form in $\mathbb{C}^6$, $\mathcal{E} = \sum_{i=0}^5 (-1)^i x_i \, dx_0 \wedge \cdots \wedge d\widehat{x}_i \wedge \cdots \wedge dx_5$, and p for a polynomial defining X (as usual, a hat over x_i means that this term is omitted). The ratio $\mathcal{E}/p^2$ is a well-defined meromorphic 5-form in P^5, with a second-order pole along X. The residue ω_p of this form is a closed 4-form on X, which is a linear combination of $(3,1)$ and $(4,0)$-forms. Its class $[\omega_p]$ is known to be non-trivial, thus, it spans $H^{3,1}(X)$.

3. Periods in the real setting

3.1 Geometric involutions

Consider a non-singular cubic fourfold X defined by a real polynomial p, and let $\mathrm{conj} : X \to X$ denote the complex conjugation map, which will also be called the *real structure on* X. The latter map induces a lattice involution $\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X)$ such that $\mathrm{conj}^*(h) = h$ and, hence, also induces a lattice involution in $\mathbb{M}^0(X)$. Denote by $\mathbb{M}^0_\pm(X)$ and $\mathbb{M}_\pm(X)$ the eigen-sublattices

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$\{x \in \mathbb{M}^0(X) \mid \mathrm{conj}^*(x) = \pm x\}$ and $\{x \in \mathbb{M}(X) \mid \mathrm{conj}^*(x) = \pm x\}$, respectively. We obviously have $\mathbb{M}_- = \mathbb{M}^0_-$ and $\sigma_-(\mathbb{M}_+(X)) = \sigma_-(\mathbb{M}^0_+(X))$, where σ_- denotes the negative index of inertia (i.e. the number of negative squares in a diagonalization).

Lemma 3.1.1. *One has* $\sigma_-(\mathbb{M}^0_\pm(X)) = 1$.

Proof. The map $w \mapsto \overline{\mathrm{conj}^* w}$ gives an anti-linear involution in $H^{3,1}(X)$. Thus, there exist non-zero elements $w \in H^{3,1}(X)$ such that $\mathrm{conj}^*(w) = \overline{w}$. In terms of the real and imaginary components of $w = u_+ + iu_-$, this identity means that $u_\pm \in \mathbb{M}_\pm(X) \otimes \mathbb{R}$. These components satisfy the relations $u_+^2 = u_-^2 = \frac{1}{2} w\overline{w} < 0$. They belong to $\mathbb{M}^0_\pm(X)$, since $wh = 0$. It remains to note that the intersection form is positive definite on $H^{2,2}(X)$. □

We call a lattice involution $c : \mathbb{M} \to \mathbb{M}$ *geometric* if $c(h) = h$ and $\sigma_-(\mathbb{M}^0_\pm(c)) = 1$, where $\mathbb{M}^0_\pm(c)$ denotes the eigen-sublattices $\{x \in \mathbb{M}^0 \mid c(x) = \pm x\}$. Let us note that all geometric involutions preserve $\mathbb{M}^0$ and the involutions induced in $\mathbb{M}^0$ belong to $\mathrm{Aut}^-(\mathbb{M}^0)$.

According to Lemma 3.1.1, all lattice involutions $c : \mathbb{M} \to \mathbb{M}$ isomorphic to an involution $\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X)$ for a non-singular real cubic X are geometric. A pair $(c : \mathbb{M} \to \mathbb{M}, h \in \mathbb{M})$ isomorphic to $(\mathrm{conj}^* : \mathbb{M}(X) \to \mathbb{M}(X), h(X))$ is called the *homological type* of X. By a *real c-marked cubic fourfold* we mean a real non-singular cubic fourfold equipped with a marking ϕ such that $\phi \circ \mathrm{conj}^* = c \circ \phi$.

Theorem 3.1.2. *For any geometric involution* $c : \mathbb{M} \to \mathbb{M}$ *the pair* (c, h) *is the homological type of some non-singular real cubic fourfold.*

This theorem is one of the results obtained in [FK08]. After fixing some notation, we give below (at the end of § 3.3) an independent proof based on the surjectivity of the period map and the global Torelli theorem.

The number of isometry classes of geometric involutions is finite. Their list can be found in [FK08] (see also Tables 8 and 9 in § 8).

Up to the end of this section we assume that c is a geometric involution.

3.2 Real parameter space $\mathcal{C}^c_\mathbb{R}$

We denote by $\mathcal{C}^c_\mathbb{R} \subset \mathcal{C}_\mathbb{R}$ the set of real cubic fourfolds of homological type c, and by $\mathcal{C}^{c\sharp}_\mathbb{R}$ the set of c-marked real cubic fourfolds. The former consists of some number of connected components of $\mathcal{C}_\mathbb{R}$. The latter can be seen as the real part of $\mathcal{C}^\sharp$ with respect to the involution $\mathrm{conj}^{c\sharp} : \mathcal{C}^\sharp \to \mathcal{C}^\sharp$, which send $(X, \phi) \in \mathcal{C}^\sharp$ to $(\mathrm{conj}(X), c \circ \phi \circ \mathrm{conj}^*)$. The forgetful map $(X, \phi) \to X$ defines a (multi-component) covering $\mathcal{C}^{c\sharp}_\mathbb{R} \to \mathcal{C}^c_\mathbb{R}$.

3.3 Real period domain $\mathcal{D}^c_\mathbb{R}$

Let us extend c to a complex linear involution on $\mathbb{M} \otimes \mathbb{C}$ and denote also by c the induced involutions on $\mathbb{M}^0 \otimes \mathbb{C}$, $P = P(\mathbb{M}^0 \otimes \mathbb{C})$, and $\widehat{\mathcal{D}}$. Note that c permutes the two components $\mathcal{D}$ and $\overline{\mathcal{D}}$ of $\widehat{\mathcal{D}}$ and, thus, $\overline{c}(\mathcal{D}) = \mathcal{D}$, where $\overline{c} : \mathbb{M}^0 \otimes \mathbb{C} \to \mathbb{M}^0 \otimes \mathbb{C}$ is the composition of c with the complex conjugation in $\mathbb{M}^0 \otimes \mathbb{C}$.

Let $\widehat{\mathcal{D}}^c_\mathbb{R}$ and $\mathcal{D}^c_\mathbb{R}$ denote the fixed point set of $\overline{c}$ restricted to $\widehat{\mathcal{D}}$ and $\mathcal{D}$. The latter consists of the lines generated by $w = u_+ + iu_-$ such that $u_\pm \in \mathbb{M}^0_\pm(c) \otimes \mathbb{R}$, $u_+^2 = u_-^2 < 0$, and the orientation u_+, u_- is the prescribed orientation. Since c is geometric, both $\mathcal{D}^c_\mathbb{R}$ and its (trivial) double covering $\widehat{\mathcal{D}^c_\mathbb{R}}$ are non-empty.

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As it follows from definitions, the period of a c-marked real cubic fourfold belongs to $\mathcal{D}^c_{\mathbb{R}} = \{x \in \mathcal{D} \mid c(x) = \overline{x}\}$. Therefore, we call $\mathcal{D}^c_{\mathbb{R}}$ the *real period domain of real c-marked cubic fourfolds.*

Proof of Theorem 3.1.2. Pick up a generic point $[w] \in \mathcal{D}^c_{\mathbb{R}}$ (so that there is no vector $v \in V_2 \cup V_6$ orthogonal to w) and apply Theorem 2.2.3. This gives a non-singular cubic fourfold X and a marking ϕ such that $\mathrm{per}(X, \phi) = [w]$. The triple $(X, X' = \overline{X}, F = \phi^{-1}c\phi)$ satisfies the assumptions of Theorem 2.2.2, which gives an antiholomorphic involution $\mathrm{conj} : X \to X$ such that $\mathrm{conj}^* = \phi^{-1}c\phi$. Clearly, $(\mathbb{M}, c)$ is the homological type of (X, conj), and it remains to note that $\mathrm{Pic}\, X = \mathbb{Z}$, $X(\mathbb{R})$ is non-empty (as it is for any real hypersurface of odd degree), and therefore any antiholomorphic involution of X is induced by the complex conjugation in suitable projective coordinates of $P^5 = P(\mathcal{O}_X(1))$. □

3.4 Refined real period map

Consider the quadratic cones $\Upsilon_\pm(c) = \{u \in \mathbb{M}^0_\pm(c) \otimes \mathbb{R} : u^2 < 0\}$ and the associated Lobachevsky (one- and two-component, respectively) spaces $\Lambda_\pm(c) = \Upsilon_\pm(c)/\mathbb{R}^*$ and $\Lambda^\sharp_\pm(c) = \Upsilon_\pm(c)/\mathbb{R}_+$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{R}_+ = (0, \infty)$.

As in § 3.3, we associate with a point in $\mathcal{D}^c_{\mathbb{R}}$ represented by $w = u_+ + iu_-$ (where $u_\pm \in \mathbb{M}^0_\pm(c) \otimes \mathbb{R}$, $u_+^2 = u_-^2 < 0$, and the orientation u_+, u_- is the prescribed orientation) the point in $\Lambda_+(c) \times \Lambda_-(c)$ represented by the pair (u_+, u_-). This gives a well-defined analytic isomorphism $\mathcal{D}^c_{\mathbb{R}} = \Lambda_+(c) \times \Lambda_-(c)$. The ambiguity in the choice of representatives gives rise to a refined real period domain $\mathcal{D}^{c\sharp}_{\mathbb{R}} \subset \Lambda^\sharp_+(c) \times \Lambda^\sharp_-(c)$, $\mathcal{D}^{c\sharp}_{\mathbb{R}} = \{(u_+\mathbb{R}_+, u_-\mathbb{R}_+) \in \Lambda^\sharp_+(c) \times \Lambda^\sharp_-(c) \mid \text{the orientation } u_+, u_- \text{ is the prescribed orientation}\}$.

To define $\mathrm{per}^{c\sharp}_{\mathbb{R}}(X, \phi) \in \mathcal{D}^{c\sharp}_{\mathbb{R}}$ for a non-singular real c-marked cubic $(X, \phi) \in \mathcal{C}^{c\sharp}_{\mathbb{R}}$, we pick up a real polynomial p defining X and consider $w = \phi([\omega_p])$ (see the end of § 2). As we have seen already, the latter splits as $w = u_+ + iu_-$, where $u_\pm \in \mathbb{M}^0_\pm(c)$, the pair (u_+, u_-) is defined uniquely by X up to a positive factor, and this pair spans a negative-definite plane with the prescribed orientation. Thus, we obtain a uniquely defined real period $\mathrm{per}^{c\sharp}_{\mathbb{R}}(X, \phi) \in \mathcal{D}^{c\sharp}_{\mathbb{R}}$ and a well-defined map $\mathrm{per}^{c\sharp}_{\mathbb{R}} : \mathcal{C}^{c\sharp}_{\mathbb{R}} \to \mathcal{D}^{c\sharp}_{\mathbb{R}}$. The above components $u_\pm\mathbb{R}_+ \in \Lambda^\sharp_\pm(c)$ of $\mathrm{per}^{c\sharp}_{\mathbb{R}}(X, \phi)$ are denoted by $u^\sharp_\pm(X, \phi)$.

3.5 Polyhedral period cells

Denote by $\mathcal{H}_\pm(c) \subset \Lambda_\pm(c)$ and $\mathcal{H}^\sharp_\pm(c) \subset \Lambda^\sharp_\pm(c)$ the union of hyperplanes orthogonal to vectors from $(V_2 \cup V_6) \cap \mathbb{M}^0_\pm(c)$. The connected components of the complement $\Lambda_\pm(c) \setminus \mathcal{H}_\pm(c)$ will be called *the cells of* $\Lambda_\pm(c)$ and the hyperplanes from $\mathcal{H}_\pm(c)$ *the walls.* As is known, these hyperplanes form a locally finite arrangement (the group generated by reflections in these hyperplanes is discrete) so that the above cells are (locally finite) polyhedra. Put

$$\mathrm{Per}^c_{\mathbb{R}} = \mathcal{D}^{c\sharp}_{\mathbb{R}} \cap ((\Lambda^\sharp_+(c) \setminus \mathcal{H}^\sharp_+(c)) \times (\Lambda^\sharp_-(c) \setminus \mathcal{H}^\sharp_-(c)))$$

and call *c-cells* the connected components of $\mathrm{Per}^c_{\mathbb{R}}$. Note that the orientation restriction involved in the definition of $\mathcal{D}^{c\sharp}_{\mathbb{R}}$ establishes a one-to-one correspondence between the halves of $\Lambda^\sharp_+(c)$ and the halves of $\Lambda^\sharp_-(c)$, and this correspondence commutes with multiplication by -1. Therefore, $\mathrm{Per}^c_{\mathbb{R}}$ spits into a union of pairs of opposite c-cells. The natural projection $\mathrm{Per}^c_{\mathbb{R}} \to \Lambda_+(c) \times \Lambda_-(c)$ establishes a one-to-one correspondence between the set of pairs of opposite c-cells and the set of products of the cells of $\Lambda_\pm(c)$.

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Given a continuous family of real c-marked cubics (X_t, ϕ_t), $t \in [0,1]$, they can be defined by a continuous family of polynomials p_t, and hence their real periods $u^\sharp_\pm(X_t, \phi_t)$ belong to the same cells of $\Lambda^\sharp_\pm(c)$. The converse is also true.

Lemma 3.5.1. *Assume that (X_i, ϕ_i), $i = 0, 1$ is a pair of real c-marked cubic fourfolds defined by real polynomials p_i. Then, X_i can be connected by a continuous family X_t of real c-marked cubic fourfolds if and only if their periods $u^\sharp_\pm(X_i, \phi_i)$ belong to the same cells of $\Lambda^\sharp_\pm(c)$* (or in other words, if and only if the periods $\mathrm{per}_\mathbb{R}(X_i, \phi_i)$ belong to the same component of $\mathrm{Per}^c_\mathbb{R}$).

Proof. The lemma follows from the description of the periods of cubic fourfolds (and the local Torelli theorem over the reals), because the vectors $v \in (V_2 \cup V_6)$ which are not from $\mathbb{M}^0_+ \cup \mathbb{M}^0_-$ define hyperplanes H_v which have intersection with $\mathbb{M}^0_\pm \otimes \mathbb{R}$ of codimension less than one. □

4. Deformations and chirality

4.1 The mirror pairs of markings

Given a real hypersurface $X \subset P^5$, we can consider its *mirror image*, $X' = R(X)$, obtained from X by a reflection $R : P^5 \to P^5$ with respect to some real hyperplane $H \subset P^n$. According to our definitions, X is *chiral* if X and X' belong to different connected components of $\mathcal{C}_\mathbb{R}$, and *achiral* if they belong to the same component.

Assume that (X, ϕ) is a marked non-singular cubic fourfold. Then the isomorphism $R^* : \mathbb{M}(X') \to \mathbb{M}(X)$ induced by R respects the Hodge structure and the polarization classes of X and X', and thus yields a marking $\phi \circ R^*$ of X'. We say that the markings ϕ and $\phi' = \phi \circ R^*$ are *mirror images of each other*, or *a mirror pair of markings*.

Lemma 4.1.1. *Assume that a non-singular real cubic fourfold X is defined by a real polynomial p and its mirror image, X', by a polynomial q. Then the period vectors $\phi[\omega_p]$ and $\phi'[\omega_q]$ are oppositely directed if X and X' are endowed with the mirror pair of markings: ϕ and $\phi' = \phi \circ R^*$.*

Proof. The form $\mathcal{E}/q^2$ representing $[\omega_q]$ (see the end of § 2) changes the direction under the action of R, because $R^*(\mathcal{E}) = -\mathcal{E}$ and $q \circ R$ differs from p by a real factor. □

As an immediate corollary of Lemmas 4.1.1 and 3.5.1 we obtain a new proof of the following theorem from [FK08].

Theorem 4.1.2 (Coarse deformation classification). *One real non-singular cubic fourfold is deformation equivalent to a projective transformation of another real non-singular cubic fourfold if and only if they are of the same homological type.*

Proof. Given a c-marking, we can compose it with lattice reflections R_v, $v \in V_2 \cap \mathbb{M}^0_\pm(c)$, and anti-reflections $-R_v$, $v \in V_6 \cap \mathbb{M}^0_\pm(c)$, to move the period into any pair of opposite cells of $\mathrm{Per}_\mathbb{R}(c)$ given in advance. When necessary, we can apply Lemma 4.1.1 and move the period into any of these opposite cells. According to Lemma 3.5.1 this means that the real non-singular cubics of homological type c are coarse deformation equivalent to each other. The 'only if' part is trivial. □

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4.2 Basic criterion of chirality for cubic fourfolds

Let us fix a geometric involution c. Given a non-singular c-marked real cubic fourfold (X, ϕ), denote by $P^\sharp(X) \subset \mathrm{Per}^c_{\mathbb{R}}$ the c-cell which contains $\mathrm{per}^{c\sharp}_{\mathbb{R}}(X, \phi)$ (in other words, the c-cell which contains $w = \phi[\omega_p]$ where, as usual, p is a real polynomial defining X).

Lemma 4.2.1. *The underlying non-singular real cubic fourfold X of a real c-marked cubic fourfold (X, ϕ) is achiral if and only if there exists a lattice isometry of $\mathbb{M}$ which:*

(1) *commutes with c;*

(2) *preserves the polarization class h;*

(3) *induces an automorphism of $\mathbb{M}^0$ which preserves the prescribed orientation; and*

(4) *sends the c-cell $P^\sharp(X)$ to the opposite c-cell, $-P^\sharp(X)$.*

Proof. Let X' denote the mirror image of X with the mirror image marking ϕ'. By Lemma 4.1.1, its period $w' = \phi'[\omega_q]$ belongs to $-P^\sharp(X)$. On the other hand, any continuous family of real non-singular cubic fourfolds connecting X with X' gives another marking of X', say ϕ', and according to Lemma 3.5.1 the period $\phi'[\omega_q]$ belongs to $P^\sharp(X)$. Comparing the two markings of X' we obtain a lattice isometry of $\mathbb{M} = \mathbb{M}(X')$ which transforms $P^\sharp(X)$ into $-P^\sharp(X)$; being a difference between two markings, it also preserves the polarization h, induces an automorphism of $\mathbb{M}^0$ which preserves the prescribed orientation, and commutes with c. Conversely, given such a lattice isometry, we can change the mirror image marking of X' and then apply Lemma 3.5.1 to deduce that X and X' both belong to the same component of $\mathcal{C}_{\mathbb{R}}$. □

4.3 Some lattice gluing lemmas

To simplify the above criterion and to reduce it to a study of $\mathrm{Aut}\mathbb{M}^0_+(c)$ we need the following results involving a technique of discriminant groups. Recall that for any non-degenerate lattice $\mathbb{L}$ of finite rank the discriminant group $\mathrm{discr}\,\mathbb{L} = \mathbb{L}/\mathbb{L}^*$ is a finite group and that, if the lattice $\mathbb{L}$ is even, this group carries a canonical finite quadratic form $\mathfrak{q}_{\mathbb{L}} : \mathrm{discr}\,\mathbb{L} \to \mathbb{Q}/2\mathbb{Z}$ defined via $\mathfrak{q}_{\mathbb{L}}(x + \mathbb{L}) = x^2 \mod 2\mathbb{Z}$. Note that any isometry, $f \in \mathrm{Aut}\mathbb{L}$, induces an automorphism of $\mathrm{discr}\,\mathbb{L}$, which preserves $\mathfrak{q}_{\mathbb{L}}$ if $\mathbb{L}$ is even. This induced automorphism is denoted by $\delta(f)$.

Theorem 4.3.1 (Nikulin's theorem [Nik79]). *Assume that $\mathbb{L}$ is an even lattice of signature $(n, 1)$, $n \geqslant 0$, whose discriminant group $\mathrm{discr}(\mathbb{L})$ is 2-periodic. Then any isometry $\delta : \mathrm{discr}(\mathbb{L}) \to \mathrm{discr}(\mathbb{L})$ is induced by some isometry $f : \mathbb{L} \to \mathbb{L}$.* □

In the present paper we deal with the three lattices: $\mathbb{M}_-(c)$, $\mathbb{M}^0_+(c)$, and the rank-one lattice $\langle h \rangle \subset \mathbb{M}$ generated by h. The first two lattices are even, and the latter is odd. The discriminant group $\mathrm{discr}\,\mathbb{M}_-(c)$ is 2-periodic, the discriminant group $\mathrm{discr}\langle h \rangle$ is a cyclic group of order three, and the discriminant group $\mathrm{discr}\,\mathbb{M}^0_+(c)$ is canonically isomorphic to the direct sum $\mathrm{discr}\,\mathbb{M}_-(c) + \mathrm{discr}\langle h \rangle$, so that $\mathrm{discr}\,\mathbb{M}_-(c)$ is identified with the 2-primary part $\mathrm{discr}_2\,\mathbb{M}^0_+(c)$ of $\mathrm{discr}\,\mathbb{M}^0_+(c)$, and $\mathrm{discr}\langle h \rangle$ with its 3-primary part $\mathrm{discr}_3\,\mathbb{M}^0_+(c)$. The canonical isomorphism $\mathrm{discr}_2\,\mathbb{M}^0_+(c) \to \mathrm{discr}\,\mathbb{M}_-(c)$ is an anti-isometry, that is, it transforms $-\mathfrak{q}_{\mathbb{M}_-(c)}$ into $\mathfrak{q}_{\mathbb{M}^0_+(c)}$ restricted to $\mathrm{discr}_2\,\mathbb{M}^0_+(c)$. (In fact, the lattice $\mathrm{discr}\langle h \rangle$, as any non-degenerate finite rank lattice with a fixed characteristic element, can be also equipped with a quadratic form, and with respect to this quadratic form the canonical isomorphism $\mathrm{discr}_3\,\mathbb{M}^0_+(c) \to \mathrm{discr}\langle h \rangle$ is also an anti-isometry.)

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The following lattice gluing lemmas are well known and their proofs are straightforward, see, e.g., [Nik79].

Lemma 4.3.2. *Any automorphism $f_+^0 \in \mathrm{Aut}(\mathbb{M}_+^0(c))$ can be uniquely extended to $\mathbb{M}_+(c)$. This extension sends the polarization class h to itself if and only if the 3-primary component $\delta_3(f_+^0)$ of $\delta(f_+^0)$ is trivial, that is $\delta_3(f_+^0) = \mathrm{id}$.* □

Lemma 4.3.3. *A pair of automorphisms $f_\pm \in \mathrm{Aut}(\mathbb{M}_\pm(c))$ are induced from $f \in \mathrm{Aut}(\mathbb{M}, c)$ if and only if $\delta(f_+) = \delta(f_-)$.* □

Automorphisms $f_\pm$ satisfying the conditions of Lemma 4.3.3 will be called *compatible*.

4.4 Lattice characterization of chirality

The *reflection group* W_+ generated in $\mathrm{Aut}(\mathbb{M}_+^0(c))$ by reflections R_v, $v \in (V_2 \cup V_6) \cap \mathbb{M}_+^0(c)$ acts transitively on the set of cells of $\Lambda_+(c)$. If $v \in V_6$, then R_v does not extends to M_+, but anti-reflection $-R_v$ does. So, we consider also the group $W_+^\# \subset \mathrm{Aut}(\mathbb{M}_+^0(c))$ generated by reflections R_v, $v \in V_2 \cap \mathbb{M}_+^0(c)$, and anti-reflections $-R_v$, $v \in V_6 \cap \mathbb{M}_+^0(c)$ (the two groups are isomorphic and induce the same action on Λ_+). Any of the cells $P_+ \subset \Lambda_+(c)$ being fixed, the group $\mathrm{Aut}(\mathbb{M}_+^0(c))$ splits into a semi-direct product $W_+ \rtimes \mathrm{Aut}(P_+)$, where $\mathrm{Aut}(P_+) = \{g \in \mathrm{Aut}(\mathbb{M}_+^0(c)) \mid g(P_+) = P_+\}$ is the stabilizer of P_+.

With $\mathbb{M}_-(c)$ the situation is even simpler: since its discriminant group is of period 2 the intersection $V_6 \cap \mathbb{M}_-(c)$ is empty. Thus, in this case we consider simply the reflection group $W_- \subset \mathrm{Aut}(\mathbb{M}_-(c))$ generated by reflections R_v, $v \in V_2 \cap \mathbb{M}_-^0(c)$. This reflection group acts transitively on the set of cells of $\Lambda_-(c)$ and, therefore, $\mathrm{Aut}(\mathbb{M}_-(c))$ splits into a semi-direct product $W_- \rtimes \mathrm{Aut}(P_-)$, where $\mathrm{Aut}(P_-) = \{g \in \mathrm{Aut}(\mathbb{M}_-(c)) | g(P_-) = P_-\}$ is the stabilizer of a cell P_- of $\Lambda_-(c)$.

The preimage of $P_\pm$ in $\Lambda_\pm^\#$ splits into two connected components: a pair of opposite *c-cells* $P_\pm^\#$ and $-P_\pm^\#$. Each $g \in \mathrm{Aut}(P_\pm)$ either permutes this pair of cells, and then we say that it is $P_\pm$*-reversing*, or it preserves both $P_\pm^\#$ and $-P_\pm^\#$, and then we call it $P_\pm$*-direct*. The subgroup of $\mathrm{Aut}(P_\pm)$ formed by $P_\pm$-direct elements will be denoted by $\mathrm{Aut}^+(P_\pm)$, while the coset of $P_\pm$-reversing elements will be denoted by $\mathrm{Aut}^-(P_\pm)$. The crucial observation for our study of chirality is that an automorphism $f \in \mathrm{Aut}(\mathbb{M})$ preserving each of $P_\pm$ belongs to $\mathrm{Aut}^+(\mathbb{M})$ if and only if its components $f_+ = f|_{\mathbb{M}_+^0}$, $f_- = f|_{\mathbb{M}_-}$ are both of the same type: either simultaneously $f_\pm \in \mathrm{Aut}^+(P_\pm)$ or simultaneously $f_\pm \in \mathrm{Aut}^-(P_\pm)$.

In the case of lattices $\mathbb{M}_+^0$, an additional characteristic of $g \in \mathrm{Aut}(\mathbb{M}_+^0)$ is its 3-primary component, $\delta_3(g)$, which may be trivial or not. In a slightly more general setting, we consider a hyperbolic lattice $\mathbb{L}$ whose discriminant splits as $\mathrm{discr}(\mathbb{L}) = \mathrm{discr}_2(\mathbb{L}) + \mathrm{discr}_3(\mathbb{L})$, where $\mathrm{discr}_2(\mathbb{L})$ is 2-periodic and $\mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$. We say that $g \in \mathrm{Aut}(\mathbb{L})$ is $\mathbb{Z}/3$*-direct* if $\delta_3(g) = \mathrm{id}$, and $\mathbb{Z}/3$*-reversing* if $\delta_3(g) \neq \mathrm{id}$ (certainly, in the latter case $\delta_3(g) = -\mathrm{id}$).

Theorem 4.4.1. *A non-singular real cubic fourfold X of homological type c is achiral if and only if the lattice $\mathbb{M}_+^0(c)$ admits an automorphism $g \in \mathrm{Aut}^-(P_+)$ which is $\mathbb{Z}/3$-direct.*

Proof. The 'only if' part is a straightforward consequence of the 'only if' part of Lemma 4.2.1.

To prove the 'if' part, let us pick up a c-marking $\phi : \mathbb{M}(X) \to \mathbb{M}$ and choose $f_+^0 \in \mathrm{Aut}^-(P_+(X))$ which is $\mathbb{Z}/3$-direct. From Lemma 4.3.2 it follows that f_+^0 extends to $f_+ \in \mathrm{Aut}\mathbb{M}_+$ preserving h. Lemma 4.3.3 and Theorem 4.3.1 imply that we can find $f_- \in \mathrm{Aut}(\mathbb{M}_-)$ compatible with f_+ and $f \in \mathrm{Aut}(\mathbb{M})$ defined by (f_+, f_-). By composing f_- (and f) with a suitable $w_- \in W_-$,

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the component f_- can be chosen in $\mathrm{Aut}(P_-) \subset \mathrm{Aut}(\mathbb{M}_-)$. If $f \in \mathrm{Aut}(\mathbb{M})$ defined by (f_+, f_-) restricts to an automorphism of $\mathbb{M}^0$ belonging to $\mathrm{Aut}^+(\mathbb{M}^0)$ (in other words, if f belongs to $\mathrm{Aut}^+(\mathbb{M})$), then f transforms $P^\sharp(X)$ into $-P^\sharp(X)$ since it preserves the prescribed orientation and $f_+ \in \mathrm{Aut}^-(P_+)$. Therefore, in this case due to Lemma 4.2.1 we are done. If f restricts to an automorphism of $\mathbb{M}^0$ belonging to $\mathrm{Aut}^+(\mathbb{M}^0)$ (in other words, if $f \in \mathrm{Aut}^-(\mathbb{M})$), then we replace f_- by $-f_-$, observe that the pair $(f_+, -f_-)$ defines an automorphism $f \circ c$ which restricts to an automorphism of $\mathbb{M}^0$ belonging to $\mathrm{Aut}^+(\mathbb{M}^0)$, and argue as before. □

5. Auxiliary arithmetics

5.1 Root systems and chirality of special hyperbolic lattices

In this section $\mathbb{L}$ is a lattice of signature $(n, 1)$, $n \geqslant 1$. Throughout this section we make two additional assumptions on $\mathbb{L}$ which are satisfied in the cases of $\mathbb{L} = \mathbb{M}^0_+(c)$ that we are concerned about. The first assumption is that the discriminant $\mathrm{discr}(\mathbb{L})$ splits as $\mathbb{Z}/3 + \mathrm{discr}_2(\mathbb{L})$, where the summand $\mathrm{discr}_2(\mathbb{L})$ is a 2-periodic group. Let $\Phi = V_2 \cup V_6$, where $V_k = \{v \in \mathbb{L} \mid v^2 = k, 2(vw/v^2) \in \mathbb{Z},\ \forall w \in \mathbb{L}\}$ (note that for $k = 2$ the condition $2(vw/v^2) \in \mathbb{Z}$ is always satisfied). Our second assumption is that the rank of Φ is equal to the rank of $\mathbb{L}$ (that is, maximal possible) and, thus, Φ is a root system in $\mathbb{L}$. This holds for $\mathbb{L} = \mathbb{M}^0_+(c)$ for all geometric involution c except one rather special case $\mathbb{M}^0_+(c) = U(2) + E_6(2)$ in which $\Phi = \varnothing$ (the complete list of $\mathbb{L} = \mathbb{M}^0_+(c)$ is given in Tables 8 and 9, in § 8, and the stated property can be easily checked on a case-by-case basis). Vectors $v \in V_k$ will be called *k-roots*.

We let $\mathbb{L}_\mathbb{R} = \mathbb{L} \otimes \mathbb{R}$, and as before, consider $\Upsilon = \{v \in \mathbb{L}_\mathbb{R} \mid v^2 < 0\}$, and the hyperbolic spaces $\Lambda = \Upsilon/\mathbb{R}^*$, along with $\Lambda^\# = \Upsilon/\mathbb{R}_+$. In this context we use notation H_v for the hyperplane $\{w \in \mathbb{L}_\mathbb{R} \mid vw = 0\}$ and $H_v^\pm$ for the half-spaces $\{w \in \mathbb{L}_\mathbb{R} \mid \pm vw \geqslant 0\}$. For $v \in \Upsilon$, H_v, and so on, we denote by $[v] \in \Lambda$, $[H_v] \subset \Lambda$, $[v]^\# \in \Lambda^\#$, $[H_v]^\# \subset \Lambda^\#$, and so on, the corresponding object after projectivization.

We distinguish the *reflection group* $W \subset \mathrm{Aut}(\mathbb{L})$ generated by the reflections $R_v \in \mathrm{Aut}(\mathbb{L})$, $x \mapsto x - 2(vx/v^2)v$, $v \in V_2 \cup V_6$, and the group $W^\# \subset \mathrm{Aut}(\mathbb{L})$ generated by the reflections R_v, $v \in V_2$, and the anti-reflections $-R_v$, $v \in V_6$. Hyperplanes $[H_v]$ (respectively, $[H_v]^\#$), $v \in \Phi$, cut Λ (respectively, $\Lambda^\#$) into open polyhedra, whose closures are called the *cells*. The cells in Λ are the fundamental chambers of W, and the pairs of opposite cells in $\Lambda^\#$ are the fundamental chambers of $W^\#$.

Let us pick up a cell $P \subset \Lambda$ and fix a covering c-cell $P^\# \subset \Lambda^\#$. Choosing any vector $p \in \Upsilon$ so that $[p]^\#$ lies in the interior of $P^\#$, we let $\Phi^\pm = \{v \in \Phi \mid \pm vp > 0\}$. The minimal subset $\Phi^b \subset \Phi^-$ such that $P^\# = \bigcap_{v \in \Phi^b} [H_v^-]^\#$ is called the *basis of* Φ *defined by* $P^\#$. The hyperplanes $[H_v]$, $v \in \Phi^b$, support n-dimensional faces of P and will be called the *walls of* P. Note that any $v \in \Phi^-$ is a linear combination of the roots in Φ^b with non-negative coefficients.

Theorem 4.4.1 motivates the following definition: $\mathbb{L}$ is called *achiral* if it admits a $\mathbb{Z}/3$-direct automorphism $g \in \mathrm{Aut}^-(P)$, for some cell P. Obviously, if $\mathbb{L}$ is achiral, then a $\mathbb{Z}/3$-direct automorphism $g \in \mathrm{Aut}^-(P)$ exists for any cell P. It is also obvious that the existence of a $\mathbb{Z}/3$-direct $g \in \mathrm{Aut}^-(P)$ is equivalent to the existence of $\mathbb{Z}/3$-reversing $h \in \mathrm{Aut}^+(P)$, since these two kinds of automorphisms just differ by sign.

5.2 Coxeter's graphs and their symmetry

The *Coxeter graph* Γ has Φ^b as the vertex set. The vertices are colored: 2-roots are white and 6-roots are black. The edges are weighted: the weight of an edge connecting vertices $v, w \in \Phi^b$

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is $m_{vw} = 4((vw)^2/v^2w^2)$, and $m_{vw} = 0$ means the absence of an edge. These weights are non-negative integers, because $2(vw/v^2), 2(vw/w^2) \in \mathbb{Z}$, and $v^2, w^2 > 0$ for any $v, w \in \Phi^b$. In the case of $m_{vw} = 1$, the angle between H_v and H_w is $\pi/3$, and $v^2 = w^2$; such edges are not labelled. The case of $m_{vw} = 2$ (which corresponds to an angle $\pi/4$) cannot happen, since $v^2, w^2 \in \{2, 6\}$. An edge of weight $m_{vw} = 3$ always connects a 2-root with a 6-root; it corresponds to an angle $\pi/6$, and will be labelled by '6'. The case of $m_{vw} = 4$ corresponds to parallel hyperplanes in Λ, and we sketch a thick edge between v and w. If $m_{vw} > 4$, then the corresponding hyperplanes in Λ are ultra-parallel (diverging), and we sketch a dotted edge.

For a subset $J \subset \Phi^b$ we may consider also the subgraph Γ_J which is formed by the vertex set J and all of the edges of Γ connecting these vertices. We say that Γ_J is the *Coxeter graph of* J. If J is finite and ordered, $J = \{v_1, \dots, v_{|J|}\}$, then we consider also the *Gram matrix*, G_J, whose (ij)-entry is $v_i v_j$.

A permutation $\sigma : J \to J$ will be called a *symmetry of* Γ_J if it preserves the weight of edges and the length of the roots, that is, $(\sigma(v))^2 = v^2$ and $m_{\sigma(v)\sigma(w)} = m_{vw}$ for all $v, w \in J$.

Theorem 5.2.1 (Existence of symmetries). *Assume that a subset $J \subset \Phi^b$ spans $\mathbb{L}$ over $\mathbb{Z}$. Then any symmetry $\sigma : J \to J$ of Γ_J is induced by an automorphism of the lattice $\mathbb{L}$ which preserves the cell $P^\#$ invariant.*

Proof. Such a symmetry preserves the Gram matrix of the vectors from J. Therefore, it is induced by an isometry of $\mathbb{L} \otimes \mathbb{Q}$. Since the vectors from J span $\mathbb{L}$ over $\mathbb{Z}$, this isometry maps $\mathbb{L}$ to $\mathbb{L}$. Assuming that it maps $P^\#$ to another cell, we observe that these two cells have J as a common set of face normal vectors. Pick up a wall separating the two cells and note that each of the normal root vectors $\pm v \neq 0$ of such a wall has non-negative product with the vectors from J, which is a contradiction, since the vectors from J generate the whole space. □

To recognize $\mathbb{Z}/3$-reversing symmetries of Γ, one can use the following observation. Considering some direct sum decomposition of $\mathbb{L}$, we notice that one of the direct summands, $\mathbb{L}_1$, has $\mathrm{discr}_3(\mathbb{L}_1) = \mathbb{Z}/3$, while the other direct summands have 2-periodic discriminants (because $\mathrm{discr}(\mathbb{L})$ obtains an induced direct sum decomposition). For any vertex w of Γ viewed as a vector in $\mathbb{L}$, we can consider its $\mathbb{L}_1$-component. Our simple observation is that *σ is $\mathbb{Z}/3$-direct if for all black vertices, $v \in V_6$, of Γ the $\mathbb{L}_1$-components of v and $\sigma(v)$ are congruent modulo $3\mathbb{L}_1$, and $\mathbb{Z}/3$-reversing if for some $v \in V_6$ we have $v - \sigma(v) \notin 3\mathbb{L}$.*

5.3 Vinberg's algorithm

Vinberg's method [Vin75] of calculation of the Coxeter graph of Φ is to pick up a vector $p \in \Upsilon$ so that $[p]^\# \in P^\#$, and then to determine a sequence of roots $v_i \in \Phi^b$, $i = 1, 2, \dots$, ordered so that the hyperbolic distance from p to the walls $H_i = H_{v_i}$ of P is increasing. Such a distance can be characterized by the (non-negative) value $2(pv_i)^2/v_i^2$, which will be called *the level of root v_i with respect to p* (the coefficient 2 here is chosen to make it an integer in the further considerations).

The level-zero vectors in Vinberg's sequence form a root basis in the root system $\{v \in \Phi | vp = 0\}$. Since choosing $[p]$ at a vertex of $[P]$ (rather than in its interior) simplifies calculations, we always try to start with such a choice of p so that the system of the level-zero roots would be of the maximal rank, namely $\dim \mathbb{L} - 1$.

If Vinberg's sequence, $v_1, \dots, v_m$, is found up to level r, then the vectors $v \in \Phi$ of higher levels should satisfy the conditions: $pv < 0$ and $vv_i \leqslant 0$ for all v_i, $1 \leqslant i \leqslant m$. If vectors v respecting these conditions do exist, then the next segment of Vinberg's sequence is constituted by all such vectors

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of the minimal level. Note that the order of Vinberg's roots within the same level is not well defined (and is inessential).

This process terminates and gives the basis Φ^b if the latter is finite, otherwise the process enumerates vectors of Φ^b in an infinite sequence. If we found Vinberg's vectors $v_1, \ldots, v_m$ up to some level r, then we can use one of Vinberg's sufficient criteria below for detecting the termination of the process.

5.4 Vinberg's termination criteria

The Gram matrix G_J and the Coxeter graph Γ_J are called *elliptic* (of rank r) if G_J is positive definite (of rank r). As is observed in [Vin75], the elliptic subgraphs of Γ of rank $n-k$ are in one-to-one correspondence with the k-dimensional faces of $[P]$. Namely, an elliptic subgraph Γ_J corresponds to the face supported by the projectivization of the linear space $H_J = \bigcap_{v \in J} H_v$.

The connected components of an elliptic graph G_J must belong to the list of the classical elliptic graphs of the root systems (see, for example, [Bou68]). In our case (since $m_{vw} = 2$ do not appear), an elliptic graph cannot be anything other than A_r, D_r, E_6, E_7, E_8, and G_2.

A connected subgraph Γ_J and its Gram matrix G_J are called *parabolic* if G_J is a positive-semi-definite matrix of rank $|J|-1$. In our case, a parabolic connected subgraph should be one of the graphs $\widetilde{A}_r$ (recall that $\widetilde{A}_1$ is just a thick edge), $\widetilde{D}_r$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, and $\widetilde{G}_2$, where the subscript always equals the *rank of the parabolic graph*, $|J|-1$. A disconnected subgraph Γ_J and its Gram matrix are called parabolic if all of the connected components of Γ_J are parabolic. The rank of such Γ_J is by definition the sum of the ranks of its components. As is observed in [Vin75], a subgraph Γ_J is parabolic of maximal possible rank, $n-1$, if and only if the intersection H_J defines a vertex of $[P]$ at infinity (on the absolute).

A matrix G_J (and its Coxeter graph Γ_J) is called *critical*, if it is not elliptic, but any submatrix $G_{J'}$, $J' \subsetneq J$, is elliptic. Such G_J is parabolic if degenerate. If a critical matrix G_J is non-degenerate, its graph G_J is called *Lannér's diagrams*. The list of Lannér's diagrams can be found, for example, in [Vin75, Vin85]. Note that the only Lannér's diagram possible under the assumptions of this section is a dotted edge (the other Lannér's diagrams all contain a pair of roots which have the ratio of length different from one and three).

Theorem 5.4.1 (Finite volume criterion [Vin75]). *Vinberg's sequence terminates at $J = \{v_1, \ldots, v_m\}$ if the polyhedron P_J bounded in $\Lambda_{\mathbb{L}}$ by the hyperplanes dual to $v \in J$ has a finite hyperbolic volume.* □

To determine the finiteness of the volume, Vinberg gives several criteria. One of them [Vin85, Proposition 4.2(1)] can be formulated (in the form of [Dol08, Proposition 2.4]) as follows.

Theorem 5.4.2 (Criterion of the finiteness of the volume). *The polyhedron P_J has a finite volume if and only if the following two conditions are satisfied:*

(1) *Γ_J contains an elliptic subdiagram of rank $n-1$ where $n = \dim \mathbb{L} - 1$;*

(2) *any elliptic subdiagram of rank $n-1$ of Γ_J can be extended to an elliptic subdiagram of rank n, or to a parabolic subdiagram of rank $n-1$; and there exist precisely two such extensions.* □

Remark. The second condition in Theorem 5.4.2 just means that any edge is adjacent to two vertices: finite or at infinity.

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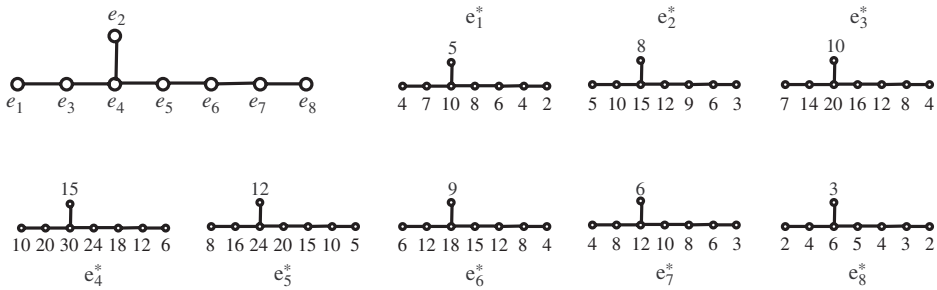

Figure 1. Coxeter graph E_8 and the vectors e_i^*.

There is another (simpler, but only sufficient) criterion which can be used if Γ_J does not contain Lannér's schemes (that is, dotted edges in our setting).

Theorem 5.4.3 (Sufficient criterion of finiteness of the volume [Vin75]). *The volume of P_J is finite if the following conditions are satisfied:*

(1) *J has rank $\dim \mathbb{L} = n+1$;*

(2) *the Coxeter graph, Γ_J, does not contain Lannér's diagrams as subgraphs;*

(3) *every connected parabolic subgraph in Γ_J is a connected component of some parabolic subgraph of rank $n-1$ in Γ.*

6. Chirality of M-cubics

6.1 Preliminaries and the main statement

A particular, characteristic, feature of M-cubics is that the lattice $\mathbb{M}$ splits into a direct sum of the eigen-lattices $\mathbb{M}_+$ and $\mathbb{M}_-$. Thus, the eigen-lattices are unimodular in the case of M-cubics, and only in this case. As it follows from the classification in [FK08] (or can be easily deduced directly from Theorem 4.1.2, Lemma 3.1.1, and the classification of unimodular lattices), there exists precisely three coarse deformation classes (equivalently, three homological types) of M-cubics. The corresponding three lattices $\mathbb{M}_+$ are $U+3I=-I+4I$, $U+3I+E_8=-I+12I$, and $U+3I+2E_8=-I+20I$. The polarization class $h\in\mathbb{M}_+$ is characteristic, of square three, and can be identified with $(1,1,1)\in 3I$. So, the primitive lattices $\mathbb{M}_+^0$ are even and isomorphic to $U+A_2$, $U+A_2+E_8$, and $U+A_2+2E_8$, respectively. The corresponding lattices $\mathbb{M}_-$ are also even and isomorphic to $U+2E_8$, $U+E_8$, and U, respectively.

Theorem 6.1.1. *Non-singular real cubic fourfolds of types $\mathbb{M}_+^0(c)=U+A_2$ and $\mathbb{M}_+^0(c)=U+A_2+E_8$ are chiral; in particular, the cubic fourfolds of each of these two types form two deformation classes. Non-singular real cubic fourfolds of type $\mathbb{M}_+^0(c)=U+A_2+2E_8$ are achiral; these cubic fourfolds form one deformation class.*

The rest of this section is devoted to a case-by-case proof of this theorem.

We fix a basis u_1, u_2 in U and a basis a_1, a_2 in A_2, so that $u_i^2=0$ $(i=1,2)$, $u_1u_2=1$, $a_i^2=2$ $(i=1,2)$, and $a_1a_2=-1$. The basis $e_1,\dots,e_8$ in E_8 is chosen as is shown on the Coxeter graph of E_8, see Figure 1 (we use the usual convention: $e_i^2=2$ for $i=1,\dots,8$ and $e_i\circ e_j=-\delta_{ij}$). This figure also presents the dual vectors e_i^*, $i=1,\dots,8$, which are also elements of E_8, because the lattice E_8 is unimodular; for example, $e_8^*=2e_8+3e_7+4e_6+5e_5+6e_4+4e_3+3e_2+2e_1$.

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Table 1. Vinberg's vectors for $\mathbb{M}^0_+(c) = U + A_2$.

	U	A_2
p	1, −1	0, 0
level 0		
v_1	1, 1	0, 0
v_2	0, 0	0, 1
v_3	0, 0	1, −1
level 1		
v_4	0, −1	−1, −1

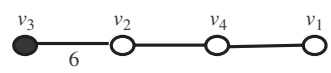

Figure 2. Coxeter's graph for $U + A_2$.

In the case of $U + A_2 + 2E_8$, the basic vectors of the additional E_8-summand will be denoted by e'_i and their duals by $(e'_i)^*$, $i = 1, \ldots, 8$.

In all of these three M-cases, to apply Vinberg's algorithm (see § 5.3) we pick $p = u_1 - u_2$. Then we choose as the set of level-zero vectors the standard bases in each of E_8-components of $\mathbb{M}^0_+(c)$ and complete them by two square-two vectors $v_1 = u_1 + u_2$ and $v_2 = a_2$, and one square-six vector $v_3 = a_1 - a_2$. This choice determines uniquely a cell P_+ in $\Lambda_+(c)$. The vectors of higher levels in Vinberg's sequence must have components $x_1u_1 + x_2u_2 + y_1a_1 + y_2a_2$ in $U + A_2$ satisfying the following relations:

$$x_2 < x_1, \quad x_1 + x_2 \leqslant 0, \quad 2y_2 \leqslant y_1, \quad y_1 \leqslant y_2.$$

Note that the vector $v_4 = -(u_2 + a_1 + a_2)$ satisfies these relation and, thus, appears in the list as a vector of level one in each of the three M-cases.

Certainly, the basic vectors of the E_8-summands also impose restrictions on the vectors of higher levels. Namely, their components in the first (respectively, second) E_8-summand should be linear combinations of $e_1^*, \ldots, e_8^*$ (respectively, $(e'_1)^*, \ldots, (e'_8)^*$) with non-positive coefficients.

6.2 The case $\mathbb{M}^0_+(c) = U + A_2$

Here, Vinberg's sequence starts from vectors v_1, v_2, v_3, v_4 given in Table 1. The Coxeter graph of the vector system $\{v_1, v_2, v_3, v_4\}$ is shown on Figure 2. The only parabolic subgraph is $\widetilde{G_2}$ (the subgraph generated by v_2 and v_3), and it has rank $2 = \dim \Lambda_+ - 1$. By Vinberg's finite volume criterion, it implies that Vinberg's sequence terminates at $\{v_1, v_2, v_3, v_4\}$, and so the polyhedron P_+ is found. Since the Coxeter graph admits no symmetries, $-\mathrm{id}$ is the only element of $\mathrm{Aut}^-(P_+)$. Thus, applying Theorem 4.4.1 we conclude that the studied cubic fourfolds are chiral.

6.3 The case $\mathbb{M}^0_+(c) = U + A_2 + E_8$

Here, the level-zero vectors are $e_1, \ldots, e_8, v_1, v_2$, and v_3. The level-one vectors are v_4 and $v_5 = -u_2 - e_8^*$ (see Table 2). This gives the Coxeter graph shown in Figure 3. This graph has only two parabolic subgraphs: $\widetilde{G_2}$ and $\widetilde{E_8}$. Vinberg's finite volume criterion is satisfied because these

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Table 2. Vinberg's vectors for $\mathbb{M}^0_+(c) = U + A_2 + E_8$.

	U	A_2	E_8
p	$1, -1$	$0, 0$	0
level 0			
v_1	$1, 1$	$0, 0$	0
v_2	$0, 0$	$0, 1$	0
v_3	$0, 0$	$1, -1$	0
level 1			
v_4	$0, -1$	$-1, -1$	0
v_5	$0, -1$	$0, 0$	$-e_8^*$

Figure 3. Coxeter's graph for $U + A_2 + E_8$.

subgraphs are disjoint and the sum of their ranks is $2 + 8 = \dim \Lambda_{\mathbb{M}^0_+} - 1$. The graph has no symmetries and arguing in a similar manner to § 6.2 we conclude that the studied cubic fourfolds are chiral.

6.4 The case $\mathbb{M}^0_+(c) = U + A_2 + 2E_8$

Here, the level-zero vectors are $e_1, \dots, e_8, e'_1, \dots, e'_8$, v_1, v_2, and v_3. The level-one consists of three 2-roots v_4, v_5, and $v'_5 = -u_2 - (e'_8)^*$. On the next level, 16, there is one 2-root $v_6 = 2(u_1 - u_2) - (a_1 + a_2) - e_1^* - (e'_1)^*$. Then, on the level 36 there is a pair of 2-roots:

$$v_7 = 3(u_1 - u_2) - (2a_1 + a_2) - e_7^* - (e'_2)^*,$$
$$v'_7 = 3(u_1 - u_2) - (2a_1 + a_2) - e_2^* - (e'_7)^*.$$

Our list of Vinberg's vectors given in Table 3 also includes a pair of 6-roots of level 48,

$$v_8 = 6(u_1 - u_2) - (4a_1 + 2a_2) - 3e_8^* - 3(e'_1)^*,$$
$$v'_8 = 6(u_1 - u_2) - (4a_1 + 2a_2) - 3e_1^* - 3(e'_8)^*.$$

The above list contains three 6-roots: v_3, v_8, and v'_8. If we drop them and consider the Coxeter subgraph formed only by the 2-roots, we obtain the hexagonal diagram shown in Figure 4. This diagram has a lot of symmetries. Consider the involution which fixes the vertices e_7, e'_4, v_6, e'_7 and permutes the vertices v_1, e_4. Since the set of vectors corresponding to the vertices of the diagram generate the lattice $\mathbb{M}^0_+(c)$, this involution is induced by a lattice involution $f : \mathbb{M}^0_+(c) \to \mathbb{M}^0_+(c)$ (see Theorem 5.2.1). Since in the whole Coxeter diagram the 6-root v_3 is connected with the 2-roots v_2, v_7, v'_7 and the 6-root v'_8 is connected with the 2-roots e_1, v_7, e'_8, the automorphism f transforms v_3 into v'_8. The A_2-components of v_3 and v'_8 are $(1, -1)$ and $(-4, -2)$, which are not congruent modulo three. This implies that f is $\mathbb{Z}/3$-reversing. By Theorem 5.2.1, f is P_+-direct, so applying Theorem 4.4.1 we conclude that this homological type is achiral.

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Table 3. Vinberg's vectors for $\mathbb{M}^0_+(c) = U + A_2 + 2E_8$.

v_i	U	A_2	E_8	E_8
p	$1, -1$	$0, 0$	0	0
level 0				
v_1	$1, 1$	$0, 0$	0	0
v_2	$0, 0$	$0, 1$	0	0
v_3	$0, 0$	$1, -1$	0	0
level 1				
v_4	$0, -1$	$-1, -1$	0	0
v_5	$0, -1$	$0, 0$	$-e_8^*$	0
v_5'	$0, -1$	$0, 0$	0	$-(e_8')^*$
level 16				
v_6	$2, -2$	$-1, -1$	$-e_1^*$	$-(e_1')^*$
level 36				
v_7	$3, -3$	$-2, -1$	$-e_7^*$	$-(e_2')^*$
v_7'	$3, -3$	$-2, -1$	$-e_2^*$	$-(e_7')^*$
level 48				
v_8	$6, -6$	$-4, -2$	$-3e_8^*$	$-3(e_1')^*$
v_8'	$6, -6$	$-4, -2$	$-3e_1^*$	$-3(e_8')^*$

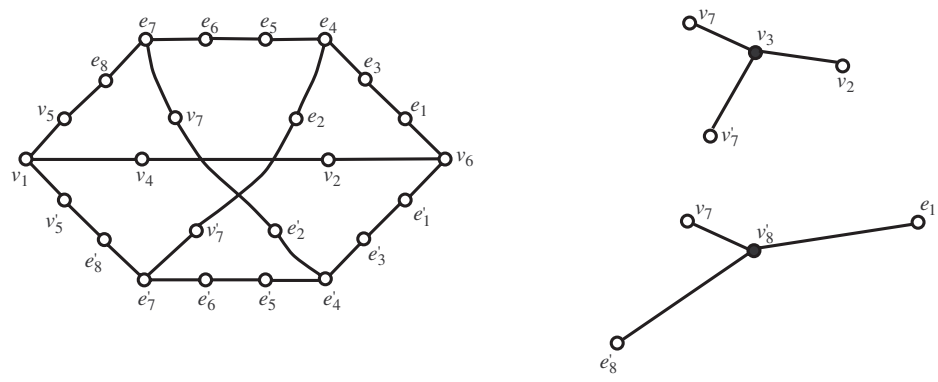

Figure 4. Hexagonal Coxeter's subgraph for $U + A_2 + 2E_8$.

Remark. This lattice, its fundamental chamber, and the complete Coxeter graph had appeared already in Vinberg's paper [Vin83] on maximally algebraic $K3$-surfaces. Note that our list contains the full set of 2-roots, and the missing 6-roots can be obtained from the 6-roots in the list by applying the symmetries of the hexagonal subgraph. The same construction is given in [Loo09].

7. Chirality of $(M-1)$-cubics

7.1 Preliminaries and the main statement

Following M-cubics, the next by their topological complexity are $(M-1)$-cubics. The deformation components of the latter, as follows from [FK08], are adjacent to the deformation

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components of M-cubics. The lattice, $\mathbb{M}$, of an $(M-1)$-cubic contains the direct sum of the eigen-lattices $\mathbb{M}_+$ and $\mathbb{M}_-$ as a sublattice of index two, and this condition characterizes $(M-1)$-cubics among all non-singular real cubic fourfolds. In the other words, the characteristic feature of $(M-1)$-cubics is that $\mathbb{M}_\pm$ have discriminant $\mathbb{Z}/2$. Using the general properties of lattices $\mathbb{M}_\pm$ (namely, that lattice $\mathbb{M}_+$ is odd with a characteristic element $h \in \mathbb{M}_+$ of square $h^2 = 3$, that lattice $\mathbb{M}_-$ is even, and that the both lattices are of index $\sigma_- = 1$), one can deduce that the $(M-1)$-cubics form precisely six homological types, see [FK08]. As usual, these types can be distinguished by sublattices M_+, as well as by sublattices M_-. The corresponding six lattices $\mathbb{M}^0_+$ are $U + A_2 + A_1 + kE_8$ and $-A_1 + A_2 + kE_8$, $k = 0, 1, 2$.

Theorem 7.1.1. *Non-singular real cubic fourfolds of types $\mathbb{M}^0_+(c) = -A_1 + A_2, U + A_2 + A_1$, and $-A_1 + A_2 + E_8$, are chiral; in particular, the cubic fourfolds of each of these three types form two deformation classes. Non-singular real cubic fourfolds of types $\mathbb{M}^0_+(c) = U + A_2 + E_8 + A_1, -A_1 + A_2 + 2E_8$, and $U + A_2 + 2E_8 + A_1$ are achiral; the cubic fourfolds of each of these three types form one deformation class.*

7.2 The case $\mathbb{M}^0_+(c) = -A_1 + A_2$

Here, Vinberg's sequence starts from vectors $\{v_1, v_2, v_3\}$ given in Table 4. The Coxeter graph of this sequence of three vectors is shown in Figure 5. It contains a unique parabolic subgraph $\widetilde{A_1}$ (a thick edge connecting v_2 and v_3). Vinberg's criteria 5.4.3 and 5.4.1 can be applied to conclude termination, since the rank of $\widetilde{A}_1$ is $1 = \dim \mathbb{M}^0_+(c) - 2$. The Coxeter graph admits no symmetries. Hence, applying Theorem 4.4.1 we deduce that the studied cubic fourfolds are chiral.

Table 4. Vinberg's vectors for $\mathbb{M}^0_+(c) = -A_1 + A_2$.

	$-A_1$	A_2
p	1	0, 0
level 0		
v_1	0	0, 1
v_2	0	1, -1
level 12		
v_3	3	-4, -2

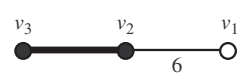

Figure 5. Coxeter's graph for $-A_1 + A_2$.

7.3 The case $\mathbb{M}^0_+(c) = U + A_2 + A_1$

Here, Vinberg's sequence starts from four level-zero vectors $\{v_1, v_2, v_3, v_4\}$ and two level-one vectors $\{v_5, v_6\}$ given in Table 5. The Coxeter graph of this sequence of six vectors is shown in Figure 6. It contains precisely two parabolic subgraphs, $\widetilde{G}_2$ (vertices v_3, v_2, v_5) and $\widetilde{A}_1$ (v_4, v_6). Vinberg's criterion is satisfied, since the rank of their union is $2 + 1 = \dim \mathbb{M}^0_+(c) - 2$. The Coxeter graph admits no symmetries. Hence, applying Theorem 4.4.1 we conclude that the studied cubic fourfolds are chiral.

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Table 5. Vinberg's vectors for $\mathbb{M}^0_+(c) = U + A_2 + A_1$.

	U	A_2	A_1
p	1,−1	0, 0	0
level 0			
v_1	1, 1	0, 0	0
v_2	0, 0	0, 1	0
v_3	0, 0	1, −1	0
v_4	0, 0	0, 0	1
level 1			
v_5	0, −1	−1, −1	0
v_6	0, −1	0, 0	−1

Figure 6. Coxeter's graph for $U + A_2 + A_1$.

7.4 The case $\mathbb{M}^0_+(c) = -A_1 + A_2 + E_8$

Here, the level-zero vectors of Vinberg's sequence are $e_1, \dots, e_8$, v_1, and v_2. They are followed by two vectors of level 4 and one vector of level 12, see Table 6. The Coxeter graph, Γ, of this sequence of 13 vectors is shown in Figure 7.

Lemma 7.4.1. *Vinberg's criterion 5.4.2 is satisfied for the Coxeter graph Γ on Figure 7.*

Proof. For $S = \{a_1, \dots, a_n\} \subset \Phi^b$, let $F_S = F_{a_1,\dots,a_n} \subset P$ denote the face of the cell P supported in the intersection of the walls $[H_v]$, where $v \in \Phi^b \setminus S$. Note that P has two vertices at infinity, F_{v_5,e_8} and F_{v_1,v_3} (because the sets $\Phi^b \setminus S$ span parabolic subgraphs of maximal possible rank $\dim(\mathbb{M}^0_+) - 2 = 9$). The other vertices of P are $F_{a,b,c}$ such that $\Phi^b \setminus \{a, b, c\}$ spans an elliptic subgraph. This subgraph cannot contain the dotted edge connecting v_3 with v_5, so the set $S = \{a, b, c\}$ should contain either v_3 or v_5 (or the both). This set should also contain at least one vertex-root from each of the parabolic subgraphs $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{G}_2$, $\widetilde{A}_1$ of Γ. If the both v_3 and v_5 are included in $S = \{a, b, c\}$, then F_S is a vertex of P only for $S = \{v_3, v_4, v_5\}$. If $a = v_5$ and $v_3 \notin S$, then b and c should be chosen from the two disjoint parabolic subgraphs $\widetilde{G_2}$ and $\widetilde{E_7}$, which gives 21 other vertices $F_{v_5,b,c}$, where $b \in \{v_2, v_1, v_4\}$ and $c \in \{e_1, \dots, e_7\}$. Similarly, if $a = v_3$ and $v_5 \notin S$, then b, c should be chosen from the two disjoint parabolic subgraphs $\widetilde{A_1}$ and $\widetilde{E_8}$, so $b = v_2$ and $c \in \{e_1, \dots, e_8, v_4\}$, which gives nine new vertices $F_{a,b,c}$. In total, Γ contains 31 finite vertices and two vertices at infinity.

The edges of P can be expressed as F_S, $S = \{a, b, c, d\}$, where $\Phi^b \setminus S$ spans an elliptic subgraph. Thus, as above, S should contain at least one of v_3 and v_5. In the edges $F_{v_3,v_5,v_4,d}$, one of the endpoints is F_{v_3,v_5,v_4}. In the cases $d \in \{e_1, \dots, e_7\}$, the other endpoint is $F_{v_5,v_4,d}$. In the cases $d = v_1$, $d = v_2$, and $d = e_8$, the other endpoint is F_{v_1,v_3}, F_{v_3,v_4,v_2}, and F_{v_5,e_8}, respectively. The edges $F_{v_3,v_5,e_8,d}$ must have $d \in \{v_1, v_2\}$ and are incident to F_{v_5,e_8}. Another endpoint is F_{v_3,v_1} for $d = v_1$, and F_{v_3,e_8,v_2} for $d = v_2$. Each of the edges $F_{v_3,v_5,c,d}$, $c \in \{v_1, v_2\}$, $d \in \{e_1, \dots, e_7\}$ has $F_{v_5,c,d}$ as one of the endpoints. The other endpoint is $F_{v_3,v_2,d}$ if $c = v_2$ and F_{v_3,v_1} if $c = v_1$.

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Table 6. Vinberg's vectors for $\mathbb{M}^0_+(c) = -A_1 + A_2 + E_8$.

	$-A_1$	A_2	E_8
p	1	0, 0	0
level 0			
v_1	0	0, 1	0
v_2	0	1, −1	0
level 4			
v_3	1	0, 0	$-e_1^*$
v_4	1	−1, −1	$-e_8^*$
level 12			
v_5	3	−4, −2	0

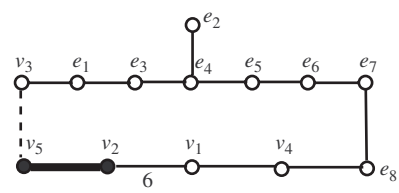

Figure 7. Coxeter's graph for $-A_1 + A_2 + E_8$.

The other edges $F_{a,b,c,d}$ have $a \in \{v_3, v_5\}$ and $b, c, d \notin \{v_3, v_5\}$. If $a = v_3$, then another vertex should be chosen from the subgraph $\widetilde{A}_1$, and we may assume that $b = v_2$ (since the case $b = v_5$ was already considered). This gives edges $F_{v_3,v_2,c,d}$ with $c \in \{e_1, \dots, e_8, v_4\}$ and $d \in \{e_1, \dots, e_8, v_4, v_1\}$. If $d \neq v_1$, then the endpoints are $F_{v_3,v_2,c}$ and $F_{v_3,v_2,d}$. The endpoints of F_{v_3,v_2,c,v_1} are $F_{v_3,v_2,c}$ and F_{v_3,v_1}. Finally, if $a = v_5$, then one of b, c, d should be chosen from $\widetilde{G}_2$, say, $b \in \{v_2, v_1, v_4\}$ and another from $\widetilde{E}_7$, say, $c \in \{e_1, \dots, e_7\}$. Then $F_{v_5,b,c,d}$ has one endpoint $F_{v_5,b,c}$. Another endpoint is $F_{v_5,c,d}$ if $d \in \{v_2, v_1, v_4\}$, and $F_{v_5,b,d}$ if $d \in \{e_1, \dots, e_7\}$. In the remaining case $d = e_8$, the second endpoint is F_{v_5,e_8}. □

The Coxeter graph admits no symmetries. Hence, applying Theorem 4.4.1 we conclude that the studied cubic fourfolds are chiral.

7.5 The case $\mathbb{M}^0_+(c) = U + A_2 + A_1 + E_8$

Here, the level-zero Vinberg's vectors are $e_1, \dots, e_8$ plus $v_1, \dots, v_4$ listed in Table 7. Then there follow three vectors v_5, v_6, v_7 of level 1 and the vector v_8 of level 48 (again see Table 7).

Consider the Coxeter subgraph formed by Vinberg's vectors $e_1, \dots, e_8, v_1, v_2, v_5, v_6, v_7$. This subgraph is shown in Figure 8. It has an evident non-trivial involution (which fixes the vertex v_7 and permutes the vertices v_2, e_1). Since the vectors $e_1, \dots, e_8, v_1, v_2, v_5, v_6, v_7$ generating this subgraph span the lattice $\mathbb{M}^0_+(c)$, this involution is induced by a P_+-direct lattice involution $f : \mathbb{M}^0_+(c) \to \mathbb{M}^0_+(c)$ (see Theorem 5.2.1). In particular, f transforms Vinberg's vector $v_3 = -v_5 - 2v_2 + v_7 + e_8^*$, into another Vinberg's vector $v_3' = -e_3 - 2e_1 + e_5 + (2e_6 + 3e_7 + 4e_8 + 5v_7 + 6v_1 + 4v_5 + 3v_6 + 2v_2)$. The A_2-component of v_3' is $4(-1, -1) + 2(0, 1) = (-4, -2)$, while the A_2-component of v_3 is $(1, -1)$. Hence, f is $\mathbb{Z}/3$-reversing and applying Theorem 4.4.1 we conclude that the type considered is achiral.

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Table 7. Vinberg's vectors for $\mathbb{M}^0_+(c) = U + A_2 + A_1 + E_8$.

	U	A_2	A_1	E_8
p	$1, -1$	$0, 0$	0	0
level 0				
v_1	$1, 1$	$0, 0$	0	0
v_2	$0, 0$	$0, 1$	0	0
v_3	$0, 0$	$1, -1$	0	0
v_4	$0, 0$	$0, 0$	1	0
level 1				
v_5	$0, -1$	$-1, -1$	0	0
v_6	$0, -1$	$0, 0$	-1	0
v_7	$0, -1$	$0, 0$	0	$-e_8^*$
level 48				
v_8	$6, -6$	$-4, -2$	-3	$-3(e_1)^*$

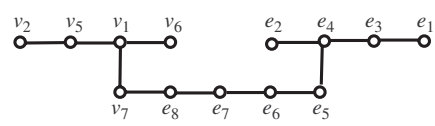

Figure 8. A symmetric fragment of Coxeter's graph for $U + A_2 + A_1 + E_8$.

7.6 The case $\mathbb{M}^0_+(c) = -A_1 + A_2 + 2E_8$

Let us start with a bit more general setting. Namely, assume that $\mathbb{L}$ is a lattice as in § 5 (for example, some of the lattices $\mathbb{M}^0_+(c)$), $P \subset \Lambda(\mathbb{L})$ is a cell, and $f \in \mathrm{Aut}^+(P)$ is an automorphism of $\mathbb{L}$ induced by some symmetry of the Coxeter graph, Γ, of $\mathbb{L}$. Suppose that a 2-root v is a vertex of Γ preserved by this symmetry, that is, $f(v) = v$. Then the sublattice $\mathbb{L}_v = \{x \in \mathbb{L} \mid xv = 0\}$ is f-invariant and we may consider an induced automorphism $f_v \in \mathrm{Aut}(\mathbb{L}_v)$.

Lemma 7.6.1. *If f is $\mathbb{Z}/3$-reversing, then f_v is also $\mathbb{Z}/3$-reversing, and P_v-direct for some cell P_v of $\mathbb{L}_v$.*

Proof. Since $\mathrm{discr}_3(\mathbb{L}_v) = \mathrm{discr}_3(\mathbb{L}) = \mathbb{Z}/3$, the automorphisms f and f_v are both $\mathbb{Z}/3$-direct or $\mathbb{Z}/3$-reversing. Furthermore, f preserves the facet $P \cap [H_v]$ of P, since it preserves both P and v. Owing to $\mathrm{discr}_3(\mathbb{L}_v) = \mathrm{discr}_3(\mathbb{L})$, each wall in $\Lambda(\mathbb{L}_v)$ is an intersection of $[H_v]$ with a wall $\Lambda(\mathbb{L})$, and thus, the facet $P \cap [H_v]$ is a part of some cell, P_v, of $\Lambda(\mathbb{L}_v)$. Such a P_v also has to be invariant. □

Corollary 7.6.2. *Lattice $-A_1 + A_2 + 2E_8$ is achiral.*

Proof. Let $\mathbb{L} = U + A_2 + 2E_8$, then for $v = v_1$ (following the notation of § 6.4) we have $\mathbb{L}_v = -A_1 + A_2 + 2E_8$. An involution of the hexagonal diagram (Figure 4) in § 6.4 is conjugate to some involution, $f \in \mathrm{Aut}^+(P)$, preserving v_1. Since f is $\mathbb{Z}/3$-reversing, we can apply Lemma 7.6.1. □

Applying Theorem 4.4.1 we can now conclude that the fourfolds with $\mathbb{M}^0_+(c) = -A_1 + A_2 + 2E_8$ are achiral.

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7.7 The case $\mathbb{M}^0_+(c) = U + A_2 + 2E_8 + A_1$

Let $\mathbb{L}$ and $\mathbb{L}_v$ be as in § 7.6. Our aim now is to obtain a criterion which is in some sense 'converse' to that in Lemma 7.6.1. Recall that lattice $\mathbb{L}$ either splits into a direct sum of $\mathbb{L}_v$ with a sublattice $A_1 = \mathbb{Z}v$, or contains this direct sum as an index-two sublattice. We show that, in the former case, achirality of $\mathbb{L}_v$ implies achirality of $\mathbb{L}$.

Lemma 7.7.1. *Assume that $\mathbb{L} = \mathbb{L}_v + A_1$, where $A_1 = \mathbb{Z}v$, and $\mathbb{L}_v$ is achiral. Then $\mathbb{L}$ is also achiral. In fact, any $\mathbb{Z}/3$-reversing automorphism $f_v \in \mathrm{Aut}^+(P_v)$ for some cell $P_v \subset \Lambda(\mathbb{L}_v)$ can be extended to a $\mathbb{Z}/3$-reversing automorphism $f \in \mathrm{Aut}^+(P)$ for some cell $P \subset \Lambda(\mathbb{L})$.*

Proof. Letting $f(v) = v$, we obtain an extension of f_v to $\mathbb{L}$ which is obviously $\mathbb{Z}/3$-reversing if f_v is.

As in Lemma 7.6.1, by the same evident reasons, P_v contains the facet $P \cap [H_v]$ of some cell P in $\Lambda(\mathbb{L})$. However, now the relation is stronger: $P \cap [H_v] = P_v$. In fact, the walls of P different from $[H_v]$ are either orthogonal to $[H_v]$ or do not intersect it. To see it, consider any wall $[H_w]$, $w \in V_2 \cup V_6$. Splitting $\mathbb{L} = \mathbb{L}_v + \mathbb{Z}v$ gives a decomposition $w = w_v + kv$, where $w_v \in \mathbb{L}_v$, $k \in \mathbb{Z}$. If $k = 0$, then $[H_w]$ is orthogonal to $[H_v]$, whereas $w_v = 0$ implies $w = v$. Otherwise we observe that $w_v^2 = w^2 - k^2v^2 \leqslant 0$, because $v^2 = 2$, and w^2 is either two or six, but in the latter case k is divisible by three. Thus, vectors perpendicular to w_v cannot have negative square, which contradicts to $P \cap [H_v] \neq \varnothing$.

The relation $P \cap [H_v] = P_v$ implies that the isometry $f = f_v \oplus \mathrm{id} : \mathbb{L} \to \mathbb{L}$ is P-direct. □

Corollary 7.7.2. *Lattice $U + A_2 + 2E_8 + A_1$ is achiral.*

Proof. According to § 6.4, the lattice $\mathbb{L}_v = U + A_2 + 2E_8$ is achiral. It remains to apply Lemma 7.7.1. □

Applying Theorem 4.4.1 we can now conclude that the cubic fourfolds with $\mathbb{M}^0_+(c) = U + A_2 + 2E_8 + A_1$ are achiral.

8. Concluding remarks

8.1 Further results

The cases of M-varieties and $(M-1)$-varieties are usually the most interesting and difficult, which explains our special interest in them in the context of the chirality problem of the cubic fourfolds. However, our methods are also applicable to the other cases. Our observations concerning the problem of chirality can be summarized as follows.

Let ρ denote the rank of the lattice $\mathbb{M}^0_+$, let $r = 22 - \rho$ denote the rank of $\mathbb{M}_-$, and let d be the discriminant rank, $\mathrm{rk}(\mathrm{discr}_2(\mathbb{M}^0_+)) = \mathrm{rk}(\mathrm{discr}(\mathbb{M}_-))$. In all of the cases studied, if $\rho + d \geqslant 14$, then $\mathbb{M}^0_+$ is achiral. In addition, the list of achiral lattices contains $\mathbb{M}^0_+ = U(2) + A_2 + D_4$ and $\mathbb{M}^0_+ = -A_1 + \langle 6 \rangle + kA_1$ with $k = 2, 3$, and 4. The other lattices that we have analyzed are chiral. (In a few cases remaining for analysis, the discriminant form is even and $\rho + d \geqslant 14$. We expect that the corresponding lattices are achiral.)

The lattices $M^0_+(c)$ of cubic fourfolds can be naturally divided into the *principal series*, which contains most of the lattices and is presented in Table 8, and several additional lattices as presented in Table 9 (see [FK08] for more details).

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Table 8. The principal series of lattices M_+^0.

$-A_1$	$+\langle 6\rangle$	$+t\ A_1,$	$0 \leqslant t \leqslant 9$
$-A_1$	$+A_2$	$+t\ A_1,$	$0 \leqslant t \leqslant 9$
U	$+A_2$	$+t\ A_1,$	$0 \leqslant t \leqslant 9$
U	$+A_2+D_4$	$+t\ A_1,$	$0 \leqslant t \leqslant 6$
$-A_1$	$+\langle 6\rangle+E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 5$
$-A_1$	$+A_2+E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 5$
U	$+A_2+E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 5$
U	$+A_2+D_4+E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 2$
$-A_1$	$+\langle 6\rangle+2E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 1$
$-A_1$	$+A_2+2E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 1$
U	$+A_2+2E_8$	$+t\ A_1,$	$0 \leqslant t \leqslant 1$

Table 9. Additional lattices M_+^0.

$\mathrm{U}(2)+E_6(2)$
$\mathrm{U}(2)+A_2$
$\mathrm{U}+E_6(2)$
$\mathrm{U}(2)+A_2+D_4$
$\mathrm{U}(2)+A_2+2D_4$
$\mathrm{U}+A_2+2D_4$
$\mathrm{U}(2)+A_2+E_8$
$\mathrm{U}(2)+A_2+D_4+E_8$
$\mathrm{U}(2)+A_2+2E_8$
$\mathrm{U}+A_2+E_8(2)$
$\mathrm{U}(2)+A_2+E_8(2)$

Table 10 describes the chirality of the principal series of cubic fourfolds in terms of the ranks r and d.

8.2 Chirality of singular cubic fourfolds

Chirality of cubic fourfolds having a nodal singularity is an interesting related problem. It is trivial to observe that any perturbation of an achiral nodal cubic provides an achiral non-singular cubic. The non-trivial part of the problem is the converse: if perturbations give only achiral cubics, can we conclude that a nodal cubic is achiral itself? To solve this, one can use the same approach as in the non-singular case, just taking into account the vanishing cycles. On the other hand, the central projective correspondence discussed in [FK08] relates chirality of nodal cubic fourfolds to a certain question about 6-polarized $K3$-surfaces. This relation can be used in the both directions.

A somewhat different kind of observation is chirality of the *discriminant cubic*,

$$\det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix} = 0,$$

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Table 10. Chirality of cubic fourfolds: the principal series.

d																					
11											a										
10										a		a									
9									a		a		a								
8								a		a		a		a							
7							a		a		a		a		a						
6						a		a		a		a		a		a					
5					a		a		a		a		a		c		a				
4				a		a		a		a		a		c		c		a			
3			a		a		a		a		a		c		c		c		c		
2		a		a		a		a		a		c		c		c		c		c	
1	a		a						a		c						c		c		
0		a								c								c			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	r

The symbol 'c' stands for the chiral deformation classes, and symbol 'a' for the achiral deformation classes.

which parameterizes the space of singular conics in the plane. The key observation is chirality of the singular locus of the discriminant cubic (this locus is the image of the Veronese map).

8.3 Explicit equations

It would be interesting to find explicit (natural) equations for representatives of each of the deformation classes. It can be helpful not only for proving achirality statements, but also for better understanding of the topology of the cubic hypersurfaces. As an example, let us consider the equations of the following type:

$$\left(\sum_1^6 x_\alpha\right)^3 - \sum_1^6 c_\alpha x_\alpha^3 = 0;$$

these equations were proposed in the late 1970s by D. Fucks (private communication to the second author) for searching the precise range of the values of the Euler characteristic of real cubic hypersurfaces in each given even dimension (a problem which, to the best of the authors' knowledge, remains open in its whole generality). Similar equations were used earlier by Klein [Kle73], and his student Rodenberg [Rod79], to find and to study explicit representatives for each of the five classes of real non-singular cubic surfaces. In fact, it is by means of these equations that Klein proved in [Kle21] the achirality of all real non-singular cubic surfaces (cf. the remark at the end of this section).

One can easily check that for c_α having all of the same value c, the topology of the hypersurface is changing at $c = 0, 4, 16$, and 36. For $c < 0$ and $c > 36$ the real part of the hypersurface is diffeomorphic to the real four-dimensional projective space, $\mathbb{R}\mathrm{P}^4$. When $c = 36$, there appears a solitary double point, so that for $16 < c < 36$ we observe $S^4 \sqcup \mathbb{R}\mathrm{P}^4$. When $c = 16$, our hypersurfaces acquire six double points of Morse index $(1, 4)$ with respect to growing c (the first, respectively second, component of the index is the number of positive, respectively

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negative, squares) and, therefore, for $4 < c < 16$ the real part of the hypersurface is diffeomorphic to the real four-dimensional projective space with five $S^1 \times S^3$-handles, that is, $\mathbb{R}\mathrm{P}^4 \# 5(S^1 \times S^3)$. Finally, when $c = 4$, one finds that there are 15 double points of the Morse index $(2, 3)$, which implies that the Euler characteristic of our hypersurfaces becomes equal to 21. According to the classification of cubics (see [FK08]), there is only one coarse deformation class with this value of Euler characteristic (in fact, it is the class studied above in § 7.7), and for the cubics of this class the real part has the homological type of $\mathbb{R}\mathrm{P}^4 \# 10(S^2 \times S^2)$. (One can also give a direct proof based on the Lefschetz trace formula and the Smith theory, which allow us to reconstruct the Betti numbers from the action of the complex conjugation in homology.)

Since for c_α having all of the same value the equation is invariant under transposition of the variables, all of these hypersurfaces represent achiral classes. In the same manner, one can show that the whole left-hand slanted border of the diagram shown in Table 10 consists exclusively of achiral classes.

8.4 Chirality in lower dimensions: quartic surfaces

When discussing the real non-singular hypersurfaces X of dimension n and degree d, it is easy to see their achirality in the trivial cases $n = 0$ (for any d), and $d \leqslant 2$ (for any n). As was pointed out in the introduction, X is also achiral if n is odd. Achirality of cubic surfaces was observed by Klein, as we mentioned in § 8.3. The next case of quartic surfaces was analyzed in [Kha84, Kha88] using a technique similar to the technique we used in this paper. It turned out that a real non-singular quartic X with a non-contractible (in $P^3(\mathbb{R})$) real locus $X(\mathbb{R})$ is chiral if and only if $X(\mathbb{R})$ has at least four spherical components, and a quartic with contractible real locus is chiral if and only if the real locus has at least three spherical components and, in addition, a component with at least three handles (see Table 11, where r is the rank of the $+1$-eigen-lattice $L_+ = \{x \in H_2(X) \mid \mathrm{conj}_* x = x\}$, d is the discriminant rank of L_+, and symbols a, or c stand as in Table 10 for achiral or, respectively, chiral deformation classes).

8.5 Reversibility

In connection with chirality, it may be worth mentioning a different but somewhat related notion of *reversibility*, which plays a non-trivial role for instance for odd-dimensional hypersurfaces. Namely, to each deformation class of real non-singular hypersurfaces $X \subset P^{n+1}$ of degree d, that is, a connected component $\mathcal{C}$ of $\mathcal{C}_{n,d} = P_{n,d}(\mathbb{R}) \setminus \Delta_{n,d}(\mathbb{R})$, we can associate its pull back $\widetilde{\mathcal{C}}$ into the sphere $\widetilde{P}_{n,d}(\mathbb{R})$ which covers $P_{n,d}(\mathbb{R})$. This pull back is either connected, or splits into a pair of opposite components. We say that C and the corresponding hypersurfaces $X \in C$ are *reversible* in the first case, and *irreversible* in the second case. In other words, X is reversible if its defining homogeneous polynomial, f, can be continuously changed into $-f$ without creating singularities in the process of deformation. One can extend the notion of reversibility to singular varieties replacing non-singular continuous families of equations by equisingular families.

If the degree d is even, then the region in $P^{n+1}(\mathbb{R})$ where $f > 0$ defines a coorientation of $X(\mathbb{R})$ and reversibility obviously means possibility to reverse this coorientation by a deformation. If n is odd, then such reversibility for non-singular hypersurfaces is impossible, because the regions where $f > 0$ and $f < 0$ are homologically different: they are distinguished by the highest dimension in which the inclusion homomorphism is non-zero. If n is even, then reversibility is possible: for example, a quadric is reversible if the signature of its equation vanishes and irreversible otherwise. Furthermore, it is not difficult to show that a real non-singular quartic surface X is irreversible if $X(\mathbb{R})$ has more than one connected components, as well as if it

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Table 11. Chirality of quartic surfaces.

d																				
10										a										
9									a		a									
8								a		a		a								
7							a		a		a		a							
6						a		a		a		a		c						
5					a		a		a		a		c		c					
4				a		a		a		a		c		c		c				
3			a		a		a		a		c		c		c		c			
2		a		a		a		a		c		c		c		c		c		
1	a		a						c		c						c		c	
0		a								c								c		
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	r

Non-contractible case

d																					
11											a										
10										a		a									
9									a		a		a								
8								a		a		a		a							
7							a		a		a		a		a						
6						a		a		a		a		a		a					
5					a		a		a		c		a		a		a				
4				a		a		a		c		c		a		a		a			
3			a		a		a		c		c		c		a		a		a		
2		a		a		a		c		c		c		c		a		a		a	
1	a		a						c		c						a		a		
0		a								c								a			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	r

Contractible case

has a single component which is contractible in $P^3(\mathbb{R})$. Conversely, if $X(\mathbb{R})$ is connected and non-contractible, then the quartic is reversible, at least if the genus of $X(\mathbb{R})$ is less than 10 (the extremal case, $g = 10$, remains unknown to the authors). Thus, we obtain nine reversible cases, more than 100 irreversible cases, and a unique unresolved case.

If the degree d is odd and n is even, then X is reversible for a trivial reason, because $-\mathrm{id}$ and id belong to the same connected component of $\mathrm{GL}(n+2, \mathbb{R})$, and $f(-x) = -x$. If both d and n are odd, then f determines an orientation of $X(\mathbb{R})$ and reversibility obviously means the possibility to alternate this orientation. If $X(\mathbb{R})$ is symmetric with respect to a mirror reflection, then such an alternation is realizable by a projective transformation, which is one of the manifestations of the similarity between the notions of reversibility and achirality. The existence of symmetric models proves, in particular, the reversibility of curves of degree at most five.

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In the case of non-singular cubic threefolds the problem of reversibility is already non-trivial. The deformation classification of such cubics obtained in [Kra06] gives nine classes. Our analysis has shown that just one of these classes is irreversible, namely, the class denoted by $\mathcal{B}(1)'_I$ in [Kra06].

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References

ACT03 D. Allcock, J. Carlson and D. Toledo, *Real cubic surfaces and real hyperbolic geometry*, C. R. Acad. Sci. Paris Ser. I **337** (2003), 185–188.

Bou68 N. Bourbaki, *Groupes et Algèbres de Lie, II* (Hermann, Paris, 1968).

Dol08 I. V. Dolgachev, *Reflection groups in algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), 1–60.

FK08 S. Finashin and V. Kharlamov, *Deformation classes of real four-dimensional cubic hypersurfaces*, J. Alg. Geom. **17** (2008), 677–707.

Kha84 V. Kharlamov, *On classification of nonsingular surfaces of degree 4 in $\mathbb{R}P^3$ with respect to rigid isotopies*, Funktsional Anal. i Prilozhen (1) **18** (1984), 49–56.

Kha88 V. Kharlamov, *On non-amphichaeral surfaces of degree 4 in $\mathbb{R}P^3$*, Lecture Notes in Mathematics, vol. 1346 (Springer, Berlin, 1988), 349–356.

Kle73 F. Klein, *Über Flachen dritte Ordnung*, Math. Ann. **6** (1873), 551–581.

Kle21 F. Klein, *Über Flachen dritte Ordnung*, Gesammelte Mathematische Abhandlungen, vol. 2 (Verlag von Julius Springer, Berlin, 1921), 11–67.

Kra06 V. Krasnov, *Rigid isotopy classification of real three-dimensional cubics*, Izv. Math. **70**(4) (2006), 731–768.

Laz07 R. Laza, *The moduli space of cubic fourfolds via the period map*, Preprint (2007), 0705.0949.

Loo09 E. Looijenga, *The period map for cubic fourfolds*, Invent. Math. **177** (2009), 213–233.

Nik79 V. V. Nikulin, *Integer quadratic forms and some of their geometrical applications*, Math. USSR – Izv. **43** (1979), 103–167.

Nik08 V. V. Nikulin, *On connected components of moduli of real polarized K3-surfaces*, Izv. Math. **72** (2008), 91–111.

Rod79 C. Rodenberg, *Zur Classification der Flächen dritter Ordnung*, Math. Ann. **14** (1879), 46–110.

Vin75 E. B. Vinberg, *Some arithmetical discrete groups in Lobačevskiî spaces*, in *Proc. Int. Coll. on Discrete Subgroups of Lie Groups and Applications to Moduli,* Bombay, 1973 (Oxford University Press, Oxford, 1975), 323–348.

Vin83 E. B. Vinberg, *The two most algebraic K3 surfaces*, Math. Ann. **265** (1983), 1–21.

Vin85 E. B. Vinberg, *Hyperbolic reflection groups*, Russian Math. Surveys **40** (1985), 31–75.

Voi86 C. Voisin, *Théorème de Torelli pour les cubiques de P^5*, Invent. Math. **86** (1986), 577–601.

Wal62 C. T. C. Wall, *On the orthogonal groups of unimodular quadratic forms*, Math. Ann. **147** (1962), 328–338.

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S. Finashin serge@metu.edu.tr
Department of Mathematics, Middle East Technical University, Ankara 06531, Turkey

V. Kharlamov kharlam@math.u-strasbg.fr
Université Louis Pasteur et IRMA (CNRS), 7 rue René Descartes, 67084 Strasbourg Cedex, France

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