21 The bosonic string

A particle moving through space sweeps out a path called a *world line*. The action of the particle is just the integral of the invariant length element along the path, up to a constant.

Suppose we want to describe the motion of a string. A string, as it moves, sweeps out a two-dimensional surface in space–time called a *world sheet*. We can parameterize the path in terms of two coordinates, one time-like and one space-like, denoted τ and σ or σ_0 and σ_1. The action should not depend on the coordinates we use to parameterize the surface. Polyakov stressed that this can be achieved by using the formalism of general relativity. Introduce a two-dimensional metric $\gamma_{\alpha\beta}$. Then an invariant action is

$$S = \frac{T}{2}\int d^2\sigma\, \sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \eta_{\mu\nu}. \tag{21.1}$$

Here our conventions are such that, for a flat space,

$$\gamma = \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{21.2}$$

(similarly, our D-dimensional space–time metric is $ds^2 = -dt^2 + d\vec{x}^2$).

This action has a large symmetry group. There are, first, general coordinate transformations of the two-dimensional surface. For a simple topology (plane or sphere), these permit us to bring the metric to the form

$$\gamma = e^{\phi}\eta. \tag{21.3}$$

In this gauge (the *conformal gauge*) the action is independent of the angle ϕ:

$$S = -\frac{T}{2}\int d^2\sigma\, \eta^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \eta_{\mu\nu}. \tag{21.4}$$

It is possible to fix this symmetry further. To motivate this gauge choice, we consider an analogous problem in field theory. In a gauge theory such as QED we can fix a covariant gauge, $\partial \cdot A = 0$. This gauge fixing, while manifestly Lorentz invariant, is not manifestly unitary. We might try to quantize covariantly by introducing creation and annihilation operators a^μ. These would obey

$$[a^\mu, a^{\dagger\nu}] = g^{\mu\nu}, \tag{21.5}$$

so that some states would seem to have a negative norm. If one proceeds in this way, it is necessary to prove that states with negative (or vanishing) norm cannot be produced in scattering amplitudes.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

One way to deal with this is to choose a non-covariant gauge. The Coulomb gauge is a familiar example, but a particularly useful description of gauge theories is obtained by choosing the *light cone gauge*. First, define light cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}). \tag{21.6}$$

We will simply denote as $\vec{X}$ the remaining, transverse, coordinates. Correspondingly, one defines the light cone momenta

$$p^{\pm} = \frac{1}{\sqrt{2}}(p^0 \pm p^{D-1}), \qquad \vec{p}. \tag{21.7}$$

Note that

$$A \cdot B = -(A^+B^- + A^-B^+) + \vec{A} \cdot \vec{B}. \tag{21.8}$$

Now we will think of x^+ as our time variable. The "Hamiltonian" generates translations in x^+; it is in fact p^-. Note that for a particle,

$$p^2 = -2p^+p^- + \vec{p}^2 \tag{21.9}$$

and the Hamiltonian is

$$H = \frac{1}{p^+}p^-. \tag{21.10}$$

Having made this choice of variables, one can then make the gauge choice $A^+ = 0$. In the Lagrangian there are no terms involving $\partial_+ A^-$, so A^- is not a dynamical field; only the $D-2$ A^is are dynamical. So we have the correct number of physical degrees of freedom. One simply solves for A^- by using its equations of motion. In the early days of QCD this description proved useful in understanding very high energy scattering. In practice, similar algebraic gauges are still very useful.

Light cone coordinates, more generally, are very helpful for identifying physical degrees of freedom. Consider the problem of counting the degrees of freedom associated with some tensor field $A^{\mu\nu\rho}$. For a massive field, one counts by going to the rest frame and restricting the indices μ, ν, ρ to be $(D-1)$-dimensional. For a massless field, the relevant group is the "little group" of the Lorentz group, $SO(D-2)$. Correspondingly, one restricts the indices to be $(D-2)$-dimensional. So, for example, for a massless vector, there are $D-2$ degrees of freedom; for a symmetric traceless tensor (the graviton), there are $[(D-2)(D-1)/2]-1$. Light cone coordinates and the light cone gauge, provide an immediate realization of this counting.

For many questions in quantum field theory, covariant methods are much more powerful than use of the light cone. Quantum field theorists are familiar with techniques for coping with covariant gauges. These involve the introduction of additional fictitious degrees of freedom (Faddeev–Popov ghosts). It is probably fair to say that most quantum field theorists do not know much about gauges such as the light cone gauge (there is almost no treatment of these topics in standard texts). But we will see in string theory that the light cone gauge is quite useful in isolating the physical degrees of freedom of strings. It lacks some of the elegance of covariant treatments but avoids the need to introduce an intricate ghost structure and, as in field theory, the physical degrees of freedom are manifest. The differences between the covariant and light cone treatments, as we will see, are most dramatic when we consider supersymmetric strings. In the light cone approach of Green and Schwarz, space–time supersymmetry is manifest. In the covariant approach, it is not at all apparent. However, for the discussion of interactions the light cone treatment tends to be rather awkward. In this chapter we will first introduce the light cone gauge and then go on to discuss aspects of the covariant formulation. The suggested readings should satisfy the reader interested in more details of the covariant treatment.

21.1 The light cone gauge in string theory

21.1.1 Open strings

In the conformal gauge, (see Eq. (21.3)) we can use our coordinate freedom to choose $X^+ = \tau$. We also can choose the coordinates such that the momentum density $\mathcal{P}^+$ is constant on the string. In this gauge, in D dimensions the independent degrees of freedom of a single string are the coordinates $X^I(\sigma, \tau)$, $I = 1, \ldots, D-2$. They are each described by the Lagrangian of a free two-dimensional field,

$$S = \frac{T}{2} \int d^2\sigma [(\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2]. \tag{21.11}$$

It is customary to define another quantity, α' (the *Regge slope*), with dimensions of length-squared:

$$\alpha' = \frac{1}{2\pi T}. \tag{21.12}$$

We will generally take a step further and use units with $\alpha' = 1/2$. In this case, the action is:

$$S = \frac{1}{2\pi} \int d^2\sigma [(\partial_\tau X^I)^2 - (\partial_\sigma X^I)^2]. \tag{21.13}$$

The reader should be alerted to the fact that there is another common choice of units, $\alpha' = 2$, and we will encounter this later. In this case, the action has a factor $1/(8\pi)$ out front.

In order to write down the equations of motion, we need to specify boundary conditions in σ. Consider, first, open strings, i.e. strings with two free ends. We want to choose boundary conditions such that when we vary the action we can ignore surface terms. There are two possible choices:

1. *Neumann boundary conditions*,

$$\partial_\sigma X^I(\tau, 0) = \partial_\sigma X^I(\tau, \pi) = 0; \tag{21.14}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

2. *Dirichlet boundary conditions*,

$$X^I(\tau,0) = X^I(\tau,\pi) = \text{ const.} \tag{21.15}$$

It is tempting to discard the second possibility, as it appears to violate translation invariance. So, for now, we will consider only Neumann boundary conditions but will return later to the Dirichlet conditions.

We want to write down a Fourier expansion for the X^Is. The normalization of the coefficients is conventionally taken to be somewhat different from that of relativistic quantum field theories:

$$X^I = x^I + p^I\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-in\tau}\cos n\sigma. \tag{21.16}$$

The α_n^μs are, up to constants, ordinary creation and annihilation operators:

$$\alpha_n^I = \sqrt{n}a_n, \quad \alpha_{-n}^I = \sqrt{n}a_n^\dagger. \tag{21.17}$$

Because we are working at finite volume (in the two-dimensional sense) there are normalizable zero modes, the x^Is and p^Is. They correspond to the coordinate and momentum of the center of mass of a string. From our experience in field theory, we know how to quantize this system. We impose the commutation relation

$$[\partial_\tau X^I(\sigma,\tau), X^J(\sigma',\tau)] = \frac{-i}{\pi}\delta^{IJ}(\sigma-\sigma'). \tag{21.18}$$

This is satisfied by

$$[x^I, p^J] = i\delta^{IJ}, \quad \left[\alpha_n^I, \alpha_{n'}^J\right] = n\delta_{n+n',0}\delta^{IJ}. \tag{21.19}$$

The states of this theory can be labeled by their transverse momenta $\vec{p}$ and by integers corresponding to the occupation numbers of the infinite set of oscillator modes. It is helpful to keep in mind that this is just the quantization of a set of free two-dimensional fields in a finite volume.

We can write down a Hamiltonian for this system. With normal ordering this is

$$H = \vec{p}^{\,2} + N + a, \tag{21.20}$$

where

$$N = \sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I \tag{21.21}$$

and a is a normal ordering constant. States can be labeled by the occupation numbers for each mode, N_{n_i} and their momentum p^I:

$$|p^I, \{N_{n_i}\}\rangle \tag{21.22}$$

The light cone Hamiltonian H generates translations in τ. It is convenient to refine the gauge choice as follows:

$$X^+ = p^+\tau.$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

Since p^- is conjugate to the light cone time x^+, we have

$$p^- = H/p^+ \tag{21.23}$$

or

$$p^+p^- = \vec{p}^2 + N + a, \quad M^2 = N + a. \tag{21.24}$$

So the quantum string describes a tower of states, of arbitrarily large mass. The constant a is not arbitrary; we will see shortly that

$$a = -1. \tag{21.25}$$

This means that the lowest state is a tachyon. We can label this state simply as

$$|T(\vec{p})\rangle = |\vec{p}, \{0\}\rangle \equiv |\vec{p}\rangle. \tag{21.26}$$

The state carries transverse momentum $\vec{p}$ and longitudinal momenta p^+ and p^- and is annihilated by the infinite tower of oscillators. The significance of this instability is not immediately clear; we will close our eyes to it for now and proceed to look at other states in the spectrum. When we study the superstring, we will often find that there are no tachyons.

The first excited state is

$$|A^I\rangle = \alpha^I_{-1}|\vec{p}\rangle. \tag{21.27}$$

Its mass is given by

$$m_A^2 = 1 + a. \tag{21.28}$$

Now we can see why $a = -1$. Here, $\vec{A}$ is a vector field with $D-2$ components. In D dimensions, a massive vector field has $D-1$ degrees of freedom; a massless vector has $D-2$ degrees of freedom. So $\vec{A}$ must be massless and $a = 1$ if the theory is Lorentz invariant. Later, we will give a fancier argument for the value of a but the content is equivalent.

At level 2 we have a number of states,

$$\alpha^I_{-2}|\vec{p}\rangle, \quad \alpha^I_{-1}\alpha^J_{-1}|\vec{p}\rangle. \tag{21.29}$$

These include a vector, a scalar and a symmetric tensor. We will not attempt here to group them into representations of the Lorentz group.

It turns out that the value of D is fixed: $D = 26$. In the light cone formulation the issue is that the light cone theory is not manifestly Lorentz invariant. To establish that the theory is Poincaré invariant, it is necessary to construct the full set of Lorentz generators and carefully check their commutators. This analysis yields the conditions $D = 26$ and $a = -1$. Later, we will discuss further the derivation of this result. In a manifestly covariant formulation such as the conformal gauge: the issue is one of unitarity, as in gauge field theories. The decoupling of negative- and zero-norm states yields, again, the condition $D = 26$.

Turning to the gauge boson, it is natural to ask: what are the fields charged under the gauge symmetry? The answer is suggested by a picture of a meson as a quark and antiquark connected by a string. We can allow the ends of the strings to carry various types of charge. These are known as Chan–Paton factors. In the case of the bosonic string these can be, for example, a fundamental and antifundamental of $SU(N)$. Then the string itself transforms as a tensor product of vector representations. Because the open strings include massless gauge bosons, this product must lie in the adjoint representation of the group. In bosonic string theory one can also have $SO(N)$ and $Sp(N)$ groups. In the case of a superstring we will see that the group structure is highly restricted. The theory will make sense only in ten flat dimensions, and then only if the group is $O(32)$.

21.2 Closed strings

We have begun with open strings, since these are in some ways the simplest, but theories of open strings by themselves are incomplete. There are always processes which will produce closed strings. For closed strings, we again have a set of fields $X^I(\sigma,\tau)$. Their action is identical to what we wrote down before, but they now obey the boundary conditions

$$X^I(\sigma+\pi,\tau) = X^I(\sigma,\tau). \tag{21.30}$$

Again, we can write a mode expansion:

$$X^I = x^I + p^I\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^I e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^I e^{-2in(\tau+\sigma)}\right). \tag{21.31}$$

The exponential terms are the familiar solutions to the two-dimensional wave equation. One can speak of modes moving to the left ("left movers") and to the right ("right movers") on the string. Again we have commutation relations:

$$[x^I, p^J] = i\delta^{IJ}, \quad \left[\alpha_n^I, \alpha_{n'}^J\right] = n\delta_{n+n'}\delta^{IJ}, \quad \left[\tilde{\alpha}_n^I, \tilde{\alpha}_{n'}^J\right] = n\delta_{n+n'}\delta^{IJ}. \tag{21.32}$$

Now the Hamiltonian is

$$H = \vec{p}^{\,2} + N + \tilde{N} + b, \tag{21.33}$$

where

$$N = \sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I, \quad \tilde{N} = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I. \tag{21.34}$$

In working out the spectrum there is an important constraint. There should be no special point on the string, i.e. translations in the σ direction should leave states alone. The generator of constant shifts of σ can be found by the Noether procedure:

$$\mathcal{P}_\sigma = \int d\sigma\, \partial_\tau X^I\, \partial_\sigma X^I = N - \tilde{N}. \tag{21.35}$$

So we need to impose the constraint $N = \tilde{N}$ on the states.

Once more, the lowest state is a scalar,

$$|T\rangle = |\vec{p}\rangle, \quad m_T^2 = b. \tag{21.36}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

Because of the constraint, the first excited state is

$$|\Psi_{IJ}\rangle = \tilde{\alpha}^I_{-1}\alpha^J_{-1}|\vec{p}\rangle. \tag{21.37}$$

We can immediately decompose these states into irreducible representations of the little group; there is a symmetric traceless tensor, a scalar (the trace) and an antisymmetric tensor. A symmetric, traceless, tensor should have, if massive, $D^2 - D - 1$ states. Here, however, we have only $D^2 - 3D + 1$ states. This is precisely the correct number of states for a massless, spin-2 particle – a graviton. The remaining states are precisely the number for a massless antisymmetric tensor field and a scalar. So we learn that $b = -2$.

This is a remarkable result. General arguments, going back to Feynman, Weinberg and others, show that a massless spin-2 particle, in a relativistic theory, necessarily couples like a graviton in Einstein's theory. So string theory is a theory of general relativity. This bosonic string is clearly unrealistic, but the presence of the graviton will be a feature of all string theories, including the more realistic ones.

21.3 String interactions

The light cone formulation is very useful for determining the spectrum of string theories, but it is somewhat more awkward for the discussion of interactions. As explained in the introduction to this chapter, string interactions are determined geometrically, by the nature of the string world sheet. Actually turning drawings of world sheets into a practical computational method is surprisingly straightforward. This is most easily done using the conformal symmetry of the string theory. So we return to the conformal gauge. There are close similarities between the treatment of open and closed strings. We will start with closed strings, for which the Green's functions are somewhat simpler. At the end of this chapter we will return to open strings.

21.3.1 String theory in conformal gauge

In conformal gauge the action is

$$S = \frac{1}{\pi}\int d^2\sigma[(\partial_\tau X^\mu)^2 - (\partial_\sigma X^\mu)^2]. \tag{21.38}$$

Introducing the two-dimensional light cone coordinates

$$\sigma_\pm = \sigma_0 \pm \sigma_1, \tag{21.39}$$

the flat world-sheet metric takes the form

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2} \tag{21.40}$$

and the action can be written as

$$S = \frac{1}{8\pi}\int d\sigma_+ d\sigma_-\, \partial_{\sigma_+}X^\mu\, \partial_{\sigma_-}X^\mu. \tag{21.41}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

At the classical level this action is invariant under a conformal rescaling of the coordinates. If we introduce light cone coordinates on the world sheet then the action is invariant under the transformations

$$\sigma_\pm \to f_\pm(\sigma_\pm). \tag{21.42}$$

Later, we will Wick-rotate and work with complex coordinates; these conformal transformations will then be the conformal transformations familiar in complex variable theory. It is well known that, by a conformal transformation, one can map the plane into a sphere, for example. In this case the regions at infinity with incoming or outgoing strings are mapped to points. The creation or destruction of strings at these points is described by local operators in the two-dimensional world-sheet theory. In order to respect the conformal symmetry these operators must, like the action, be integrals over the world sheet of local dimension-two operators. These operators are known as vertex operators, $V(\sigma,\tau)$.

In conformal gauge the action also contains Faddeev–Popov ghost terms, associated with fixing the world-sheet general coordinate invariance. We will discuss some of their features later. But we will focus on the fields X^μ first. If we simply write down mode expansions for the fields (taking closed strings, for definiteness),

$$X^\mu = x^\mu + p^\mu\tau + i\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\right), \tag{21.43}$$

then we will encounter difficulties. The α^μs will now obey the commutation relations

$$[x^\mu, p^\nu] = ig^{\mu\nu}, \quad \left[\alpha_n^\mu, \alpha_{n'}^\nu\right] = \left[\tilde{\alpha}_n^\mu, \tilde{\alpha}_{n'}^\nu\right] = n\delta_{n+n'}g^{\mu\nu}. \tag{21.44}$$

If we proceed naively, for $\mu = \nu = 0$ the minus sign from g^{00} means that we will have states in the spectrum of negative or zero norm.

The appearance of negative-norm states is familiar in gauge field theory. The resolution of the problem, there, is gauge invariance. One can either choose a gauge in which there are no states with negative norm or one can work in a covariant gauge in which the negative-norm states are projected out. In a modern language, this projection is implemented by the BRST procedure. But it is not hard to check that, in a covariant gauge, low-order diagrams in QED, for example, give vanishing amplitudes to produce negative- or zero-norm states (i.e. photons with time-like or light-like polarization vectors). In gauge theories it is precisely the gauge symmetry which accounts for this. In string theory it is another symmetry, the residual conformal symmetry of the conformal gauge.

In Chapter 17 on general relativity we learned that differentiation of the matter action with respect to the metric gives the energy–momentum tensor. In Einstein's theory, differentiating the Einstein term as well gives Einstein's equations. In the string case the world-sheet metric has no dynamics (the Einstein action in two dimensions is a total derivative), and the Euler–Lagrange equation for γ yields an equation starting that the energy–momentum tensor vanishes. Quantum mechanically, these become constraint equations. The components of the energy–momentum tensor are

$$T_{10} = T_{01} = \partial_0 X \cdot \partial_1 X, \quad T_{00} = T_{11} = \frac{1}{2}[(\partial_0 X)^2 + (\partial_1 X)^2]. \tag{21.45}$$

The energy–momentum tensor is traceless. This is a consequence of conformal invariance; you can show that the trace is the generator of conformal transformations. In terms of the light cone coordinates, the non-vanishing components of the stress tensor are

$$T_{++} = \partial_+ X \cdot \partial_+ X, \quad T_{--} = \partial_- X \cdot \partial_- X. \tag{21.46}$$

Note that $T_{+-} = T_{-+} = 0$. Energy–momentum conservation then says that

$$\partial_- T_{++} = 0, \quad \partial_+ T_{--} = 0. \tag{21.47}$$

As a result, any quantity of the form $f(x^+)T_{++}$ or $f(x^-)T_{--}$ is also conserved. Integrating over the world sheet, this gives an infinite number of conserved charges.

We want to impose the condition of vanishing stress tensor as a condition on states. There is an obstacle, however, and this leads to one way of understanding the origin of the critical dimension, 26. The obstacle is an anomaly, similar to the anomalies we encountered in the first part of this text. One can see the problem if one takes the mode expansions for the X^μs and works out the commutators for the Ts. We will show in the next section that

$$\begin{aligned} &[T_{++}(\sigma), T_{++}(\sigma')] \\ &\quad = \frac{i}{24}(26 - D)\delta'''(\sigma - \sigma') + i[T_{++}(\sigma) + T_{++}(\sigma')]\delta'(\sigma - \sigma'), \end{aligned} \tag{21.48}$$

and a similar equation holds for T_{--}. The first term in Eq. (21.48) is clearly an obstruction to imposing the constraint unless $D = 26$. The number 26 arises from the energy–momentum tensor of the Faddeev–Popov ghosts. Were it not for the ghosts, strings would *never* make sense quantum mechanically. One can calculate this commutator painstakingly by decomposing in modes. But there are simpler methods, which also provide important insights into string theory and which we will develop in the next section.

21.4 Conformal invariance

The analysis of conformal invariance is enormously simplified by passing to Euclidean space. Define

$$w = \tau + i\sigma, \quad \bar{w} = \tau - i\sigma. \tag{21.49}$$

The ws describe a cylinder. Again, in this section $\alpha' = 2$. This choice will make the coordinate space Green's functions for the X^μs very simple. The Euclidean action is now

$$S = \frac{1}{8\pi}\int d^2w\, \partial_w X^\mu\, \partial_{\bar{w}} X^\mu. \tag{21.50}$$

In complex coordinates the non-vanishing components of the energy–momentum tensor are

$$T_{ww} = -\frac{1}{2}\partial_w X \cdot \partial_w X, \quad T_{\bar{w}\bar{w}} = -\frac{1}{2}\partial_{\bar{w}} X \cdot \partial_{\bar{w}} X. \tag{21.51}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

We saw in the previous section that the string action, in Minkowski coordinates, is invariant under the transformations

$$\sigma^+ \to f(\sigma^+), \quad \sigma_- \to g(\sigma^-). \tag{21.52}$$

In terms of the complex coordinates this becomes invariance under the transformations

$$w \to f(w), \quad \bar{w} \to f^*(\bar{w}). \tag{21.53}$$

These are conformal transformations of the complex variable and, as a result of this symmetry, we are able to bring all the machinery of complex analysis to bear on this problem. One particularly useful conformal transformation is the mapping of the cylinder onto the complex plane

$$z = e^w, \quad \bar{z} = e^{\bar{w}}. \tag{21.54}$$

Under this mapping, surfaces of constant τ on the cylinder are mapped into circles in the complex plane; $\tau \to -\infty$ is mapped into the origin and $\tau \to \infty$ is mapped to ∞. Surfaces of constant τ are mapped into circles.

It is convenient to write our previous expression for X^μ in terms of the variable z. First, we write down our previous expressions again:

$$\begin{aligned} X^\mu &= x^\mu + p^\mu \tau + i \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n^\mu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}\right) \\ &= X_{\mathrm{L}}^\mu + X_{\mathrm{R}}^\mu, \end{aligned} \tag{21.55}$$

where

$$X_{\mathrm{L}}^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau - \sigma) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}, \tag{21.56}$$

$$X_{\mathrm{R}}^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau + \sigma) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau+\sigma)}. \tag{21.57}$$

Here X_{L} is holomorphic (analytic) in z and X_{R} is antiholomorphic:

$$\partial X_{\mathrm{L}} = -i\alpha_n^\mu z^{-n-1}, \quad \bar{\partial} X_{\mathrm{R}} = -i\tilde{\alpha}_n^\mu \bar{z}^{-n-1}, \tag{21.58}$$

where $\partial \equiv \partial/\partial z, \sim \bar{\partial} \equiv \partial/\partial\bar{z}$ and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}p^\mu$.

Let us evaluate the propagator of the xs in coordinate space. The Xs are just two-dimensional quantum fields. Their kinetic term, however, is somewhat unconventional. Because we are working with units $\alpha' = 2$, the action has a factor $1/(8\pi)$ out front. Accounting for the extra 4π, the coordinate-space propagator is (in Euclidean space)

$$\langle X^\mu(\sigma) X^\nu(0)\rangle = 4\pi\delta^{\mu\nu}\int \frac{d^2k}{(2\pi)^2}\frac{e^{i\sigma\cdot k}}{k^2}. \tag{21.59}$$

The right-hand side is logarithmically divergent in the infrared. We can use this fact to our advantage, cutting off the integral at scale μ and isolating the $\ln(\mu|z - z'|)$ factor. The logarithmic dependence can be seen almost by inspection of the integral:

$$\langle X^\mu(z) X^\nu(z')\rangle = 2g^{\mu\nu}\ln(|z - z'|\mu) = g^{\mu\nu}\left[\ln(z - z') + \ln(\bar{z} - \bar{z}') + \ln\mu^2\right]. \tag{21.60}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

As we will see shortly, the infrared cutoff drops out of the physically interesting quantities, so we will suppress it in the following.

In the covariant formulation, conformal invariance is crucial to the quantum theory of strings. To understand the workings of two-dimensional conformal invariance, we can use techniques of complex variable theory and the operator product expansion (OPE). We have discussed the OPE previously, in the context of two-dimensional gauge anomalies. It is also important in QCD in the analysis of various short-distance phenomena. The basic idea is that, for two operators, $\mathcal{O}(z_1)$ and $\mathcal{O}(z_2)$, when $z_1 \to z_2$ we have

$$\mathcal{O}_i(z_1)\mathcal{O}_j(z_2) \underset{z_1 \to z_2}{\to} \sum_k C_{ijk}(z_1 - z_2)\mathcal{O}_k(z_1). \tag{21.61}$$

The coefficients C_{ijk} are, in general, singular as $z_1 \to z_2$. The singularity is determined by the conformal dimension of $\mathcal{O}_i$ defined below (Eq. (21.75)).

To implement this rather abstract statement one can insert the above two operators into a Green's function with other operators located at some distance from z_1. In other words, one studies

$$\langle \mathcal{O}_i(z_1)\mathcal{O}_j(z_2)\Psi(z_3)\Psi(z_4)\cdots\rangle. \tag{21.62}$$

The operators in $\mathcal{O}(z_1)$ can be contracted with those in $\mathcal{O}(z_2)$, giving expressions which are singular as $z_1 \to z_2$, or with the other operators, giving non-singular expressions. The leading term in the OPE comes from the term with the maximum number of operators at z_1 contracted with operators at z_2; less singular operators arise when we contract fewer operators.

As an example which will be useful shortly, consider the product $\partial X^\mu(z)\partial X^\nu(w)$. If this appears in a Green's function, the most singular term as $z \to w$ will be that where we contract $\partial X(z)$ with $\partial X(w)$. The result will be equivalent to the insertion of the unit operator at a point times the singular function $1/(z-w)^2$, so we can write:

$$\partial X^\mu(z)\partial X^\nu(\omega) \sim \frac{g^{\mu\nu}}{(z-w)^2} + \cdots. \tag{21.63}$$

A somewhat more non-trivial, and important, set of operator product expansions is provided by the stress tensor and derivatives of X:

$$T(z)\partial X^\nu(w) = \partial X^\mu(z)\partial X^\mu(z)\partial X^\nu(w). \tag{21.64}$$

Now the most singular term arises when we contract the $\partial X(w)$ factor with one of the $\partial X(z)$ factors in $T(z)$. The other $\partial X(z)$ is left alone; in Green's functions, it must be contracted with other away operators that are further away. So we are left with

$$T(z)\partial X(w) \approx \frac{1}{(z-w)^2}\partial X(w) + \frac{1}{z-w}\partial^2 X(w) + \cdots. \tag{21.65}$$

Another important set of operators will turn out to be exponentials of x:

$$T(z)e^{ik\cdot x} = \frac{k^2}{(z-w)^2}e^{ik\cdot x} + \cdots. \tag{21.66}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

To get some sense of the utility of conformal invariance and OPEs, we will compute the commutators of the α^μs. Start with

$$\alpha_n^\mu = \oint \frac{dz}{2\pi} z^n \partial X^\mu, \tag{21.67}$$

where the contour is taken about the origin. Now use the fact that, on the complex plane, time ordering becomes radial ordering, So, for $|z| > |w|$,

$$T\langle \partial X^\mu(z) \partial X^\nu(w)\rangle = \langle \partial X^\mu(z) \partial X^\nu(w)\rangle. \tag{21.68}$$

For $|z| < |w|$,

$$T\langle \partial X^\mu(z) \partial X^\nu(w)\rangle = \langle \partial X^\nu(w) \partial X^\mu(z)\rangle. \tag{21.69}$$

Thus we have

$$\left[\alpha_m^\mu, \alpha_n^\nu\right] = \left(\oint \frac{dz}{2\pi} z^m \oint \frac{dw}{2\pi} w^n - \oint \frac{dw}{2\pi} w^n \oint \frac{dz}{2\pi} z^m\right) T\left(\partial X^\mu(z) \partial X^\nu(w)\right), \tag{21.70}$$

where the contour can be taken to be a circle about the origin. In the first term, we take $|z| > |w|$, and in the second, $|w| > |z|$. Now, to evaluate the integral, we do, the z integral first, say. For fixed w, deform the z contour so that it encircles w (Fig. 21.1). Then

$$\begin{aligned}\left[\alpha_m^\mu, \alpha_n^\nu\right] &= \oint \frac{dw}{2\pi} w^n \oint \frac{dz}{2\pi} z^m \frac{1}{(z-w)^2} g^{\mu\nu} \\ &= m\delta_{m+n} g^{\mu\nu}.\end{aligned}$$

Let us now return to the stress tensor. We expect that the stress tensor is the generator of conformal transformations and that its commutators should contain information about the dimensions of operators. What we have just learned, by example, is that the operator products of operators encode the commutators. We could show by the Noether procedure

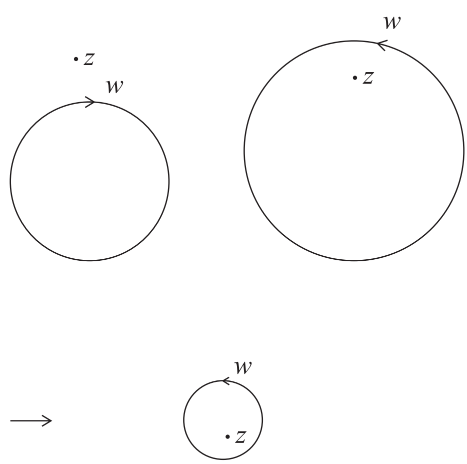

Fig. 21.1 Contour integral manipulations used to evaluate commutators in conformal field theory.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

that the stress tensor is the generator of conformal transformations. But we can verify this directly. Consider the transformation

$$z \to z + \epsilon(z). \tag{21.71}$$

We expect that the generator of this transformation is

$$\oint dz\, T(z)\epsilon(z). \tag{21.72}$$

Let us take the special case of an overall conformal rescaling:

$$\epsilon(z) = \lambda z. \tag{21.73}$$

Now suppose that we have an operator $\mathcal{O}(w)$ and that

$$T(z)\mathcal{O}(w) = \frac{h}{(z-w)^2}\mathcal{O}(w) + \text{ less singular terms.} \tag{21.74}$$

Then

$$\left[\frac{1}{2\pi i}\oint T(z)\epsilon(z), \mathcal{O}(w)\right] = \frac{1}{2\pi i}\oint dz \frac{\lambda z h \mathcal{O}(w)}{(z-w)^2}$$
$$= \lambda h \mathcal{O}(w). \tag{21.75}$$

This means that, under the conformal rescaling, we have $\mathcal{O} \to h\mathcal{O}$, just as we would expect for an operator of dimension h. As an example, consider $\mathcal{O} = (\partial)^n X$. This should have dimension n, and the leading term in its OPE is just of the form of Eq. (21.74), with $h = n$.

More precisely, an operator is called a primary field of dimension d if

$$T(z)\mathcal{O}(w) = \frac{d\mathcal{O}}{(z-w)^2} + \frac{\partial\mathcal{O}}{z-w}. \tag{21.76}$$

Note that $\partial X(z)$ is an example; $e^{ip\cdot x}$ is another. However, $(\partial)^n X$ is not, in general, as the $1/(z-w)$ term does not have quite the right form. A particularly interesting operator is the stress tensor itself. Naively, this has dimension two, but it is not a primary field. In the OPE, the most singular term arises from the contraction of all the derivative terms. This is proportional to the unit operator. The first subleading term, where one contracts just one pair of derivatives, gives a contribution proportional to the stress tensor itself:

$$T(z)T(w) = \frac{D}{(z-w)^4} + \frac{1}{(z-w)^2}T(w). \tag{21.77}$$

When one includes the Faddeev–Popov ghosts, one finds that they give an additional contribution, changing D to $D - 26$.

The algebra of the Fourier modes of T is known as the Virasoro algebra, and is important in string theory, conformal field theory and mathematics. In string theory it provides important constraints on states. Define the operators

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z). \tag{21.78}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

In terms of these we have

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \tag{21.79}$$

and similarly for $\bar{z}$. Because the stress tensor is conserved, we are free to choose any time (i.e. radius for the circle). The operator product (21.77) is equivalent to the commutation relations above. Proceeding as we did for the commutators of the αs gives

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n}. \tag{21.80}$$

Using expression (21.16) we can construct the L_ns:

$$L_m = \frac{1}{2}\sum : \alpha^{\mu}_{m-n}\alpha_{\mu\ n} :, \quad \tilde{L}_m = \frac{1}{2}\sum : \tilde{\alpha}^{\mu}_{m-n}\tilde{\alpha}_{\mu\ n} :, \tag{21.81}$$

where the colons indicate normal ordering. Only when $m = 0$ is this significant. In this case we have to allow for the possibility of a normal-ordering constant. This constant is related to the constant we found in the Hamiltonian in light cone gauge,

$$L_0 = \sum_{n=0}^{\infty} \alpha^{\mu}_{-n}\alpha_{\mu n} - a, \quad \tilde{L}_0 = \sum_{n=0}^{\infty} \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{\mu n} - a. \tag{21.82}$$

Now we want to consider the constraint on states corresponding to the classical vanishing of the stress tensor. Because of the commutation relations, we cannot require all of Ls annihilate physical states. We require instead that

$$L_m|\Psi\rangle = 0 \tag{21.83}$$

for $m \geq 0$. Since $L^{\dagger}_m = L_{-m}$, this ensures that

$$\langle\Psi|L_n|\Psi\rangle = 0 \quad \forall\, n. \tag{21.84}$$

The constraint (21.35) in the light cone of invariance under translations along the string now becomes the condition $L_0 = \tilde{L}_0$. At the first excited level we have the state:

$$|\epsilon\rangle = \epsilon_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|p^{\mu}\rangle. \tag{21.85}$$

The $L_n s$, for $n > 1$, trivially annihilate the state. For $n = 1$ we have

$$L_1|\epsilon\rangle = \alpha^{\mu}_0\epsilon_{\mu\nu}|p^{\nu}\rangle. \tag{21.86}$$

Taking into account also $\tilde{L}_1$, we have the conditions

$$p_{\mu}\epsilon^{\mu\nu} = 0 = p_{\nu}\epsilon^{\mu\nu}. \tag{21.87}$$

This is similar to the condition $k_{\mu}\epsilon^{\mu}$ familiar in covariant gauge electrodynamics and it eliminates the negative-norm states. Consider, now, L_0:

$$L_0|\epsilon\rangle = (p^2 - a + 1)|\epsilon\rangle. \tag{21.88}$$

So, if $a = 1$ then the constraint is $p^2 = 0$, as we expect from Lorentz invariance. For open strings there is an analogous construction.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

21.5 Vertex operators and the S-matrix

We have argued that, when the cylinder is mapped to the plane, the creation or destruction of states is described by local operators known as *vertex operators*. In this section we discuss the properties of these operators and their construction. We explain how the space–time S-matrix is obtained from correlation functions of these operators, and compute a famous example.

21.5.1 Vertex operators

There is a close correspondence between states and operators: $z \to 0$ corresponds to $t \to -\infty$. So consider, for example,

$$\partial_z X^\mu |0\rangle; \tag{21.89}$$

as $z \to 0$ we have

$$\partial_z X(z \to 0)|0\rangle = -i \sum_{m=-1}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}} |0\rangle. \tag{21.90}$$

All terms but the term $m = -1$ annihilate the state to the right. Combining this with a similar left-moving operator creates a single-particle state.

More generally, in conformal field theories there is a one-to-one correspondence between states and operators. This is the realization of the picture discussed in the introduction. By mapping the string world sheet to the plane the incoming and outgoing states have been mapped to points, and the production or annihilation of particles at these points is described by local operators.

The construction of the S-matrix in string theory relies on this connection between states and operators. The operators which create and annihilate states are known as vertex operators. What properties should a vertex operator possess? The production of the particle should be represented as an integral over the string world sheet (so that there is no special point along the string). The expression

$$\int d^2z\, V(z, \bar{z}) \tag{21.91}$$

should be invariant under conformal transformations. This means that the operator should possess dimension two; more precisely, it should possess dimension one with respect to both the left- and the right-moving stress tensors, so that

$$T(z)V(w, \bar{w}) = \frac{1}{(z-w)^2} V(w, \bar{w}) + \frac{1}{z-w} \partial_w V(w, \bar{w}) + \cdots \tag{21.92}$$

and similarly for $\bar{T}$. An operator with this property is called a $(1, 1)$ operator.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

A particularly important operator in two-dimensional free-field theory (i.e. the string theories we have been describing up to now) is constructed from the exponential of the scalar field:

$$\mathcal{O}_p = e^{ip\cdot x}. \tag{21.93}$$

This has dimension

$$d = p^2 \tag{21.94}$$

with respect to the left-moving stress tensor, and similarly for the right-moving part.

With these ingredients, we can construct operators of dimension $(1, 1)$. These are in one-to-one correspondence with the states we found in the light cone construction, as follows.

1. *The tachyon*:

$$e^{ip\cdot x}, \quad p^2 = 1. \tag{21.95}$$

2. *The graviton, antisymmetric tensor, and dilaton*:

$$\epsilon_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ip\cdot x}, \quad p^2 = 0. \tag{21.96}$$

The operator product

$$\partial X^\rho(z)\partial X_\rho(z)\epsilon_{\mu\nu}(p)\bar{\partial}X^\mu(w)\partial X^\nu(w)e^{ip\cdot x}(w) \tag{21.97}$$

contains terms which go as $1/(z-w)^3$ and have come from contracting one derivative in the stress tensor with $e^{ip\cdot x}$ and one with ∂X^μ. Examining Eq. (21.92), this leads to the requirement

$$p^\mu \epsilon_{\mu\nu}(p) = 0, \tag{21.98}$$

which we expect for massless spin-2 states. In our earlier operator discussion, this was one of the Virasoro conditions.

3. *Massive states*:

$$\epsilon_{\mu_1\cdots\mu_n}(p)\partial X^{\mu_1}\partial X^{\mu_2}\cdots\partial X^{\mu_n}e^{ip\cdot x}, \quad p^2 = 1-n. \tag{21.99}$$

Obtaining the correct OPE with the stress tensor now gives a set of constraints on the polarization tensor; again these are just the Virasoro constraints. Without worrying about degeneracies, we have a formula for the masses:

$$M_n^2 = n - 1. \tag{21.100}$$

This is what we found in the light cone gauge. Traditionally, the states were organized in terms of their spins. States of a given spin all lie on straight lines, known as *Regge trajectories*. These results are all in agreement with the light cone spectra we found earlier.

21.5.2 The S-matrix

Now we will make a guess as to how to construct an S-matrix. Our vertex operators, integrated over the world-sheet, are invariant under reparameterizations and conformal transformation of the world-sheet coordinates. We have seen that they correspond to the creation and annihilation of states in the far past and far future. We will normalize the vertex operators in such a way that

$$V_i(z)V_j(w) \sim \frac{\delta_{ij}}{|z-w|^4}. \tag{21.101}$$

So, we need to study correlation functions of the form

$$\mathcal{A} = \int d^2z_1 \cdots d^2z_n \langle V_1(z_1,p_1)\cdots V_n(z_n p_n)\rangle. \tag{21.102}$$

We will include a coupling constant g with each vertex operator.

Before evaluating this expression in special cases, let us consider the problem of evaluating the correlation functions of exponentials

$$\left\langle \exp\left(i\sum p_i\cdot X(z_i)\right)\right\rangle. \tag{21.103}$$

An easy way to evaluate this expression is to work in the path integral framework. Then the exponential has the structure

$$\int d^2z\, J_\mu(z)X(z), \tag{21.104}$$

where

$$J_\mu(z) = \sum_i p_{i\mu}\delta^2(z-z_i). \tag{21.105}$$

But we know that the result of such a path integral is

$$\exp\left(i\int d^2z d^2z' J_\mu(z)J^\mu(z')\Delta(z-z')\right) = \exp\left(\sum p_i\cdot p_j \ln|(z_i-z_j)|^2\mu^2\right), \tag{21.106}$$

where we have made a point of restoring the infrared cutoff.

We will consider the infrared cutoff first. Overall, we have a factor:

$$\mu^{(\sum p_i)^2}. \tag{21.107}$$

This vanishes as $\mu \to 0$ unless $\sum p_i = 0$, i.e. *unless momentum is conserved*. This result is related to the Mermin–Wagner–Coleman theorem, which states that there is no spontaneous breaking of global symmetries in two dimensions. Translational invariance is a global symmetry of the two-dimensional field theory; $e^{ip\cdot x}$ transforms under this symmetry. The only non-vanishing correlation functions are translationally invariant.

This correlation function also has an ultraviolet problem, coming from the $i=j$ terms in the sum. Eliminating these corresponds to the normal ordering of the vertex operators, and we will do this in what follows (we can, if we like, introduce an explicit ultraviolet cutoff; this gives a factor which can be absorbed into the definition of the vertex operators).

There is one more set of divergences with which we need to deal. These are associated with a part of the conformal invariance that we have not yet fixed. The operators L_0, L_1 and L_{-1} form a closed algebra. On the plane they generate overall rescalings (L_0), translations (L_1) and more general transformations (L_{-1}) which can be unified in $SL(2, C)$, the Möbius group. It transforms coordinates z to coordinates z^1, where

$$z = \frac{\alpha z' + \beta}{\gamma z' + \delta}. \tag{21.108}$$

Such transformations have the feature that they map the plane once into itself. It is necessary to fix this symmetry and divide by the volume of the corresponding gauge group. We can choose the location of three of the vertex operators, say z_1, z_2, z_3. These location are conventionally taken to be $0, 1, \infty$. It is necessary also to divide by the volume of this group; the corresponding factor is

$$\Omega_M = |z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2. \tag{21.109}$$

One can simply accept that this factor emerges from a Faddeev–Popov condition or it can be derived in the exercises at the end of the chapter. Finally, it is necessary to divide by g_s^2. This ensures that a three-particle process is proportional to g_s, a four-particle process to g_s^2 and so on.

Using these results we can construct particular scattering amplitudes. While it is physically somewhat uninteresting, the easiest case to examine is simply the scattering of tachyons. Let us specialize to the case of two incoming and two outgoing particles. Putting together our results above we have (remembering that $z_3 \to \infty$) the amplitute for particle scattering takes the form

$$\mathcal{A} = \frac{1}{\Omega_M} \int d^2 z_4 \, |z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2$$
$$|z_3|^{p_3 \cdot (p_1 + p_2 + p_3)} |z_1 - z_2|^{p_1 \cdot p_2} |z_4|^{p_4 \cdot p_1} |z_4 - 1|^{p_4 \cdot p_2}. \tag{21.110}$$

Using momentum conservation, the z_4-independent contributions cancel out in the limit and we are left with

$$\mathcal{A} = \int d^2 z \, |z|^{2p_1 \cdot p_4} |z - 1|^{2p_2 \cdot p_4}. \tag{21.111}$$

Now we need an integral table to obtain

$$I = \int d^2 z \, |z|^{-A} |1 - z|^{-B}$$
$$= \beta \left(1 - \frac{A}{2}, 1 - \frac{B}{2}, \frac{A + B}{2} - 1\right). \tag{21.112}$$

The beta function is defined by

$$\beta = \pi \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a + b)\Gamma(b + c)\Gamma(c + a)}. \tag{21.113}$$

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

We can express this result in terms of the Mandelstam invariants for $2 \to 2$ scattering, $s = -(p_1+p_2)^2$, $t = -(p_2-p_3)^2$ and $u = -(p_1-p_4)^2$. Using the mass shell conditions,

$$p_4 \cdot p_1 = \frac{1}{2}\left[u + \left(p_1^2 - p_4^2\right)\right],$$

$$p_4 \cdot p_2 = -(p_3+p_2+p_1)\cdot p_2 = \frac{1}{2}(-s-t+2m^2), \tag{21.114}$$

gives

$$\mathcal{A} = \frac{\kappa^2}{4\pi}\beta(-4s+1, -4t+1, -4u+1). \tag{21.115}$$

This is the Virasoro–Shapiro amplitude. There are a number of interesting features of this amplitude. It has singularities at precisely the locations of the masses of the string states. It should be noted, also, that we have obtained this result by an analytic continuation. The original integral is only convergent for a range of momenta, corresponding, essentially, to rules sitting below the threshold for the tachyon in the intermediate states.

We will not develop the machinery of open-string amplitudes here, but it is similar. One again needs to compute correlation functions of vertex operators. The vertex operators are somewhat different. Also, the boundary conditions for the two-dimensional fields, and thus the Green's functions, are different. The scattering amplitude for open-string tachyons is known as the Veneziano formula (see Section 21.6).

21.5.3 Factorization

The appearance of poles in the S-matrix at the masses of the string states is no accident. We can understand it in terms of our vertex operator and OPE analysis. Suppose that particles one and two, with momenta p_1 and p_2, have $s = (p_1+p_2)^2 = -m_n^2$, the mass-squared of a physical state of the system. Consider the OPE of their vertex operators:

$$e^{ip_1\cdot X(z_1)}e^{ip_2\cdot X(z_2)} \approx e^{i(p_1+p_2)\cdot X(z_2)}|z_1-z_2|^{2p_1\cdot p_2}. \tag{21.116}$$

So, in the S-matrix, fixing $z_2 = 0$, $z_3 = 1$ and $z_4 = \infty$, we encounter:

$$\int d^2z\, |z_1|^{2p_1\cdot p_2}\left\langle e^{i(p_1+p_2)\cdot X(z_2)}e^{ip_3\cdot X(z_3)}e^{ip_4\cdot X(z_4)}\right\rangle. \tag{21.117}$$

Using momentum conservation and the on-shell conditions for p_1 and p_2 we obtain

$$2p_2\cdot p_1 = q^2 - 8, \tag{21.118}$$

where $q = p_1 + p_2$. So the z-integral gives a pole,

$$\mathcal{A} \sim \frac{1}{4-q^2} \tag{21.119}$$

i.e. it vanishes when the intermediate state is an on-shell tachyon.

This is general. Poles appear in the scattering amplitude when intermediate states go on-shell. The coefficients are precisely the couplings of the external states to the (nearly) on-shell physical state; this follows from the OPE.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

21.6 The S-matrix versus the effective action

The Virasoro–Shapiro and Veneziano amplitudes are beautiful formulas. Analogous formulas for the case of massless particles can be obtained. These are particularly important for the superstring. For many of the questions which interest us, we are not directly interested in the S-matrix. One feature of the string S-matrix construction is that it involves on-shell states; the momenta appearing in the exponential factors satisfy $p^2 = -m^2$, where m is the mass of the state. So one cannot calculate, for example, the effective potential for the tachyon, since this requires that all momenta vanish. For massless particles things are better, since $p = 0$ is the limiting case of an on-shell process. But the S-matrix is not precisely the effective action. Instead, given the S-matrix, it is usually a straightforward matter to determine a low-energy effective action which will reproduce it. At tree level, one just needs to subtract massless particle exchanges. In loops, one must be more careful.

It is particularly easy to extract three-point couplings of massless particles at tree level. One just needs to study an "S-matrix" for three particles (one could also be a little could also more careful and study a four-particle amplitude, isolating the coefficient of the massless propagator). From our previous analysis, we need

$$\mathcal{A} = \frac{1}{\Omega_M} \langle V_1(z_1) V_2(z_2) V_3(z_3) \rangle, \tag{21.120}$$

where we do not integrate over the locations of the vertex operators. We are free to take z_1 and z_2 arbitrarily close to one another. Then the operator product will involve

$$V_1(z_1) V_2(z_2) \approx C_{123} \frac{1}{|z_1 - z_2|^2} V_3(z_2). \tag{21.121}$$

The final correlation function follows from the normalization of the vertex operators and cancels the Möbius volume. So the net result is that $g_s C_{123}$ is the coupling.

As an example, consider the coupling of two gravitons in the bosonic string. The vertex operator is

$$V_1 = \epsilon_{\mu\nu}(k_1)\, \partial X^\mu(z) \bar{\partial} X^\nu(z)\, e^{ik_1 \cdot X(z)}, \tag{21.122}$$

and similarly for V_2 and V_3. So the operator product has the following structure:

$$V_1(z) V_2(w) = \frac{1}{|z-w|^4} + \epsilon_{\mu\nu}(k_1) \epsilon_{\rho\sigma}(k_2) e^{i(k_1+k_2)\cdot X(z)} \left(k_1^\nu k_2^\sigma \frac{1}{|z-w|^2} \partial X^\mu(z) \bar{\partial} X^\rho(z) + \cdots \right). \tag{21.123}$$

Here the first term arises from the contraction of all the ∂X terms with each other. Loosely speaking, it is related to the production of off-shell tachyons. We will ignore it. The second term that we have indicated explicitly comes from contracting the first $\bar{\partial} X$ factor with the second exponential and the second ∂X factor with the first exponential. The ellipses indicate a long set of contractions. The complete vertex is precisely the on-shell coupling of three gravitons in Einstein's theory, along with couplings to the antisymmetric tensor and dilaton. We will not worry with the details here. When we discuss the heterotic string, we will show that the theory completely reproduces the Yang–Mills vertex in much the same way. We should not be surprised that it is difficult to define off-shell Green functions. In gravity, apart from the S-matrix it is in general hard to define coordinate-invariant observables.

21.7 Loop amplitudes

So far, we have considered tree amplitudes. Closed or open strings interact by splitting and joining. Once we allow for quantum fluctuations, strings in intermediate states can split and join too. Because of conformal invariance, the only invariant characteristic of these diagrams is their topology (for closed-strings, the tree level world sheet has the topology of a sphere). In the closed-string case, each additional loop adds a handle to the world sheet. In general, the theory of string loops is complicated, but the description of one-loop diagrams is rather simple and exposes important features of the theory not apparent in tree diagrams. In the case of closed strings, requiring that the one-loop amplitude be sensible places strong constraints on the theory. Invariance under certain (global) two-dimensional general coordinate transformations, known as modular transformations, accounts for many features of both the bosonic and superstring theories. In space–time, satisfying these constraints is a necessary condition for the unitarity of the scattering amplitude. In this section we provide only a brief introduction. We will leave for later the discussion of open-string loops.

The one-loop amplitude has the topology of a donut, or torus. A simple representation of a torus is as indicated in Fig. 21.2. In this figure, the world sheet is flat and of finite size. We can think of this torus as living in the complex plane. It is (up to conformal transformations) the world sheet appearing in the Euclidean path integral. The two possible periods of the torus are translated into two complex periods, λ_1 and λ_2. We require that the fields are periodic under

$$z \to z + m\lambda_1 + n\lambda_2. \tag{21.124}$$

We can transform λ_1 and λ_2 by a transformation in the modular group, $SL(2, Z)$,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} \tag{21.125}$$

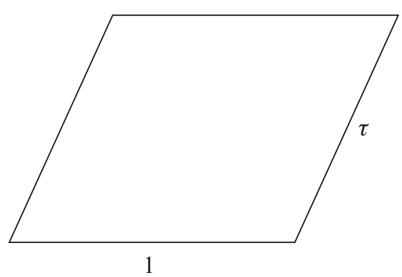

Fig. 21.2 A simple representation of a torus.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

with a, b, c and d integers satisfying $ad - bc = 1$, provided that we also transform the integers n and m by the inverse matrix,

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} m' \\ n' \end{pmatrix} \tag{21.126}$$

Now rescale z by λ_1, and set $\tau = \lambda_2/\lambda_1$. Then z has the periodicities 1 and τ. Under modular transformations, τ transforms as follows:

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{21.127}$$

The modular transformations are general coordinate transformations of the world-sheet theory, but they are not continuously connected to the identity. In order that one-loop string amplitudes make sense, we require that they be invariant under this transformation. The general amplitude will be a correlation function

$$\langle V(z_1)V(z_2)\cdots\rangle_{\text{torus}}, \tag{21.128}$$

evaluated on the torus, as indicated. The simplest amplitude is that with no vertex operators inserted. (At tree level this amplitude vanishes owing to the division by the infinite Möbius volume.) For the bosonic string, we can evaluate the amplitude in light cone gauge. We simply need to evaluate the functional determinant. As these are free fields on a flat space, this is not too difficult. It is helpful to remember some basic field theory facts. The path integral, with initial configuration $\phi_{\rm i}(x)$ and final configuration $\phi_{\rm f}(x)$, computes the quantum mechanical matrix element:

$$\langle\phi_{\rm f}|e^{-iHT}|\phi_{\rm i}\rangle. \tag{21.129}$$

If we take the time to be Euclidean, impose periodic boundary conditions and sum (integrate) over all possible $\phi_{\rm i}$, we will have computed

$$\text{Tr}\, e^{-HT} \tag{21.130}$$

i.e. the quantum mechanical partition function. As described in Appendix C, this observation is the basis of the standard treatments of finite-temperature phenomena in quantum field theory. In the present case the periodicity is in the τ direction. So we compute

$$\text{Tr}\, e^{-H_{\rm lc}\tau}. \tag{21.131}$$

It is convenient to rewrite the light cone Hamiltonian, $H_{\rm lc,}$ in terms of L_0 and $\bar{L}_0$. Introducing

$$q = e^{2\pi i\tau}, \quad \bar{q} = e^{-2\pi i\bar{\tau}} \tag{21.132}$$

we want to evaluate

$$\text{Tr}\left(q^{L_0}, \bar{q}^{\bar{L}_0}\right). \tag{21.133}$$

From any oscillator with oscillator number n, just as in quantum mechanics we obtain $(1 - q^n)^{-1}$; so, allowing for the different values of n and the $D - 2$ transverse directions,

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press

we have

$$\prod q^{D/24}\bar{q}^{D/24}(1-q^n)^{2-D}(1-\bar{q}^n)^{2-D}. \tag{21.134}$$

This is conveniently expressed in terms of a standard function, the Dedekind η-function,

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{21.135}$$

We also need the contribution of the zero modes. This is

$$\int \frac{d^{D-2}p}{(2\pi)^{D-2}} e^{-\tau_2 p^2} \propto \tau_2^{D-2}. \tag{21.136}$$

In the final expression, we need to integrate over τ. The measure for this can be derived from the Faddeev–Popov ghost procedure, but it can be guessed from the requirement of modular invariance. It is easy to check that

$$\int \frac{d^2\tau}{\tau_2^2} \tag{21.137}$$

is invariant. So, in 26 dimensions, we finally have

$$Z \propto \int \frac{d^2\tau}{\tau_2^2}\tau_2^{-12}|\eta(\tau)|^{-48}. \tag{21.138}$$

Now, to check that this is modular invariant we note, first, that the full modular group is generated by the transformations

$$\tau \to \tau + 1, \quad \tau \to -1/\tau. \tag{21.139}$$

Under these transformations, as we said, the measure is invariant. The Dedekind η function transforms as

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau), \quad \eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau). \tag{21.140}$$

Since $\tau_2 \to \tau_2/\tau_1^2 + \tau_2^2$, under $\tau \to -1/\tau$ we have that Z is invariant. Here we see that the bosonic string makes sense only in 26 dimensions.

Suggested reading

More detail on the material in this chapter can be found in Green *et al.* (1987) and in Polchinski (1998). The light cone treatment described here is nicely developed in Peskin (1985).

Exercises

(1) Enumerate the states of the bosonic closed string at the first level with positive mass-squared. Don't worry about organizing them into irreducible representations, but list their spins.

(2) OPEs: explain why X^μ and X^ν do not have a sensible operator product expansion. Work out the OPE of ∂X^μ and ∂X^ν as in the text. Verify the commutator of α^μ and α^ν, as in the text.

(3) Work out the Virasoro algebra, starting with the operator product expansion for the stress tensor and using the contour method.

(4) The Mermin–Wagner–Coleman theorem: consider a free two-dimensional quantum field theory with a single, massless, complex field ϕ. Describe the conserved $U(1)$ symmetry. Show that correlation functions of the form

$$\left\langle e^{iq_1\phi(x_1)} \dots e^{iq_n\phi(x_n)} \right\rangle \tag{21.141}$$

are non-vanishing only if $\sum q_i = 0$. Argue that this means that the global symmetry is not broken. From this construct an argument that global symmetries are never broken in two dimensions.

(5) Show that the factor Ω_M of Eq. (21.109) is invariant under the Möbius group. You might want to proceed by analogy with the Faddeev–Popov procedure in gauge theories.

(6) Show that the factorization of tree level S-matrix elements is general, i.e. that if the kinematics are correctly chosen for two incoming particles 1 and 2, so that $(p_1+p_2)^2 \approx m_n^2$, that the amplitude is approximately a product of the coupling of particles 1 and 2 to particle n, times a nearly on-shell propagator for the n.

https://doi.org/10.1017/9781009290883.028 Published online by Cambridge University Press