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PROPERTIES OF EQUIVALENT CAPACITIES

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0. Introduction. Various definitions of capacity of a subset of a domain in Euclidean space have been used in recent times to shed light on the solvability and spectral theory of elliptic partial differential equations and to establish properties of the Sobolev spaces in which these equations are studied. In this paper we consider two definitions of the capacity of a closed set E in a domain G. One of these capacities measures, roughly speaking, the amount by which the set of function in $C^\infty(G)$ which vanish near E fails to be dense in the Sobolev space $W^{m,p}(G)$. It is generalization of the classical capacity of Weiner and has been used (see e.g. Maz'ja [5]) in the study of Dirichlet problems. The second capacity measures the degree to which functions in $C^\infty(G)$ which vanish near E are forced to have small L^p norms when their derivatives do. The nonvanishing of this capacity is a Poincaré condition and the capacity has been used to determine the solvability of certain Dirichlet problems [4] and to formulate [1], [2] a condition on an unbounded domain G which is necessary and sufficient for the compactness of certain Sobolev space imbeddings on G.

In §1 we shall define the capacities and establish some relationships between them. In particular we show that they vanish simultaneously. In §2 we consider some consequences for the Sobolev spaces $W^{m,p}(G)$ and $W_0^{m,p}(G)$ of the vanishing or nonvanishing of the capacities, and illustrate these with applications to strongly elliptic differential operators on G. In §3 we establish a geometric condition on G guaranteeing the nonvanishing of the capacities.

1. Definitions and basic properties of capacity. Let G be an open, bounded set in Euclidean n-space, E_n, $n>1$. For certain of our results we shall require also that $\bar{G}$ be diffeomorphic to a compact, convex set. The Sobolev space $W^{m,p}(G)$ [resp. $W_0^{m,p}(G)$] denotes the completion of $C^\infty(\bar{G})$ [resp. $C_0^\infty(G)$] with respect to the norm $(p\geq 1, m=1, 2, \ldots)$

$$\|u\|_{m,p,G} = \left\{\sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{0,p,G}^p\right\}^{1/p},$$

where $\|u\|_{0,p,G}$ is the norm of u in $L^p(G)$. Here, as usual, $\alpha=(\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers; $|\alpha|=\sum \alpha_i$; $D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n}$ where $D_i=\partial/\partial x_i$.

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For a closed subset E of $\bar{G}$ we consider two definitions of "(m, p)-capacity" of E in G, specifically

$$(1) \qquad K_G^{m,p}(E) = \inf_{\substack{\phi \in C^\infty(\bar{G}) \\ \phi(x)=1 \text{ near } E}} \|\phi\|_{m,p,G}^p,$$

$$(2) \qquad C_G^{m,p}(E) = \inf_{\substack{\phi \in C^\infty(\bar{G}) \\ \phi(x)=0 \text{ near } E}} \frac{\sum_{1 \le |\alpha| \le m} \|D^\alpha \phi\|_{0,p,G}^p}{\|\phi\|_{0,p,G}^p}.$$

Certain properties of these capacities follow trivially from the definitions, namely

$$(3) \qquad K_G^{m,p}(E) \le K_G^{m+1,p}(E); \qquad C_G^{m,p}(E) \le C_G^{m+1,p}(E)$$

$$(4) \qquad K_G^{m,p}(E) \le K_G^{m,p}(F); \qquad C_G^{m,p}(E) \le C_G^{m,p}(F) \quad \text{for } E \subset F \subset \bar{G}$$

$$(5) \qquad K_G^{m,p}(E) \le \lambda(G)$$

where λ is Lebesgue measure in E_n. In fact equality can hold in (5) with $p > 1$ only if $E = \bar{G}$ for otherwise let $f \in C_0^\infty(G-E)$ satisfy $0 \le f(x) \le 1, f(x) \not\equiv 0$. Then if $\psi_\varepsilon(x) = 1 - \varepsilon f(x)$ we have

$$\begin{aligned} K_G^{m,p}(E) &\le \inf_{0<\varepsilon<1} \|\psi_\varepsilon\|_{m,p,G}^p \\ &\le \inf_{0<\varepsilon<1} \left\{ \lambda(G) - \varepsilon\left[\int_{G-E} f(x)\,dx - \varepsilon^{p-1} \sum_{1 \le |\alpha| \le m} \|D^\alpha f\|_{0,p,G}^p\right]\right\} \\ &< \lambda(G). \end{aligned}$$

If $p=1$ it is easily checked that $K_G^{m,p}(E) = \lambda(G)$ whenever $\bar{G} - E \subset G$, diam. $(\bar{G} - E) \le 1$.

Strictly speaking (2) does not define $C_G^{m,p}(E)$ for $E = \bar{G}$. However if $\bar{G} - E \subset G$ and diam. $(\bar{G} - E) \le \varepsilon$ then by Poincaré's inequality every $\phi \in C^\infty(\bar{G})$ vanishing near E satisfies $\|\phi\|_{0,p,G} \le \varepsilon \|\text{grad } \phi\|_{0,p,G}$ so that $C_G^{m,p}(E) \ge C_G^{1,p}(E) \ge \varepsilon^{-p}$. Thus (4) forces $C_G^{m,p}(\bar{G}) = \infty$. Since clearly $C_G^{m,p}(E) = \infty$ only if $E = \bar{G}$ we have for $p > 1$

$$K_G^{m,p}(E) = \lambda(G) \Leftrightarrow C_G^{m,p}(E) = \infty \Leftrightarrow E = \bar{G}.$$

Example. Let $G = B_r(0)$ the ball of radius r and centre the origin in E_n. Let $E = \{0\}$. It can be shown for $n \ge p > 1$ that

$$K_G^{1,p}(E) = C_G^{1,p}(E) = 0,$$

while for $p > n > 1$

$$C_G^{1,p}(E) \ge Cr^{-p},$$

where C is a positive constant depending on n and p. By Lemma 1 below we have as well in this case

$$K_G^{1,p}(G) \ge \lambda(G) C r^{-p} \{1 + r^{-1} C^{1/p}\}^{-p}.$$

The remainder of this section is devoted to the proof of the following theorem.

THEOREM 1. *If $\bar{G}$ is diffeomorphic to a compact, convex set then*

$$C_G^{m,p}(E) = 0 \Leftrightarrow K_G^{m,p}(E) = 0. \tag{6}$$

For the proof we require the following lemmas.

LEMMA 1. $C_G^{m,p}(E) \le K_G^{m,p}(E)[(\lambda(G))^{1/p} - (K_G^{m,p}(E))^{1/p}]^{-p}$.

Proof. It is sufficient to consider the case $E \ne \bar{G}$. Let $K = K_G^{m,p}(E) < \lambda(G)$ and let $\varepsilon > 0$ be small enough that $K+\varepsilon < \lambda(G)$. There exists $\phi \in C^\infty(\bar{G})$, $\phi(x) = 1$ near E such that $\|\phi\|_{m,p,G}^p \le K+\varepsilon$. Let $\psi(x) = 1-\phi(x) = 0$ near E. Since

$$(\lambda(G))^{1/p} = \|1\|_{0,p,G} \le \|\phi\|_{0,p,G} + \|\psi\|_{0,p,G}$$

we have

$$C_G^{m,p}(E) \le \frac{\sum_{1 \le |\alpha| \le m} \|D^\alpha \psi\|_{0,p,G}^p}{\|\psi\|_{0,p,G}^p}$$

$$\le \frac{K+\varepsilon}{[(\lambda(G))^{1/p} - (K+\varepsilon)^{1/p}]^p}$$

and the lemma follows since ε is arbitrary.

LEMMA 2. *Let G be a convex, bounded, open set in E_n and let $\eta > 0$. There exist constants $c_1(\eta)$ and c_2 (depending on n, p, G) such that for any real-valued $u \in C^1(\bar{G})$*

$$\lambda(A)\|u\|_{0,p,G}^p \le (1+\eta)\lambda(G)\|u\|_{0,p,A}^p + c_1(\eta)\|\text{grad } u\|_{0,p,G}^p \tag{7}$$

$$\lambda(B) \le c_1(\eta)\|\text{grad } u\|_{0,p,G}^p\{k^p\lambda(G) - (1+\eta)\|u\|_{0,p,G}^p\}^{-1} \tag{8}$$

$$\lambda(R)\{\lambda(S)\}^{1-1/n} \le c_2\|\text{grad } u\|_{0,p,G}^p \tag{9}$$

where A is any measurable subset of G,

$$B = \{x \in G : |u(x)| \ge k,\ k^p\lambda(G) > (1+\eta)\|u\|_{0,p,G}^p\}$$
$$R = \{x \in G : u(x) \ge \tfrac{1}{2}\},\ S = \{x \in G : u(x) \le -\tfrac{1}{2}\}.$$

Proof. For $x, y \in G$ we have

$$u(x) = u(y) + \int_0^{|x-y|} \nu \cdot \text{grad } u \, ds \tag{10}$$

where $\nu = (x-y)|x-y|^{-1}$ and s denotes distance along the line from y to x. Using well-known inequalities we obtain

$$|u(x)|^p \le (1+\eta)|u(y)|^p + c_3(\eta, p)|x-y|^{p-1}\int_0^{|x-y|} |\text{grad } u|^p \, ds. \tag{11}$$

Let (r, σ) denote spherical coordinates in E_n with pole at y so that the boundary of G is given by $r = f(\sigma) \le \text{diam. } G$, $\sigma \in \Sigma$ the unit sphere. Integrating (11) with respect to x over G we obtain

$$\|u\|_{0,p,G}^p \le (1+\eta)\lambda(G)|u(y)|^p + \frac{c_3\,(\text{diam. } G)^{n+p-1}}{n+p-1}\int_G \frac{|\text{grad } u|^p}{|x-y|^{n-1}}\,dx.$$

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Integration with respect to y over A now yields (7) since

$$\int_A \frac{dy}{|x-y|^{n-1}} = \text{const. } [\lambda(A)]^{1/n} \le \text{const. } [\lambda(G)]^{1/n}$$

Now take $x \in B$ in (11) and integrate with respect to y over G to obtain

$$k^p\lambda(G) \le (1+\eta)\|u\|^p_{0,p,G} + \frac{c_3\,(\text{diam } G)^{n+p-1}}{n+p-1} \int_G \frac{|\text{grad } u|^p}{|x-y|^{n-1}}\,dx.$$

Integration over B now yields (8) since $k^p\lambda(G) > (1+\eta)\|u\|^p_{0,p,G}$.

Finally, take $x \in R$, $y \in S$ in (10) so that, via Hölder's inequality

$$1 \le |u(x)-u(y)|^p \le |x-y|^{p-1} \int_0^{|x-y|} |\text{grad } u|^p\,ds.$$

Integrating x over R we obtain

$$\lambda(R) \le \frac{(\text{diam } G)^{n+p-1}}{n+p-1} \int_G \frac{|\text{grad } u|^p}{|x-y|^{n-1}}\,dx$$

and integrating y over S then produces (9).

Lemma 3. *If G is bounded, open and convex then*

$$C_G^{m,p}(E) = 0 \Rightarrow K_G^{m,p}(E) = 0.$$

Proof. Let $0 < \varepsilon < \frac{1}{2}$ and suppose $C_G^{m,p}(E) = 0$. Then for any $\delta > 0$ there exists $\phi \in C^\infty(\bar{G})$ such that $\phi(x) = 0$ near E, $\|\phi\|^p_{0,p,G} = \lambda(G)$ and $\sum_{1 \le |\alpha| \le m} \|D^\alpha\phi\|^p_{0,p,G} \le \delta\lambda(G)$. Without loss of generality we may assume ϕ is real-valued. Let

$$\begin{aligned}
A &= \{x \in G : |\phi(x)| \le 1-\varepsilon\},\\
B &= \{x \in G : |\phi(x)| \ge 1+\varepsilon\},\\
R &= \{x \in G : 1-\varepsilon < \phi(x) < 1+\varepsilon\},\\
S &= \{x \in G : -1-\varepsilon < \phi(x) < -1+\varepsilon\}.
\end{aligned}$$

Let $\eta > 0$ be small enough that $(1+\eta)(1-\varepsilon)^p < 1$ and $(1+\varepsilon)^p > 1+\eta$. We have, by Lemma 2,

$$\begin{aligned}
\lambda(R)+\lambda(S) &= \lambda(R \cup S) = \lambda(G) - \lambda(A \cup B) \ge \lambda(G) - c_4\delta\\
\lambda(R)[\lambda(S)]^{1-1/n} &\le c_2\delta.
\end{aligned}$$

Since $n > 1$ there exists δ_0 with $0 < \delta_0 < \varepsilon$ such that if $\delta < \delta_0$ then $\lambda(T) > \lambda(G) - \varepsilon$ and $\lambda(G-T) < \varepsilon$ where T is either R or S. Since $\|\phi\|^p_{0,p,G} = \lambda(G)$

$$\begin{aligned}
\|\phi\|^p_{0,p,G-T} &= \lambda(G) - \|\phi\|^p_{0,p,T}\\
&\le \lambda(G) - (1-\varepsilon)^p(\lambda(G)-\varepsilon).
\end{aligned}$$

If $T = R$ let $\psi = 1-\phi$; if $T = S$ let $\psi = 1+\phi$. In either case $\psi(x) = 1$ near E, $|\psi(x)| < \varepsilon$

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on T, $|\psi(x)|<1+|\phi(x)|$ on $G-T$, and $\sum_{1\le|\alpha|\le m}\|D^\alpha\psi\|_{0,p,G}^p<\delta\lambda(G)<\varepsilon\lambda(G)$. Thus $|\psi(x)|^p=c_5(1+|\phi(x)|^p)$ on $G-T$ and so

$$\begin{aligned} K_G^{m,p}(E) &\le \sum_{0\le|\alpha|\le m}\|D^\alpha\psi\|_{0,p,G}^p \\ &\le \varepsilon\lambda(G)+\|\psi\|_{0,p,T}^p+c_5\lambda(G-T)+c_5\|\phi\|_{0,p,G-T}^p \\ &< \varepsilon\lambda(G)+\varepsilon^p\lambda(T)+c_5\varepsilon+c_5[\lambda(G)-(1-\varepsilon)^p(\lambda(G)-\varepsilon)] \end{aligned}$$

which tends to zero with ε whence the lemma.

Proof of Theorem 1. Let Ω be an open, bounded, convex set in E_n and g a diffeomorphism of $\bar\Omega$ onto $\bar G$. Let $F=g^{-1}(E)$. Since $\bar\Omega$ is compact $|\det g'|$ and $|\det(g^{-1})'|$ are bounded on $\bar\Omega$ and $\bar G$ respectively. Since $C_G^{m,p}(E)=0$ for any $\varepsilon>0$ there exists $\phi\in C^\infty(\bar G)$ with $\phi(y)=0$ near E, $\|\phi\|_{0,p,G}=1$ and $\sum_{1\le|\alpha|\le m}\|D^\alpha\phi\|_{0,p,G}^p<\varepsilon$. Then $\psi=\phi\circ g\in C^\infty(\bar\Omega)$ satisfies $\psi(x)=0$ near F, $\|\psi\|_{0,p,\Omega}>\text{const.}>0$ and $\sum_{1\le|\alpha|\le m}\|D^\alpha\psi\|_{0,p,\Omega}^p<\text{const.}\ \varepsilon$ whence $C_\Omega^{m,p}(F)=0$. Since Ω is convex $K_\Omega^{m,p}(F)=0$ and by an argument similar to that given above $K_G^{m,p}(E)=0$. This completes the proof.

REMARKS. (1) Theorem 1 ought to be true for a larger class of domains than that for which we have proved it.

(2) We have in fact shown that for sets of small capacity both capacities are equivalent.

2. Equality of Sobolev spaces and applications to differential equations. Let G be an open, bounded set in E_n and E a closed subset of $\bar G$. We denote by $W^{m,p}(G,E)$ the completion of $C^\infty(\bar G)\cap\{u\colon u(x)=0 \text{ near } E\}$ with respect to the norm $\|\cdot\|_{m,p,G}$ so that $W_0^{m,p}(G)=W^{m,p}(G,\partial G)$.

THEOREM 2. (a) $W^{m,p}(G)=W^{m,p}(G,E)\Leftrightarrow K_G^{m,p}(E)=0$.

(b) $W_0^{m,p}(G)=W_0^{m,p}(G-E)\Leftrightarrow K_G^{m,p}(F)=0$ *for every closed* $F\subset E\cap G$.

Proof. We prove only (b); (a) is similar. Clearly $W_0^{m,p}(G-E)\subset W_0^{m,p}(G)$. Suppose $K_G^{m,p}(F)=0$ for every closed $F\subset E\cap G$. Let $\psi\in C_0^\infty(G)$ and let $\varepsilon>0$. Let $F=E\cap\operatorname{supp}\psi$. Then there exists $\omega\in C^\infty(\bar G)$ with $\omega(x)=1$ near F and $\|\omega\|_{m,p,G}<\varepsilon$. Let $\phi=\psi(1-\omega)\in C_0^\infty(G-E)$. Then for some constant c depending on m, n, and p we have

$$\|\psi-\phi\|_{m,p,G} = \|\omega\phi\|_{m,p,G} \le \sup_{|\alpha|\le m}\sup_{x\in G}|D^\alpha\psi(x)|\varepsilon.$$

Since ε is arbitrary, $\psi\in W_0^{m,p}(G-E)$ so $W_0^{m,p}(G)=W_0^{m,p}(G-E)$.

Conversely, suppose $W_0^{m,p}(G)\subset W_0^{m,p}(G-E)$. Let $\varepsilon>0$ and let $F\subset E\cap G$ be closed. Let $\psi\in C_0^\infty(G)$ satisfy $\psi(x)=1$ near F. There exists $\omega\in C_0^\infty(G-E)$ such that $\|\psi-\omega\|_{m,p,G}<\varepsilon$. Since $\psi-\omega=1$ near F it follows that $K_G^{m,p}(F)<\varepsilon$ whence $K_G^{m,p}(F)=0$.

REMARKS. (1) The condition $C_G^{1,p}(E)\neq 0$ (or $K_G^{1,p}(E)\neq 0$ if $\bar{G}$ is diffeomorphic to a compact, convex set) is a Poincaré inequality for functions $\phi\in C^\infty(\bar{G})$ which vanish near E; specifically, for such ϕ

$$\|\phi\|_{0,p,G} \le [C_G^{1,p}(E)]^{-1/p}\,\|\text{grad}\,\phi\|_{0,p,G}$$
$$\le \text{const.} \sum_{|\alpha|=m} \|D^\alpha\phi\|_{0,p,G}.$$

It follows that the Dirichlet form $l(u, v)$ of the polyharmonic operator $(-\Delta)^m$ is positive definite on $W^{m,2}(G, E)$ whenever $C_G^{1,2}(E)\neq 0$ and so the linear operator L in $W^{m,2}(G, E)$ defined by

$$l(u, v) = (Lu, v)_{W^{m,p}(G)}, \quad u, v\in W^{m,2}(G, E)$$

is a homeomorphism of $W^{m,2}(G, E)$ onto itself. This leads at once to the existence of weak solutions of mixed boundary-value problems for

$$(-\Delta)^m u = f \text{ in } G, \quad f\in L^2(G)$$

with Dirichlet boundary values on $E\subset\partial G$. The same is true for the operators $(-\Delta)^{m-j+1}+(-\Delta)^m$ and $\sum_{k=m-j+1}^{m}(-\Delta)^k$ provided $C_G^{j,2}(E)\neq 0$.

(2) If $K_G^{m,2}(E)=0$ the L^2 realizations corresponding to null Dirichlet boundary data of uniformly strongly elliptic operators of order $2m$ (such as $(-\Delta)^m$) over G and $G-E$ have the same domains, i.e.

$$W_0^{m,2}(G)\cap\{u\in L^2(G)\colon \Delta^m u\in L^2(G)\}$$
$$= W_0^{m,2}(G-E)\cap\{u\in L^2(G-E)\colon \Delta^m u\in L^2(G-E)\}.$$

Consequently the spectrum of such a realization is unchanged by the removal of E from $\bar{G}$, as are weak solutions of corresponding boundary-value problems. Even for very regular G, however, the regularity theorem $u\in W_0^{m,2}(G)\cap C^{m-1}(\bar{G})$ $\Rightarrow |D^\alpha u|=0$ on ∂G, $|\alpha|\le m-1$ cannot be extended to $G-E$.

3. Extensions and a geometric condition for nonvanishing capacity. It is often convenient to refer the capacity of a subset E of $\bar{G}$ to a larger set $H\supset G$ with simple geometric properties (e.g. a cube). We shall say that G has the (m, p)-extension property if there exists a continuous linear operator P from $W^{m,p}(G)$ to $W^{m,p}(E_n)$ with $Pu(x)=u(x)$ for $x\in\bar{G}$. This will be the case if, for example, G has the uniform cone property (Calderon [3], p. 45). If in addition $\bar{G}$ and $\bar{H}\supset\bar{G}$ are diffeomorphic to compact convex sets then

$$C_G^{m,p}(E) = 0 \Leftrightarrow K_G^{m,p}(E) = 0 \Leftrightarrow K_H^{m,p}(E) = 0 \Leftrightarrow C_H^{m,p}(E) = 0.$$

For such sets we are able to give a simple geometric condition on E which guarantees the nonvanishing of the capacity of E.

THEOREM 3. *Let G have the (m, p)-extension property and be diffeomorphic to a*

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convex, open, bounded set in E_n. Suppose there exists an integer k, $1 \le k \le n$ with $mp > k$ and an $(n-k)$-dimensional plane Q in E_n such that the projection of E onto Q has positive $(n-k)$-measure (Lebesgue). Then $C_G^{m,p}(E) > 0$ and $K_G^{m,p}(E) > 0$.

Proof. Let H be a cube containing G with one $(n-k)$-face parallel to Q. The projection of E onto this face has positive $(n-k)$-measure. It is known (see [1], Theorem 2) that in this circumstance $C_H^{m,p}(E) > 0$ whence the result.

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