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ON JOINT SPECTRA OF NON-COMMUTING HYPONORMAL OPERATORS

A. SOŁTYSIAK

We show that the left joint spectrum of an arbitrary n-tuple of hyponormal Hilbert space operators can be obtained from the spectral set γ introduced by McIntosh and Pryde. A dual statement for cohyponormal operators is also true. The result is a generalisation of a theorem proved by Pryde and the author for normal operators.

Let H be a complex Hilbert space and let $\mathcal{B}(H)$ denote the Banach algebra of all (bounded linear) operators on H. For an n-tuple $T = (T_1, \dots, T_n)$ of operators on H a *spectral set* $\gamma(T)$ is defined as follows:

$$\gamma(T) = \left\{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{j=1}^{n} (T_j - \lambda_j)^2 \text{ is not invertible in } \mathcal{B}(H) \right\}.$$

(Here we write as usual $T_j - \lambda_j$ instead of $T_j - \lambda_j \operatorname{id}_H$.) This set was introduced by McIntosh and Pryde [**4, 5**] and has proved useful not only in the spectral theory of self-adjoint operators but also in comparing various types of joint spectra of commuting families of operators (see [**6**]). One advantage of the set $\gamma(T)$ over other joint spectra is that it can be easily computed. In [**7**] it was shown that this set is also useful in the multiparameter spectral theory of normal operators.

In this paper we generalise one of the results proved in [**7**] to n-tuples of (not necessarily commuting) hyponormal operators.

We recall some necessary definitions. An operator $T \in \mathcal{B}(H)$ is *hyponormal* (*co-hyponormal*) if $\|T^*x\| \le \|Tx\|$ ($\|Tx\| \le \|T^*x\|$ respectively) for all $x \in H$ (see [**2**]). Clearly if an operator T is hyponormal, then T^* is cohyponormal. Moreover an operator T is normal if it is both hypo- and cohyponormal.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of operators. A point $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is not in the *left* (*joint*) *spectrum* of T if there exist operators $U_1, \dots, U_n \in \mathcal{B}(H)$ such that $\sum_{j=1}^{n} U_j(T_j - \lambda_j) = \operatorname{id}_H$. The left spectrum of T will be denoted by $\sigma_l(T)$. The

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right spectrum, $\sigma_r(T)$, is defined analogously. The *Harte spectrum* of T $\big($in $\mathcal{B}(H)\big)$, denoted by $\sigma_H(T)$, is the union of the left and right joint spectra, that is

$$\sigma_H(T) = \sigma_l(T) \cup \sigma_r(T).$$

All these spectra are compact (possibly empty) subsets of $\mathbb{C}^n$ (see [**3**]). Notice that for a single operator T the Harte spectrum $\sigma_H(T)$ coincides with the usual spectrum $\sigma(T)$.

It is well-known (see [**3**, Theorems 2.5 and 2.4]) that

$$\sigma_l(T) = \left\{\lambda \in \mathbb{C}^n : \inf_{\|x\|=1} \sum_{j=1}^{n} \big\|(T_j - \lambda_j)x\big\| = 0\right\}$$

(the *approximate point spectrum*) and

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C}^n : \sum_{j=1}^{n} \big((T_j - \lambda_j)(H)\big) \neq H\right\}$$

(the *defect spectrum*).

In this paper we use "non-commutative polynomials" (see [**3**, pp.98–99]). By $\mathcal{P}^{(n)}$ we denote the algebra of all polynomials over $\mathbb{C}$ in non-commutative indeterminates $X_1, \dots, X_n$. In other words, $\mathcal{P}^{(n)}$ is the free associative complex unital algebra generated by the symbols $X_1, \dots, X_n$. An n-tuple of operators $(T_1, \dots, T_n) \in \mathcal{B}(H)^n$ induces a homomorphism $f \mapsto f(T_1, \dots, T_n)$ from $\mathcal{P}^{(n)}$ to $\mathcal{B}(H)$ which preserves the identity and sends each X_j to the corresponding T_j $(j = 1, \dots, n)$. A system $(f_1, \dots, f_m) \in \big(\mathcal{P}^{(n)}\big)^m$ will be identified with a polynomial map $f : \mathcal{B}(H)^n \to \mathcal{B}(H)^m$ which sends $(T_1, \dots, T_n)$ to $\big(f_1(T_1, \dots, T_n), \dots, f_m(T_1, \dots, T_n)\big)$. The restriction of this mapping to the scalar multiples of the unit $\mathbb{C}^n \subset \mathcal{B}(H)^n$ takes its values in $\mathbb{C}^m \subset \mathcal{B}(H)^m$ and reduces to the system of "numerical" polynomials.

It is well-known that the left, right, and Harte spectrum satisfy the one-way spectral mapping theorem (see [**3**, Theorem 3.2]), that is

$$f\big(\sigma_*(T)\big) \subset \sigma_*\big(f(T)\big), \tag{1}$$

where σ_* denotes one of the above-mentioned spectra, T is an arbitrary n-tuple of operators, and f is any polynomial map.

Let us introduce the following notation. For a single operator T symbols $\operatorname{Re} T$ and $\operatorname{Im} T$ will denote as usual its real and imaginary part. Hence $T = \operatorname{Re} T + i \operatorname{Im} T$. If $T = (T_1, \dots, T_n)$ is an n-tuple of operators, then $T^* = (T_1^*, \dots, T_n^*)$, $\operatorname{Re} T = (\operatorname{Re} T_1, \dots, \operatorname{Re} T_n)$, $\operatorname{Im} T = (\operatorname{Im} T_1, \dots, \operatorname{Im} T_n)$, and $\Pi(T) = (\operatorname{Re} T, \operatorname{Im} T)$.

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Moreover for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ we write $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_n)$ ($\overline{\lambda}_j$ is the complex conjugate of λ_j), $\operatorname{Re}\lambda = (\operatorname{Re}\lambda_1, \dots, \operatorname{Re}\lambda_n)$ and $\operatorname{Im}\lambda = (\operatorname{Im}\lambda_1, \dots, \operatorname{Im}\lambda_n)$. The letter p will denote the polynomial map $p(z_1, \dots, z_{2n}) = (z_1 + iz_{n+1}, \dots, z_n + iz_{2n})$. Notice that $\lambda = p(\operatorname{Re}\lambda, \operatorname{Im}\lambda)$.

For convenience of the reader we shall state the following result (see [**1**, Lemma 2.4]) which is essential for the paper.

Dash's lemma. *Let $T = (T_1, \dots, T_n)$ be an arbitrary n-tuple of operators. Then*

(a) $\lambda \in \sigma_l(T)$ *if and only if* $0 \in \sigma\Big(\sum_{j=1}^{n} (T_j - \lambda_j)^*(T_j - \lambda_j)\Big)$;

(b) $\lambda \in \sigma_r(T)$ *if and only if* $0 \in \sigma\Big(\sum_{j=1}^{n} (T_j - \lambda_j)(T_j - \lambda_j)^*\Big)$.

We shall also need the following (see [**7**, Proposition 1]):

Proposition 1. *If $T = (T_1, \dots, T_n)$ is an arbitrary n-tuple of self-adjoint operators, then*

$$\sigma_l(T) = \sigma_r(T) = \sigma_H(T) = \gamma(T).$$

Now we make the following observation. For $T = (T_1, \dots, T_n) \in \mathcal{B}(H)^n$ and $\lambda \in \mathbb{C}^n$ let us denote

$$S = \sum_{j=1}^{n} (T_j - \lambda_j)^*(T_j - \lambda_j) + \sum_{j=1}^{n} (T_j - \lambda_j)(T_j - \lambda_j)^*.$$

Then the operator S is self-adjoint and by Dash's lemma we see that $(\lambda, \overline{\lambda}) \in \sigma_l(T, T^*)$ if and only if $0 \in \sigma(S)$ and this is true if and only if $(\lambda, \overline{\lambda}) \in \sigma_r(T, T^*)$. Therefore for an arbitrary n-tuple $T = (T_1, \dots, T_n)$ we have

$$\sigma_l(T, T^*) = \sigma_r(T, T^*) = \sigma_H(T, T^*).$$

Lemma. *If $T = (T_1, \dots, T_n)$ is an arbitrary n-tuple of operators, then*

$$\sigma_H(T, T^*) = \{(\lambda, \overline{\lambda}) \in \mathbb{C}^{2n} : \lambda \in \sigma_l(T) \cap \sigma_r(T)\}.$$

Proof: By [**3**, Theorem 3.4 (i)] we have

$$\sigma_H(T, T^*) \subset \{(\lambda, \overline{\lambda}) \in \mathbb{C}^{2n} : \lambda \in \sigma_H(T)\}.$$

Suppose $\lambda \notin \sigma_l(T) \cap \sigma_r(T)$. If $\lambda \notin \sigma_l(T)$ there exist operators U_j, $j = 1, \dots, n$, such that $\sum_{j=1}^{n} U_j(T_j - \lambda_j) = \mathrm{id}_H$. This gives $(\lambda, \overline{\lambda}) \notin \sigma_l(T, T^*) = \sigma_H(T, T^*)$. A similar argument shows that $\lambda \notin \sigma_r(T)$ implies $(\lambda, \overline{\lambda}) \notin \sigma_H(T, T^*)$.

To show the other inclusion suppose $(\lambda, \overline{\lambda}) \notin \sigma_H(T, T^*)$. This means $0 \notin \sigma(S)$. Then there exists $\delta > 0$ such that $\|Sx\| \geq \delta\|x\|$ for every $x \in H$. This implies

$$\left\|\sum_{j=1}^{n}(T_j - \lambda_j)^*(T_j - \lambda_j)x\right\| + \left\|\sum_{j=1}^{n}(T_j - \lambda_j)(T_j - \lambda_j)^*x\right\| \geq \delta\|x\|.$$

Therefore either

$$\left\|\sum_{j=1}^{n}(T_j - \lambda_j)^*(T_j - \lambda_j)x\right\| \geq \frac{1}{2}\delta\|x\| \tag{2}$$

or

$$\left\|\sum_{j=1}^{n}(T_j - \lambda_j)(T_j - \lambda_j)^*x\right\| \geq \frac{1}{2}\delta\|x\|. \tag{3}$$

If (2) occurs, then $0 \notin \sigma\Big(\sum\limits_{j=1}^{n}(T_j - \lambda_j)^*(T_j - \lambda_j)\Big)$ and by Dash's lemma $\lambda \notin \sigma_l(T)$.

Suppose (3) holds true. Then $0 \notin \sigma\Big(\sum\limits_{j=1}^{n}(T_j - \lambda_j)(T_j - \lambda_j)^*\Big)$ and by Dash's lemma $\lambda \notin \sigma_r(T)$. Therefore in both cases $\lambda \notin \sigma_l(T) \cap \sigma_r(T)$ which completes the proof. ∎

Corollary 1. *$\sigma_l(T) \cap \sigma_r(T) = \emptyset$ if and only if $\sigma_H(T, T^*) = \emptyset$.*

Proposition 2. *If $T = (T_1, \dots, T_n)$ is an arbitrary n-tuple of hyponormal (cohyponormal) operators, then $\sigma_l(T) \subset \sigma_r(T)$ ($\sigma_r(T) \subset \sigma_l(T)$ respectively).*

Proof: Let $T = (T_1, \dots, T_n)$ be an n-tuple of hyponormal operators. Suppose that $\lambda \notin \sigma_r(T)$. By Dash's lemma we get $0 \notin \sigma\Big(\sum\limits_{j=1}^{n}(T_j - \lambda_j)(T_j - \lambda_j)^*\Big)$. Therefore there exists $\delta > 0$ such that

$$\left\|\sum_{j=1}^{n}(T_j - \lambda_j)(T_j - \lambda_j)^*x\right\| \geq \delta\|x\| \quad \text{for } x \in H.$$

Let $M = \max\{\|T_j - \lambda_j\| : j = 1, \dots, n\}$. Then we have

$$\begin{aligned}\delta\|x\| &\leq \left\|\sum_{j=1}^{n}(T_j - \lambda_j)(T_j - \lambda_j)^*x\right\| \leq \sum_{j=1}^{n}\|(T_j - \lambda_j)\|\,\|(T_j - \lambda_j)^*x\| \\ &\leq M\sum_{j=1}^{n}\|(T_j - \lambda_j)^*x\| \leq M\sum_{j=1}^{n}\|(T_j - \lambda_j)x\|,\end{aligned}$$

which gives $\lambda \notin \sigma_l(T)$.

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The proof for cohyponormal operators is similar. ∎

By Proposition 2, the Lemma, and the one-way spectral mapping property of the left and right spectra (see also [**3**, Theorem 3.4 (ii)]) we get

COROLLARY 2. *If $T = (T_1, \dots, T_n)$ is an n-tuple of hyponormal (cohyponormal) operators, then for every polynomial map $f \in \left(\mathcal{P}^{(2n)}\right)^m$*

$$\{f(\lambda, \overline{\lambda}) : \lambda \in \sigma_l(T)\} \subset \sigma_l(f(T, T^*)) \tag{4}$$

(and respectively

$$\{f(\lambda, \overline{\lambda}) : \lambda \in \sigma_r(T)\} \subset \sigma_r(f(T, T^*))\Big). \tag{5}$$

REMARKS. 1. It is well-known that inclusions in (1) cannot be replaced by equalities (see [**3**, p.101]). To see that they can be proper even for self-adjoint operators take the following 2 by 2 matrices:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\sigma(A_1) = \{0, 1\}$, $\sigma(A_2) = \{-1, 1\}$, $\sigma(A_1 A_2) = \{0\}$, and $\sigma_H(A_1, A_2) = \emptyset$. If f is the polynomial $f(X_1, X_2) = X_1 X_2$, then $f(\sigma_H(A_1, A_2)) = \emptyset$ and $\sigma(f(A_1, A_2)) = \{0\}$.

2. If we take the following 2 by 2 matrices:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

then $A_2^* = A_1$, and $\sigma_l(A_1, A_2) = \sigma_r(A_1, A_2) = \emptyset$ (see [**3**, Example 1.6]). Let f be the polynomial map $f(X_1, X_2) = (X_1, X_2)$. Then the right-hand sides of both (4) and (5) are empty while the left-hand sides are equal to the set $\{(0, 0)\}$. This shows that Corollary 2 is not true when operators are neither hyponormal nor cohyponormal.

3. The simplest example of a hyponormal operator (which is not normal) is the unilateral shift on the sequence space ℓ_2, $U(\xi_0, \xi_1, \xi_2, \dots) = (0, \xi_0, \xi_1, \xi_2, \dots)$. It is well-known that $\sigma_l(U) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, $\sigma_r(U) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$, and $\sigma_H(U, U^*) = \{(\lambda, \overline{\lambda}) : |\lambda| = 1\}$. Taking the same map f as before we see that (5) is not true for hyponormal operators (and (4) is not true for cohyponormal ones).

THEOREM. *If $T = (T_1, \dots, T_n)$ is an arbitrary n-tuple of hyponormal (cohyponormal) operators, then*

$$\sigma_l(T) = p\Big(\gamma(\Pi(T))\Big)$$

(and respectively

$$\sigma_r(T) = p\Big(\gamma(\Pi(T))\Big)\Big).$$

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PROOF: We give the proof for hyponormal operators. The argument for cohyponormal operators is analogous.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of hyponormal operators. By the one-way spectral mapping property of the left spectrum and Proposition 1 we get

$$p\Big(\gamma\big(\Pi(T)\big)\Big) = p\Big(\sigma_l\big(\Pi(T)\big)\Big) \subset \sigma_l\Big(p\big(\Pi(T)\big)\Big) = \sigma_l(T).$$

On the other hand, if $\lambda \in \sigma_l(T)$, then by Corollary 2 we obtain $(\operatorname{Re}\lambda, \operatorname{Im}\lambda) \in \sigma_l\big(\Pi(T)\big)$. Therefore

$$\lambda = p(\operatorname{Re}\lambda, \operatorname{Im}\lambda) \in p\Big(\sigma_l\big(\Pi(T)\big)\Big) = p\Big(\gamma\big(\Pi(T)\big)\Big)$$

which was to be proved. $\square$

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Faculty of Mathematics and Computer Science
Adam Mickiewicz University
ul. Matejki 48/49
60–769 Poznań
Poland
e-mail: asoltys@amu.edu.pl