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KAC'S THEOREM FOR EQUIPPED GRAPHS AND FOR MAXIMAL RANK REPRESENTATIONS

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Abstract We give two generalizations of Kac's Theorem on representations of quivers. One is to representations of equipped graphs by relations, in the sense of Gelfand and Ponomarev. The other is to representations of quivers in which certain of the linear maps are required to have maximal rank.

Keywords: representations of quivers; representations of graphs; Kac's Theorem

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1. Introduction

Recall that Gabriel's Theorem [**1**] asserts that a connected quiver has only finitely many indecomposable representations up to isomorphism if and only if the underlying graph is a Dynkin diagram, and that in this case the indecomposables are in one-to-one correspondence with positive roots for the corresponding root system. Kac's Theorem [**3**, **4**] extends the latter part to arbitrary quivers, saying that the possible dimension vectors of indecomposable representations are the positive roots, with a unique indecomposable for real roots and infinitely many for imaginary roots.

In [**2**], Gelfand and Ponomarev announced a generalization of Gabriel's Theorem to representations of what they called *equipped graphs*, which may be thought of as generalizations of quivers, in which one is allowed not just arrows $\bullet \longrightarrow \bullet$ but also edges with two heads $\bullet \longleftrightarrow \bullet$ or two tails $\bullet$ —— $\bullet$. Representations are given by a vector space at each vertex and a linear relation of a certain sort for each edge. One motivation, perhaps, was that by taking all edges to be two headed or two tailed, one can study the representation theory of a graph without needing to choose an orientation.

In this paper we generalize Kac's Theorem to equipped graphs. In order to obtain our proof, we study representations of quivers in which the linear maps associated with certain arrows are required to have maximal rank, and also prove an analogue of Kac's Theorem in this setting. The case of real roots has been dealt with by Wiedemann [**5**].

2. Maximal rank representations

Let $Q = (Q_0, Q_1, h, t)$ be a quiver and let $\alpha \in \mathbb{N}^{Q_0}$ be a dimension vector. Recall that if q is a prime power, $A_{Q,\alpha}(q)$ denotes the number of isomorphism classes of absolutely indecomposable representations of Q of dimension vector α over the finite field with q elements. By Kac's Theorem [**4**, § 1.15], $A_{Q,\alpha}(q)$ is given by a polynomial in $\mathbb{Z}[q]$, which is non-zero if and only if α is a positive root, and then is monic of degree $1 - q_Q(\alpha)$, where q_Q is the associated quadratic form.

If M is a subset of Q_1, we say that a representation of Q is M-*maximal* if the arrows in M are represented by linear maps of maximal rank. We denote by $A^M_{Q,\alpha}(q)$ the number of isomorphism classes of absolutely indecomposable M-maximal representations of Q over the field with q elements.

Theorem 2.1. *$A^M_{Q,\alpha}(q)$ is given by a polynomial in $\mathbb{Z}[q]$, which is non-zero if and only if α is a positive root, and then is monic of degree $1 - q_Q(\alpha)$. Moreover, this polynomial does not depend on the orientation of Q.*

Proof. If M were empty, this would be Kac's Theorem, so suppose here that M is not empty. We prove the assertion for all quivers, dimension vectors and subsets M by induction, firstly on $m = \max\{\min(\alpha_{t(a)}, \alpha_{h(a)})\colon a \in M\}$, and then on the size of the set $\{a \in M\colon \min(\alpha_{t(a)}, \alpha_{h(a)}) = m\}$.

Choose $a \in M$ with $\min(\alpha_{t(a)}, \alpha_{h(a)}) = m$ and let $M' = M \setminus \{a\}$. Let Q' be the quiver obtained from Q by adjoining a new vertex v and replacing a by arrows $b\colon t(a) \to v$ and $c\colon v \to h(a)$. By factorizing a linear map as the composition of a surjection followed by an injection, one sees that there is a one-to-one correspondence, up to isomorphism, between representations of Q in which a has rank d, and representations of Q' in which the vector space at v has dimension d, and b and c have rank d. Moreover, this correspondence respects absolute indecomposability. For any $0 \leqslant d < m$, let α_d be the dimension vector for Q' obtained from α by setting $\alpha_d(v) = d$. Using that, in any representation, if a does not have rank m, then it must have rank $d < m$, we obtain

$$A^M_{Q,\alpha}(q) = A^{M'}_{Q,\alpha}(q) - \sum_{d=0}^{m-1} A^{M' \cup \{b,c\}}_{Q',\alpha_d}(q).$$

By induction, all terms on the right-hand side are independent of orientation and are in $\mathbb{Z}[q]$, and (as used in [**5**])

$$\begin{aligned} q_{Q'}(\alpha_d) &= q_Q(\alpha) + \alpha_{h(a)}\alpha_{t(a)} + d^2 - d\alpha_{t(a)} - \alpha_{h(a)}d \\ &= q_Q(\alpha) + (\alpha_{h(a)} - d)(\alpha_{t(a)} - d) > q_Q(\alpha), \end{aligned}$$

so that only $A^{M'}_{Q,\alpha}(q)$, which is monic of degree $1 - q_Q(\alpha)$, contributes to the leading term of $A^M_{Q,\alpha}(q)$. □

Let K be an algebraically closed field. The theorem, together with standard arguments as used in the proof of Kac's Theorem, gives the following result.

Corollary 2.2. *There is an indecomposable M-maximal representation of Q of dimension vector α over K if and only if α is a positive root. If α is a real root, there is a unique such representation up to isomorphism, while if α is an imaginary root, there are infinitely many such representations.*

3. Equipped graphs

As we will allow our equipped graphs to have loops, our definitions differ slightly from those of Gelfand and Ponomarev [**2**].

Recall that a graph may be thought of as a pair $(G, *)$ consisting of a quiver $G = (G_0, G_1, h, t)$, together with a fixed-point free involution $*$ of the set of arrows G_1 that exchanges heads and tails, so $h(a) = t(a^*)$ for all $a \in G_1$. By an *equipped graph* we mean a collection $((G, *), \phi)$ consisting of a graph $(G, *)$ and a mapping $\phi\colon G_1 \to \{0, 1\}$.

One may depict an equipped graph by drawing a vertex for each element of G_0, an arrow $a\colon t(a) \longrightarrow h(a)$ for each $a \in G_1$ with $\phi(a) = 1$ and $\phi(a^*) = 0$, a double-headed edge $a\colon t(a) \longleftrightarrow h(a)$ for each $*$-orbit $\{a, a^*\}$ with $\phi(a) = \phi(a^*) = 0$, and a double-tailed edge $a\colon t(a)$ —— $h(a)$ for each $*$-orbit $\{a, a^*\}$ with $\phi(a) = \phi(a^*) = 1$.

Recall that a *linear relation* between vector spaces V and W is a subspace $R \subseteq V \oplus W$. The *opposite relation* is $R^{\mathrm{op}} = \{(w, v)\colon (v, w) \in R\} \subseteq W \oplus V$. Considering graphs of linear maps leads one to define the following subspaces: the kernel $\operatorname{Ker} R = \{v \in V\colon (v, 0) \in R\}$, the domain of definition $\operatorname{Def} R = \{v \in V\colon (v, w) \in R \text{ for some } w \in W\}$, the indeterminacy $\operatorname{Ind} R = \{w \in W\colon (0, w) \in R\}$ and the image $\operatorname{Im} R = \{w \in W\colon (v, w) \in R \text{ for some } v \in V\}$. There is an induced isomorphism $\operatorname{Def} R/\operatorname{Ker} R \cong \operatorname{Im} R/\operatorname{Ind} R$, and we call the dimension of this space the *rank* of R.

Following Gelfand and Ponomarev, if $r, s \in \{0, 1\}$, we say that R is an *equipped relation of type* (r, s) if, in V, either $\operatorname{Ker} R = 0$ if $r = 0$ or $\operatorname{Def} R = V$ if $r = 1$, and, in W, either $\operatorname{Ind} R = 0$ if $s = 0$ or $\operatorname{Im} R = W$ if $s = 1$. As observed by Gelfand and Ponomarev, an equipped relation of type $(1, 0)$ is the same as the graph of a linear map, and one of type $(0, 1)$ is the opposite of a graph of a linear map. On the other hand, specifying an equipped relation of type $(0, 0)$ is the same as specifying a pair of subspaces $\operatorname{Def} R \subseteq V$ and $\operatorname{Im} R \subseteq W$ and an isomorphism $\operatorname{Def} R \cong \operatorname{Im} R$, and specifying an equipped relation of type $(1, 1)$ is the same as specifying a pair of subspaces $\operatorname{Ker} R \subseteq V$ and $\operatorname{Ind} R \subseteq W$ and an isomorphism $V/\operatorname{Ker} R \cong W/\operatorname{Ind} R$.

By a *representation* X of an equipped graph $((G, *), \phi)$ we mean a collection consisting of a vector space X_i for each vertex $i \in G_0$ and an equipped relation $X_a \subset X_{t(a)} \oplus X_{h(a)}$ of type $(\phi(a), \phi(a^*))$ for each arrow $a \in G_1$, such that $X_{a^*} = X_a^{\mathrm{op}}$ for all a.

A *morphism* of representations $\theta\colon X \to X'$ is given by a collection of linear maps $\theta_i\colon X_i \to X'_i$ for each $i \in G_0$, such that $(\theta_{t(a)}(x), \theta_{h(a)}(y)) \in X'_a$ for all $a \in G_1$ and $(x, y) \in X_a$. In this way one obtains an additive category of representations of the equipped graph $((G, *), \phi)$ over any given field. It is easy to see that this category has split idempotents, and we can clearly also define the notion of an absolutely indecomposable representation. The *dimension vector* of a representation X is the tuple $(\dim X_i) \in \mathbb{N}^{G_0}$.

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Table 1. *Equipped relations.*

type	configuration	corresponding equipped relation
$(0,0)$	$V \overset{a}{\hookleftarrow} U \overset{b}{\hookrightarrow} W$	$R = \{(a(u), b(u)) \colon u \in U\}$
$(0,1)$	$V \overset{a}{\hookleftarrow} U \overset{b}{\twoheadleftarrow} W$	$R = \{(a(b(w)), w) \colon w \in W\}$
$(1,0)$	$V \overset{a}{\twoheadrightarrow} U \overset{b}{\hookrightarrow} W$	$R = \{(v, b(a(v))) \colon v \in V\}$
$(1,1)$	$V \overset{a}{\twoheadrightarrow} U \overset{b}{\twoheadleftarrow} W$	$R = \{(v, w) \in V \oplus W \colon a(v) = b(w)\}$

Theorem 3.1. *The number of isomorphism classes of absolutely indecomposable representations of an equipped graph $((G,*),\phi)$ of dimension vector $\alpha \in \mathbb{N}^{G_0}$, over a finite field with q elements, is equal to $A_{Q,\alpha}(q)$, where Q is any quiver with underlying graph $(G,*)$.*

Proof. We use the following observation. If V and W are vector spaces, and U is a vector space of dimension d, then equipped relations $R \subseteq V \oplus W$ of type (r,s) and rank d are in one-to-one correspondence with $\mathrm{GL}(U)$-orbits of pairs of linear maps a, b of rank d in the configurations shown in Table 1.

Let Δ be the quiver whose vertex set is the disjoint union $\Delta_0 = G_0 \cup \bar{G}_1$, where $\bar{G}_1$ denotes the set of $*$-orbits $[a]$ in G_1, and where each arrow $a \in G_1$ defines an arrow a' in Δ_1, with either $h(a') = t(a)$, $t(a') = [a]$ if $\phi(a) = 0$, or $h(a') = [a]$, $t(a') = t(a)$ if $\phi(a) = 1$.

The observation above leads to an equivalence between the category of representations of $((G,*),\phi)$ and the category of representations of Δ in which arrows with their heads in $\bar{G}_1$ are represented by surjective linear maps and arrows with their tails in $\bar{G}_1$ are represented by injective linear maps.

It follows that the number of isomorphism classes of absolutely indecomposable representations of $((G,*),\phi)$ of dimension vector $\alpha \in \mathbb{N}^{G_0}$ over a finite field with q elements is

$$n_{((G,*),\phi),\alpha}(q) = \sum_{\alpha'} A^{\Delta_1}_{\Delta,\alpha'}(q),$$

where α' runs over all dimension vectors for Δ with the property that the restriction of α' to G_0 is equal to α, and with $\alpha'_{[a]} \leqslant \min(\alpha_{t(a)}, \alpha_{h(a)})$ for all $a \in G_1$.

Changing the equipping function ϕ changes the orientation of Δ but, by the results in the previous section, this does not change the sum above, so $n_{((G,*),\phi),\alpha}(q)$ is independent of ϕ.

In particular, identifying G_1 with the disjoint union $Q_1 \cup Q_1^*$, and defining ϕ' by $\phi'(a) = 1 \Leftrightarrow a \in Q_1$, representations of $(G,*,\phi')$ are essentially the same as representations of Q, so $n_{((G,*),\phi),\alpha}(q) = n_{((G,*),\phi'),\alpha}(q) = A_{Q,\alpha}(q)$. $\square$

If K is an algebraically closed field, we deduce the following.

Corollary 3.2. *There is an indecomposable representation of an equipped graph $((G,*),\phi)$ of dimension vector $\alpha \in \mathbb{N}^{G_0}$ over K if and only if α is a positive root for the underlying graph $(G,*)$. If α is a real root, there is a unique such representation up to isomorphism, while if α is an imaginary root, there are infinitely many such representations.*

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