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A Note on Uniformly Bounded Cocycles into Finite Von Neumann Algebras

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Abstract. We give a short proof of a result of T. Bates and T. Giordano stating that any uniformly bounded Borel cocycle into a finite von Neumann algebra is cohomologous to a unitary cocycle. We also point out a separability issue in their proof. Our approach is based on the existence of a non-positive curvature metric on the positive cone of a finite von Neumann algebra.

1 Statement of the Main Result

Let Γ be a discrete countable group acting on a standard probability space (S,μ) with μ being quasi-invariant and ergodic. Let $\mathcal{M}$ be a von Neumann algebra and denote by $GL(\mathcal{M})$ its invertible group, equipped with strong operator topology. A Borel map $\alpha\colon \Gamma\times S\to GL(\mathcal{M})$ is called a *cocycle* if for any $g,h\in\Gamma$, for almost all $s\in S$,

$$\alpha(gh,s)=\alpha(g,hs)\alpha(h,s).$$

A cocycle is said to be *uniformly bounded* if there exists $c>0$ such that for any $g\in\Gamma$, for almost all $s\in S$, $\|\alpha(g,s)\|\le c$. Two cocyles $\alpha,\beta\colon\Gamma\times S\to GL(\mathcal{M})$ are cohomologous if there exists a Borel map $\phi\colon S\to GL(\mathcal{M})$ such that for all $h\in\Gamma$ and almost all $s\in S$,

$$\beta(h,s)=\phi(hs)\alpha(h,s)\phi(s)^{-1}.$$

In this note, we give a new proof of the following result due to T. Bates and T. Giordano, this Theorem generalizes results of both F.-H. Vasilescu and L. Zsidó [6] and R. J. Zimmer [7].

Theorem 1.1 ([4, Theorem 3.3]) *Let Γ be a discrete countable group acting on (S,μ) standard Borel space with probability measure μ that is quasi-invariant and ergodic, and let $\mathcal{M}$ be a finite von Neumann algebra with separable predual. Let $\alpha\colon\Gamma\times S\to GL(\mathcal{M})$ be a uniformly bounded Borel cocycle. Then α is cohomologous to a cocycle valued in the unitary group of $\mathcal{M}$.*

Their approach is based on adapting the Ryll–Nardzewski fixed point theorem. However, it seems that there is a gap in the argument, and we were not able to determine to what extend this gap was fillable; see Remark 3.1. We take a different approach, based on a more geometric property of finite von Neumann algebras in the spirit of [5].

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2 Circumcenter and Non-positive Curvature

Let (X, d) be a metric space and let $B \subset X$ be a non-empty bounded subset of X. The circumradius of B is the real number

$$r(B) := \inf_{x \in X} \sup_{y \in B} d(x, y).$$

A point $x \in X$ is called a *circumcenter* of B if the closed ball centered at x and with radius $r = r(B)$ contains B. Note that, in general, a circumcenter does not always exist and is not necessarily unique.

A geodesic metric space (X, d) is called a CAT(0)-*space* if it satisfies the *semi-parallelogram law*: for any $x_1, x_2 \in X$, there exists $z \in X$ such that for all $x \in X$,

$$d(x_1, x_2)^2 + 4d(z, x)^2 \leq 2d(x_1, x)^2 + 2d(x_2, x)^2.$$

In a complete CAT(0)-space, every non-empty bounded subset always admits a unique circumcenter; see, for instance, [1, Theorem 11.26]. Of course it is not always the case that a subset contains its circumcenter, but its closed convex hull does [1, Theorem 11.27].

The important point for us is that the set of positive elements in a finite von Neumann algebra can be endowed with a metric satisfying the semi-parallelogram law; see [3]. Let $\mathcal{M}$ be a finite von Neumann algebra with finite trace τ. For $x \in \mathcal{M}$, its L_2-norm is denoted by $\|x\|_2 := \tau(x^*x)^{1/2}$. Denote by $GL(\mathcal{M})_+$ the set of positive invertible elements. For $a, b \in GL(\mathcal{M})_+$, set

$$d(a, b) := \| \ln(a^{-1/2} b a^{-1/2}) \|_2.$$

This defines a metric on $GL(\mathcal{M})_+$. Here are the main features of this metric; for more details, we refer the reader to [5] and the references therein.

(i) For any $g \in GL(\mathcal{M})$, $d(a, b) = d(g^* a g, g^* b g)$.

(ii) The metric d satisfies the semi-parallelogram law.

(iii) For all $c > 1$, the metric d is equivalent to $\| \cdot \|_2$ on the set

$$GL(\mathcal{M})_c := \{x \in GL(\mathcal{M})_+, c^{-1} \leq x \leq c\}.$$

In fact, for all $c > 1$, the space $(GL(\mathcal{M})_c, d)$ is a (geodesic) CAT(0)-space, which is bounded, complete, and separable (this is not the case of $GL(\mathcal{M})_+$).

Consequently, for all $c > 1$, every non-empty subset $B \subset GL(\mathcal{M})_c$ admits a unique circumcenter x, which lies in $GL(\mathcal{M})_c$.

3 Proof of the Theorem

For each $s \in S$, denote $B_s = \{\alpha(g, s)^* \alpha(g, s),\ g \in \Gamma\}$. Since α is a uniformly bounded cocycle, there exists $c > 1$ and a conull Borel set $S_0 \subset S$ such that for all $s \in S_0$ and all $h \in \Gamma$,

$$B_s \subset GL(\mathcal{M})_c \qquad \text{and} \qquad \alpha(h, s)^* B_{hs} \alpha(h, s) = B_s.$$

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For every $s \in S_0$, denote by $\gamma(s) \in GL(\mathcal{M})_c$ the unique circumcenter of B_s. By [1, Theorem 11.27] and [1, Lemma 11.28], $\gamma(s)$ is also characterized as the unique circumcenter of B_s inside $GL(\mathcal{M})$. Combining this uniqueness with property (i) above, we get

$$\alpha(h,s)^*\gamma(hs)\alpha(h,s) = \gamma(s), \text{ for all } s \in S_0.$$

We claim that the map $s \in S \mapsto \gamma(s)^{1/2} \in \mathcal{M}$ (with γ arbitrarily defined on $S \setminus S_0$) almost surely coincides with a Borel map φ. After we prove this claim, we will get that the Borel map

$$\beta\colon (h,s) \in \Gamma \times S \longmapsto \varphi(hs)\alpha(h,s)\varphi(s)^{-1} \in GL(\mathcal{M})$$

is a unitary cocycle cohomologous to α, giving the theorem.

To prove the claim we follow the argument in [2, Lemma 3.18]. As the cocycle α is a Borel map and Γ is countable, for all $v \in GL(\mathcal{M})_c$, the map

$$s \in S \longmapsto r(v, B_s) := \sup_{g \in \Gamma} d\big(v, \alpha(g,s)^*\alpha(g,s)\big)$$

is Borel. By continuity of the maps $v \mapsto r(v, B_s)$, $s \in S$, and separability (iii) of $GL(\mathcal{M})_c$, we deduce that the map

$$s \in S \longmapsto \inf_{v \in GL(\mathcal{M})_c} r(v, B_s) = r(B_s)$$

coincides with an infimum over a countable subset of $GL(\mathcal{M})_c$, and hence is Borel. For $n \geq 1$, the following set D_n is a Borel bundle over S:

$$D_n = \big\{ (s,v) \in S \times GL(\mathcal{M})_c : r(v,B_s)^2 \leq r(B_s)^2 + n^{-1} \big\}.$$

By [8, Theorem A.9], there exist Borel maps $\xi_n\colon S \to GL(\mathcal{M})_c$ such that $(s, \xi_n(s)) \in D_n$, for all s in some conull subset $S_1 \subset S$. For all $s \in S_0 \cap S_1$, the semi-parallelogram law implies that the sequence $(\xi_n(s))_n$ converges to $\gamma(s)$. More precisely, for all fixed n, with $x_1 = \gamma(s)$ and $x_2 = \xi_n(s)$, there exists $z \in GL(\mathcal{M})_c$ such that for all $x \in B_s$,

$$d(x_1,x_2)^2 + 4d(z,x)^2 \leq 2d(x_1,x)^2 + 2d(x_2,x)^2.$$

Taking the supremum over $x \in B_s$, we get

$$d\big(\gamma(s), \xi_n(s)\big)^2 + 4r(z,B_s)^2 \leq 2r(\gamma(s),B_s)^2 + 2r\big(\xi_n(s), B_s\big)^2 \leq 4r(B_s)^2 + 2n^{-1}.$$

Since $r(z, B_s) \geq r(B_s)$, this readily gives the desired convergence. Therefore, γ almost surely coincides with the Borel map $\lim_n \xi_n$, which completes the proof. ∎

Remark 3.1 We point out a gap in the proof of the main result by T. Bates and T. Giordano [4, Theorem 3.3]. With the notations of that proof, at the bottom of p. 747, it is not clear why a countable cover of X should exist. For instance, in the case where $\mathcal{M}$ is the trivial algebra $\mathcal{M} = \mathbb{C}$, $\widetilde{\mathcal{M}}_C$ and B_ε are simply balls (in the $\|\cdot\|_\infty$-norm) inside $\widetilde{\mathcal{M}} = L^\infty(S)$, of radii C and ε, respectively. Of course, there exists an ultraweakly dense sequence $(\phi_n)_n$ of X, but B_ε has empty interior (for the ultraweak topology) inside $L^\infty(S)$, so $(\phi_n + B_\varepsilon)_n$ has a priori no reason to cover X.

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