Bull. Aust. Math. Soc. **93** (2016), 350–352
doi:10.1017/S0004972715001148

ADDENDUM TO 'ON SETS OF PP-GENERATORS OF FINITE GROUPS'

JAN KREMPA and AGNIESZKA STOCKA✉

(Received 18 June 2015; accepted 24 June 2015; first published online 5 October 2015)

Abstract

This note completes the proof of the structure theorem for pp-matroid groups which was stated in our earlier paper J. Krempa and A. Stocka ['On sets of pp-generators of finite groups', *Bull. Aust. Math. Soc.* **91** (2015), 241–249].

2010 *Mathematics subject classification*: primary 20F05.

Keywords and phrases: finite group, generating set, independent set, solvable group.

In [3, Theorem 4.5] we gave a description of pp-matroid groups. However, the proof was given only in the Frattini-free case. Thus, in [3], we proved in fact only the result below.

THEOREM 1. *Let H be a Frattini-free group. Then H is a pp-matroid group if and only if one of the following holds:*

(1) *H is an elementary abelian p-group for some prime p;*
(2) *$H = P \rtimes Q$ is a scalar extension for primes $p \neq q$, where $q|(p-1)$ and Q is cyclic of order q;*
(3) *H is a direct product of groups given in (1) and (2) with coprime orders.*

Now we can prove a structure theorem for all pp-matroid groups and the full version of [3, Theorem 4.5].

THEOREM 2. *Let G be a group. Then G is a pp-matroid group if and only if one of the following holds:*

(1) *G is a p-group for some prime p*;
(2) *G is an indecomposable pp-matroid $\{p, q\}$-group;*
(3) *G is a direct product of groups given in (1) and (2) with coprime orders.*

PROOF. Let G be a pp-matroid group. We know, by Theorem 1, that G is solvable. Hence, by [2, Theorem VI.2.3], there exist Sylow p_i-subgroups P_i, for $i = 1, \ldots, n$, such that $G = P_1 \cdots P_n$, and every product P_iP_j is a subgroup of G. We shall use the bar notation for subgroups of the quotient $\bar{G} = G/\Phi(G)$. Then $\bar{G} = \bar{P}_1 \cdot \bar{P}_2 \cdots \bar{P}_n$ and, by [2, Theorem III.3.8], $\bar{P}_i$ is nontrivial for $i = 1, \ldots, n$. If $\bar{P}_i$ is a normal subgroup of $\bar{G}$ for some $i = 1, \ldots, n$, then $G = N_G(P_i)$, by a Frattini argument. Hence, $P_i \lhd G$.

If, for example, $\bar{P}_1$ is not normal in $\bar{G}$, then, with the help of Theorem 1, we can assume that $\bar{P}_2$ and $\bar{P}_1\bar{P}_2 = \bar{P}_2 \rtimes \bar{P}_1$ are normal in $\bar{G}$. Since $\bar{P}_1\bar{P}_2 \lhd \bar{G}$, we have $P_1P_2\Phi(G) \lhd G$. From $\bar{P}_2 \lhd \bar{G}$, we also know that $P_2 \lhd G$, so $\bar{P}_1\bar{P}_2 = \bar{P}_2 \rtimes \bar{P}_1$. Moreover, again by a Frattini argument, $G = N_G(P_1)P_2$. From this, by straightforward calculation, we obtain $P_1P_2 \lhd G$. Thus, $G = H_1 \times H_2 \times \cdots \times H_m$, for some $m \le n$, where H_i are either p-groups or indecomposable $\{p, q\}$-groups. Thus, H_j satisfy either condition (1) or (2) of the theorem. The proof can be finished as in [3]. □

From the above theorem and [1] it follows that G is a matroid group if and only if G is an indecomposable pp-matroid group. Corollary 5.2 from [1] is a description of such groups, in particular of indecomposable $\{p, q\}$-groups, which are pp-matroid.

Proposition 3. *Let G be a Frattini-free pp-matroid group. Then:*

(1) *every proper subgroup of G and any homomorphic image of G are also pp-matroid, Frattini-free groups;*
(2) *if $H < G$ is a subgroup and $a \in G \setminus H$ is a pp-element, then H is of prime index in $\langle H, a\rangle$;*
(3) *let $g_1, \ldots, g_n$ be any sequence of pp-elements in G. Then $\langle g_1\rangle \ne \langle g_1, g_2\rangle \ne \cdots \ne \langle g_1, g_2, \ldots, g_n\rangle = G$ if and only if $\{g_1, g_2, \ldots, g_n\}$ is a pp-base of G.*

Proof. (1) According to Theorem 1, let $G = G_1 \times \cdots \times G_n$ be a direct product with factors of coprime orders, where every G_i is either a p-group or a scalar extension satisfying the conditions in (2) of Theorem 1. If $H \subseteq G$ is a subgroup, then, as is well known, $H = (H \cap G_1) \times \cdots \times (H \cap G_n)$. From this observation, claim (1) follows.

(2) It is visible from our assumption that if $X \subseteq G$ is a subset of pp-elements, then $X \subseteq \bigcup_{i=1}^{n}(X \cap G_i)$. Thus, we can assume that G is an indecomposable pp-matroid group.

Assume first that $G = P \rtimes Q$ is a pp-matroid $\{p, q\}$-group. Then every element of G is of order either p or q. Moreover, every element of order p generates a normal subgroup in G.

Let $H \subseteq G$ be a subgroup and $a \in G \setminus H$. If either H or $\langle a\rangle$ is a p-group, then it is normal and the claim about $H \le \langle H, a\rangle$ is clear. If $x \in H$ and $a \in G$ are of order q, then $a = x^iy$ for some $y \in P$ and $i \ge 1$. In this case we have $\langle H, a\rangle = \langle H, y\rangle$ and we are back in the previous case.

If we assume that G is a p-group for a prime p, then G is elementary abelian and thus the claim (2) is evident.

(3) This claim is an easy consequence of (2). □

It is easy to observe that the formal conditions for being a pp-matroid group are weaker than these from [4].

https://doi.org/10.1017/S0004972715001148 Published online by Cambridge University Press

Using Theorem 2, we show now that pp-matroid groups are matroids in the sense of [4] when as bases one takes pp-bases. It is enough to prove the following proposition.

Proposition 4. *If G is a pp-matroid group and B_1, B_2 are two pp-bases of G, then, for every $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\}$ is also a pp-base of G.*

Proof. Let G be a pp-matroid group and $G = G_1 \times \cdots \times G_n$ be a decomposition as in Theorem 2. If B is a pp-base of G, then $B = B_1 \cup \cdots \cup B_n$, where B_i is a pp-base of G_i for $i = 1, \ldots, n$. Hence, we can assume that G is an indecomposable pp-matroid group. Using properties of pp-bases, we can also assume that G is Frattini-free and apply Proposition 3. □

As an immediate consequence of matroid theory, we obtain the following corollary.

Corollary 5. *Let G be a pp-matroid group and $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_{k+1}\}$ be pp-independent subsets in G. Then there exists $y \in Y$ such that the set $X \cup \{y\}$ is also pp-independent.*

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JAN KREMPA, Institute of Mathematics,
University of Warsaw, Banacha 2,
02-097 Warszawa, Poland
e-mail: jkrempa@mimuw.edu.pl

AGNIESZKA STOCKA, Institute of Mathematics,
University of Białystok, Ciołkowskiego 1M,
15-245 Białystok, Poland
e-mail: stocka@math.uwb.edu.pl