Compositio Mathematica **106:** 203–210, 1997.

Smooth curves linked to thick curves

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Received 27 June 1996; accepted in final form 13 February 1996

Abstract. In this paper we construct smooth irreducible space curves C which link geometrically by surfaces of minimal degree containing C to curves Γ of generic embedding dimension three. This produces interesting behavior with respect to both C and Γ. The curves Γ link to smooth connected curves by surfaces of low degree but cannot link to smooth connected curves by surfaces of high degree. The curves C give counterexamples to a conjecture of Martin-Deschamps and Perrin.

Mathematics Subject Classifications (1991): 14H50, 14M06.

Key words: Linkage, smooth connected curves, superficial curves.

1. Introduction

In their paper on smoothing curves and linkage, Martin-Deschamps and Perrin propose the following ([8], Conjecture 2.4):

CONJECTURE 1.1. Let C be a smooth connected curve in $\mathbf{P}^3$, F a surface of minimal degree containing C, G a surface of minimal degree containing C and which is not a multiple of F. Then the curve Γ linked to C by $F \cap G$ is superficial.

A curve is said to be *superficial* if it has generic embedding dimension $\leqslant 2$. In this short note we construct counterexamples to this conjecture by producing smooth connected curves C which link via surfaces of minimal degree to a multiplicity structure Γ on a smooth curve which has generic embedding dimension three.

Most of the examples produced here (see Corollary 2.3) have the property that the complete intersection $F \cap G$ of minimal degree containing C is unique. In Proposition 3.2 this is not the case, but the general minimal degree complete intersection containing C *does* link C to a superficial curve. This leaves the following question unanswered: if C is a smooth connected curve which lies on a moving family of least degree complete intersections, then must C link to a superficial curve via the general such complete intersection?

Throughout this paper we work with projective schemes over an algebraically closed field k of arbitrary characteristic. All curves will be assumed to be locally Cohen-Macaulay. In Section 2, the main linking theorem is presented, along with the calculation of invariants of the associated curves. Section 3 is devoted to examples produced by the method of Section 2. Special attention is given to the smallest example, which is given by a rational quintic curve not lying on a quadric surface.

2. A linking theorem

THEOREM 2.1. *Let $Y \subset \mathbf{P}^3$ be a smooth curve, $d > 1$ an integer and $Y^{(d)}$ be the subscheme defined by ${\mathcal{I}_Y}^d$. Let n be an integer such that $\mathcal{I}_{Y^{(d)}}(n)$ is generated by global sections. If $s, t > n$, then for a general pair of surfaces S, T of degrees s, t containing $Y^{(d)}$ we have*

$$S \cap T = \Gamma \cup C,$$

where $Y^{(d)} \subset \Gamma$, $\mathrm{Supp}(\Gamma) = Y$, C is a smooth irreducible curve, and C meets Γ properly. In other words, $S \cap T$ geometrically links a smooth irreducible curve C to a multiplicity structure on Y which contains $Y^{(d)}$.

Proof. Let $W = Y^{(d)}$ and consider the blow-up $\widetilde{\mathbf{P}}^3 \xrightarrow{\pi} \mathbf{P}^3$ at W. Since $\mathcal{I}_W = \mathcal{I}_Y^d$, the blow-up at W is isomorphic to the blow-up at Y ([5], II, Exercise 7.11(a)), and it's not hard to see that $E_W = dE_Y$ via this isomorphism, where E_W, E_Y denote the respective exceptional divisors (In fact, raising a local equation of E_Y to the dth power gives a local equation for E_W). Since $Y \subset \mathbf{P}^3$ is a smooth curve, $\widetilde{\mathbf{P}}^3$ is a smooth irreducible threefold and E_Y is a ruled surface over Y. We have a diagram

$$\begin{array}{ccccc} E_Y & \subset & E_W & \subset & \widetilde{\mathbf{P}}^3 \\ \downarrow & & \downarrow & & \downarrow \pi \\ Y & \subset & W & \subset & \mathbf{P}^3 \end{array}.$$

Following the proof of [12], Proposition 4.1, we see that the invertible sheaf $\mathcal{I}_{E_W} \otimes \pi^*(\mathcal{O}(s))$ is very ample on $\widetilde{\mathbf{P}}^3$ and gives a closed immersion $\sigma_s : \widetilde{\mathbf{P}}^3 \hookrightarrow \mathbf{P}I_s$, where $I_s = H^0(\mathcal{I}_W(s))$. For $f \in I_s$, let H_f denote the associated hyperplane in $\mathbf{P}I_s$ and $Z(f)$ denote the surface in $\mathbf{P}^3$ with equation f. In this case we have that $\sigma_s^{-1}(H_f)$ and $\pi^{-1}(Z(f))$ are equal away from the exceptional divisor E_W. Similarly we have a closed immersion $\sigma_t : \widetilde{\mathbf{P}}^3 \hookrightarrow \mathbf{P}I_t$ such that $\sigma_t^*(\mathcal{O}(1)) = \mathcal{I}_{E_W} \otimes \pi^*(\mathcal{O}(t))$. For $g \in I_t$, we have that $\sigma_t^{-1}(H_g) = \pi^{-1}(Z(g))$ away from E_W.

Applying the standard Bertini theorem to the closed immersion σ_s, we can find $f \in I_s$ such that $\sigma_s^{-1}(H_f)$ is a smooth surface S in $\widetilde{\mathbf{P}}^3$ and $\sigma_s^{-1}(H_f) \cap E_Y$ is a smooth connected curve D containing no fibre π_y. Now consider the general hyperplane section of σ_t. Let

$$W = \{(H_g, y) \in (\mathbf{P}I_t)^* \times Y : \mathrm{length}(D \cap \sigma_t^{-1}(H_g) \cap \pi_y) \geqslant 2\}.$$

W is a closed subscheme of $(\mathbf{P}I_t)^* \times Y$, and composition with the second projection gives a dominant morphism $W \xrightarrow{q} Y$ (the fibres over Y are nonempty because

$d > 1$). Noting that $(H_g, y) \in W$ if and only if H_g contains a secant line to $D \cap \pi_y$, we see that each fibre $q^{-1}(y) \subset (\mathbf{P}I_t)^*$ is of codimension $\geqslant 2$, because $D \cap \pi_y$ has at most finitely many secant lines in $\mathbf{P}I_t$. It follows that the image of W in $(\mathbf{P}I_t)^*$ is of codimension $\geqslant 1$. In particular, if $g \in I_t$ is sufficiently general, then $\sigma_t^{-1}(H_g) \cap S$ is a smooth connected curve $\widetilde{C}$ which meets the fibres of π in schemes of length at most 1. In this case the composite map $\widetilde{C} \subset \widetilde{\mathbf{P}}^3 \to \mathbf{P}^3$ is a closed immersion (it is projective and has fibres of finite length $\leqslant 1$). Letting $C = \pi(\widetilde{C})$ denote the image curve, we see that $Z(f) \cap Z(g)$ consists of C and a curve Γ containing W and supported on Y.

The theorem above gives smooth connected curves which geometrically link to curves Γ of generic embedding dimension three. On $\mathbf{P}^3$, the curves Γ and C are produced somewhat indirectly. Fortunately the construction of $\widetilde{C}$ on $\widetilde{\mathbf{P}}^3$ is fairly straightforward, so we can compute the invariants of these curves. This is the content of the following proposition.

PROPOSITION 2.2. *Let Y, C, Γ be as in Theorem 2.1, and assume further that Y is connected. Then the degrees and genera of C and Γ are as follows.*

$$\begin{aligned}
&\deg C = st - d^2 \deg Y\\
&2\operatorname{genus} C - 2 =\\
&\quad st(s+t-4) - 2d^2(2d-1)\\
&\qquad -[(3d^2-d)(s+t) - 8d^3]\deg Y + 2d^2(2d-1)\operatorname{genus} Y\\
&\deg \Gamma = d^2 \deg Y\\
&\operatorname{genus} \Gamma = \operatorname{genus} C + [2d^2 \deg Y - st](\tfrac{s+t}{2} - 2)
\end{aligned}$$

Proof. Let h denote the hyperplane class $c_1(\pi^* \mathcal{O}_{\mathbf{P}^3}(1))$ in the Chow ring $A(\widetilde{\mathbf{P}}^3)$ and let y denote the class of the exceptional divisor E_Y. We first compute the triple intersection numbers for these classes. The intersection $h^3 = 1$ can be computed on $\mathbf{P}^3$ by [2], Example 8.3.9. $h^2 y = 0$ because a general line does not meet Y. For $y^2 h$ and y^3, we compute the intersection numbers yh and y^2 on the ruled surface E_Y.

By [5], Theorem 8.2.4, the exceptional divisor E_Y is isomorphic to $\mathbf{P}_Y(\mathcal{N}^\vee)$, where $\mathcal{N} = \mathcal{N}_{Y, \mathbf{P}^3}$. If $j : E_Y \hookrightarrow \widetilde{\mathbf{P}}^3$ is the natural inclusion and ξ represents the relative $\mathcal{O}_{\mathbf{P}}(1)$ from the **P**roj construction of $\mathbf{P}_Y(\mathcal{N}^\vee)$, then $j^*(y) = -\xi$ under this correspondence. By [2], Example 8.3.4, we have that $A(\mathbf{P}_Y(\mathcal{N}^\vee)) \cong A(Y)[\xi]/(\xi^2 + c_1(\mathcal{N})\xi + c_2(\mathcal{N}))$ as graded rings. Since $c_2(\mathcal{N}) = 0$, the identification of graded pieces of degree two gives

$$A^2(\mathbf{P}_Y(\mathcal{N}^\vee)) \cong (\xi \operatorname{Pic} Y \oplus \xi^2 \mathbf{Z})/(\xi^2 + c_1(\mathcal{N})\xi).$$

https://doi.org/10.1023/A:1000161724052 Published online by Cambridge University Press

In other words, $A^2(\mathbf{P}_Y(\mathcal{N}^\vee)) \cong \xi \operatorname{Pic} Y$ where $\xi^2 = -\xi c_1(\mathcal{N})$. Thus $yh = -\xi h$ corresponds to $-h$ in $\operatorname{Pic} Y$, which has degree $-\deg Y$ while $y^2 = \xi^2 = -c_1(\mathcal{N})\xi$ corresponds to $-c_1(\mathcal{N})$ in $\operatorname{Pic} Y$, which has degree $-\deg_Y \mathcal{N}$. On E_Y, it follows that we have the intersection numbers $yh = -\deg Y$ and $y^2 = -\deg_Y \mathcal{N}$.

From the previous paragraph, we find the triple intersection numbers $y^2 h = -\deg Y$ and $y^3 = \deg_Y \mathcal{N}^\vee$. We can use the standard exact sequences (see [5], II, Theorems 8.13 and 8.17)

$$0 \to \mathcal{N}_Y^\vee \to \Omega_{\mathbf{P}^3}|_Y \to \omega_Y \to 0$$

$$0 \to \Omega_{\mathbf{P}^3}|_Y \to \mathcal{O}_Y(-1)^4 \to \mathcal{O}_Y \to 0$$

to show that $y^3 = -4 \deg Y - 2 \operatorname{genus} Y + 2$.

To compute the invariants of C and Γ, recall that C was constructed as the isomorphic image of $\widetilde{C} \subset \widetilde{\mathbf{P}}^3$ under π, where $\widetilde{C}$ is represented by $(sh-dy).(th-dy)$ in $A(\widetilde{\mathbf{P}}^3)$ ($\widetilde{C}$ was the intersection of two surfaces represented by $sh-dy$ and $th-dy$). The degree of C is given by the intersection number $(sh-dy).(th-dy).h$. The canonical sheaf on $\widetilde{\mathbf{P}}^3$ is given by $-4h+y$ ([5], II, Example 8.5b). Applying [5], II, Proposition 8.20, we find that the canonical sheaf $\omega_{\widetilde{C}}$ is given by restriction of the class $sh - dy + th - dy - 4h + y$ to $\widetilde{C}$. In particular, we have

$$2 \operatorname{genus} \widetilde{C} - 2 = (sh-dy).(th-dy).((s+t-4)h - (2d-1)y).$$

Substituting the intersection numbers of the previous paragraph gives the invariants for C. The invariants for Γ are computed from those of C using the linkage between them ([6], Remark 4.7.1).

COROLLARY 2.3. *Let $Y \subset \mathbf{P}^3$ be a smooth connected curve, $d > 1$ an integer, and assume that $\mathcal{I}_{Y^{(d)}}(n)$ is generated by global sections. If* $\max\{n, d^2 \deg Y\} <$ *$s \leqslant t$, then the general surfaces S, T of degrees s, t containing $Y^{(d)}$ link a smooth connected curve C to a multiplicity structure Γ on Y and $S \cap T$ is the unique complete intersection of least degree which contains C.*

Proof. Theorem 2.1 gives the linkage $\Gamma \sim C$ via the complete intersection $S \cap T$. By Proposition 2.2, we have that $\deg \Gamma = d^2 \deg Y$. Since $d > 1$, Γ has generic embedding dimension three, and hence is not planar. It now follows from a result of Martin-Deschamps and Perrin ([9], Corollary 2.4) that $e(\Gamma) \leqslant d^2 \deg Y - 4$, where $e(\Gamma)$ denotes the exceptionality of Γ. Now consider the exact sequence

$$0 \to \mathcal{I}_{S \cap T} \to \mathcal{I}_C \to \omega_\Gamma(4-s-t) \to 0.$$

The inequality $s > d^2 \deg Y$ shows that $h^0(\omega_\Gamma(4-s)) = 0$. Hence twisting the sequence above by t and taking global sections shows that the total ideals of C and $S \cap T$ agree up to degree t, hence $S \cap T$ is the unique least degree complete intersection which contains C.

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Remark 2.4. The curves C produced by Corollary 2.3 give counterexamples to Conjecture 1.1. The curves Γ are interesting in their own right: they link to smooth connected curves by surfaces of low degree, but cannot link to smooth curves by surfaces of high degree. This is because the singular locus of $\widetilde{\mathbf{P}}^3(\Gamma)$ is two dimensional (see [11], Proposition 4.1.7), hence the general intersection of two surfaces from very ample linear systems is a singular curve in $\widetilde{\mathbf{P}}^3$.

3. Examples

To obtain counterexamples to Conjecture 1.1, we can apply Corollary 2.3 with $Y \subset \mathbf{P}^3$ any curve, $d > 1$, and s and t sufficiently large. In this section, we discuss the simplest application, when $Y \subset \mathbf{P}^3$ is a straight line and $1 < d^2 < s \leqslant t$. In this way we obtain smooth connected curves C which are contained in a unique complete intersection $X = S \cap T$ of surfaces of degrees s and t. The complete intersection X links C to a curve Γ which is supported on Y and contains $Y^{(d)}$, and hence cannot be superficial. The counterexample of smallest degree (see Proposition 3.2 below) can be found in any smooth rational quintic curve which does not lie on a quadric surface.

EXAMPLE 3.1. If we apply Corollary 2.3 with $Y \subset \mathbf{P}^3$ a line, $d = 2, s = t = 5$, we obtain a complete intersection of quintics which links a smooth connected curve C of degree 21 and genus 46 to a multiplicity four structure Γ on Y of arithmetic genus -5. This is the smallest degree application of Corollary 2.3. The conclusion of the corollary holds if we take $Y \subset \mathbf{P}^3$ a line, $d = 2$ and $s = t = 4$ as well. In this case we get a smooth connected curve C of degree 12 and genus 13 which links to a multiplicity four structure Γ on Y of arithemetic genus -3.

The last example above is still not the counterexample of smallest degree. It is easily checked that smooth connected curves of degree $\leqslant 4$ cannot give counterexamples to the conjecture. Indeed, the minimal degree surfaces containing these curves are either planes or integral quadric surfaces: for each such surface, the singular locus is at most zero dimensional, so that any curve lying on these surfaces is superficial. Thus the following proposition gives the counterexample of least degree.

PROPOSITION 3.2. *Let $C \subset \mathbf{P}^3$ be a smooth connected rational quintic curve which does not lie on a quadric surface. Then C has a unique* 4*-secant line Y and*

(a) *C is linked by a complete intersection X of cubic surfaces to a multiplicity* 4*-structure Γ on Y which contains $Y^{(2)}$. In particular, Γ is not superficial.*
(b) *If $X' \neq X$ is another complete intersection of cubic surfaces containing C, then X' links C to a curve Γ' which is superificial.*

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Proof. Let C be a smooth connected rational quintic curve which does not lie on a quadric surface. That C has a unique 4-secant line is a result of Migliore ([10], Theorem 5.2). We reprove this here in order to set notation needed later. To construct the 4-secant line Y, we first note that $h^0(\mathcal{I}_C(l)) = h^2(\mathcal{I}_C(l)) = 0$ for $0 \leqslant l \leqslant 2$ and that $h^1(\mathcal{I}_C) = 0, h^1(\mathcal{I}_C(1)) = 2, h^1(\mathcal{I}_C(2)) = 1$. The ideal sheaf $\mathcal{I}_C$ is 4-regular by ([3], Theorem 1.1), so $h^i(\mathcal{I}_C(n)) = 0$ for $n \geqslant 3$ and $i > 0$. Since $h^1(\mathcal{I}_C(n)) = 0$ for $n < 0$, the Rao module of C has type $(2, 1)$.

Let $R = k[X_0, X_1, X_2, X_3]$ denote the homogeneous coordinate ring of $\mathbf{P}^3$. Up to choice of homogeneous coordinates, the Rao module $M_C = H^1_*(\mathcal{I}_C)$ is isomorphic to $k^2(-1) \oplus k(-2)$ (a Buchsbaum module), $R/(X_0, X_1, X_2, X_3^2)(-1) \oplus k(-1)$, or $(R/(X_0, X_1, X_2^2, X_2X_3, X_3^2)(2))^*$. Using the algorithm of Martin-Deschamps and Perrin ([7], IV Section 6), we find that the minimal curves for these modules have respective degrees $20, 9$ and 5, hence $M_C = (R/(X_0, X_1, X_2^2, X_2 X_3, X_3^2)(2))^*$ and C is minimal in its even linkage class. The linear annihilators X_0, X_1 of M give the equations of the line Y. If we restrict C first to the plane $H = Z(X_0)$ and then we restrict $Z = C \cap H$ to the line $Y = H \cap Z(X_1)$, we find that $h^1(\mathcal{I}_{C\cap Y,Y}(2)) = 1$. It follows that the ideal sheaf $\mathcal{I}_{C\cap Y,Y}$ on Y is $\mathcal{O}_Y(-4)$ and hence Y is a 4-secant line for C. If L is any other line, restricting C to a plane H which contains L but does not contain Y gives $H^1(\mathcal{I}_{C\cap H,H}(2)) = 0$, hence L cannot be a 4-secant line.

Applying Theorem 2.1 to Y with $d = 2$ and $s = t = 3$, we find that a general pair S, T of cubic surfaces containing $Y^{(2)}$ give a complete intersection $S \cap T$ which links a 4-structure Γ on Y to a smooth connected curve D. Proposition 2.2 shows that D is a rational quintic curve and that $p_a(\Gamma) = -1$. Since $Y^{(2)} \subset \Gamma$, the first Cohen-Macaulay filtrant of Γ is $Y^{(2)}$ and we have an exact sequence

$$0 \to \mathcal{I}_\Gamma \to \mathcal{I}_{Y^{(2)}} \to \mathcal{L} \to 0,$$

where $\mathcal{L}$ is a line bundle on Y (see [1]). The arithmetic genus of $Y^{(2)}$ is 0, so the sequence shows that $\mathcal{L} = \mathcal{O}_Y$. Since $H^1_*(\mathcal{I}_{Y^{(2)}}) = 0$, $H^1_*(\mathcal{I}_\Gamma)$ is a quotient of $H^0_*(\mathcal{O}_Y)$. The linkage between Γ and D shows that $h^1(\mathcal{I}_\Gamma(2)) = h^1(\mathcal{I}_D) = 0$, hence the Rao module of Γ can be obtained by truncating $H^0_*(\mathcal{O}_Y)$ in degree 2. We conclude that $H^1_*(\mathcal{I}_\Gamma) \cong R/(X_0, X_1, X_2^2, X_2X_3, X_3^2) \cong M^*(2)$ and hence D and C have the same cohomology and isomorphic Rao modules M.

For D constructed as above, $D \cup Y^{(2)}$ links to Y by the complete intersection $S \cap T$. The standard exact sequence

$$0 \to \mathcal{I}_{S\cap T} \to \mathcal{I}_{D\cup Y^{(2)}} \to \omega_Y(-2) \to 0$$

shows that $h^0(\mathcal{I}_{D\cup Y^{(2)}}(3)) = 2$, so that $S \cap T$ is the only complete intersection of cubic surfaces which contains D and $Y^{(2)}$. It follows that the family of rational quintic curves D produced in this way is parametrized by an open subset $U \subset \mathrm{Grass}_2(H^0(\mathcal{I}_{Y^{(2)}}(3)))$, the Grassmann variety of 2 dimensional subspaces. In

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particular, there is an injective morphism $U \xrightarrow{\psi} H_{\gamma,M}$, where $H_{\gamma,M}$ is the Hilbert scheme of curves with the same cohomology and Rao module as D ($\gamma = \gamma_D$). Since $H^0(\mathcal{I}_{Y^{(2)}}(3))$ is a vector space of dimension 10, the associated Grassmann variety has dimension 16 and so does U. On the other hand, the Hilbert scheme $H_{\gamma,M}$ also has dimension 16: applying [7], IX, Corollary 3.9 with

$$\gamma(l) = \gamma_D(l) = -\binom{l}{0} + 4\binom{l-3}{0} - 2\binom{l-4}{0} - 2\binom{l-5}{0} + \binom{l-6}{0}$$

we compute that $\delta_\gamma = 20, \epsilon_{\gamma,\rho} = -3$ and $h_M = 1$. Thus the image of ψ contains a dense open set. The set of all curves D such that $H^0(\mathcal{I}_D(3))$ contains a 2-dimensional subspace $V \subset H^0(\mathcal{I}_{Y^{(2)}}(3))$ is a closed subset of $H_{\gamma,M}$. Since $H_{\gamma,M}$ is irreducible, we conclude that this closed set (which contains the image of ψ, hence an open set) is all of $H_{\gamma,M}$.

Now we prove the proposition. Since $C \in H_{\gamma,M}$, the last part of the preceding paragraph shows C is contained in a complete intersection X of cubic surfaces which contains $Y^{(2)}$. In particular, the linked curve Γ contains $Y^{(2)}$. Further, since $C \cup \Gamma$ is a local complete intersection and $Y^{(2)}$ is not, Γ must contain at least a 4-structure on Y, and hence must be a 4-structure on Y for degree reasons. This proves part (a).

Now suppose that $X' \neq X$ is another complete intersection of cubic surfaces which contains C and links C to Γ'. Since Y is a 4-secant line to C, $Y \subset X'$ and hence $Y \subset \Gamma'$. On the other hand, $Y^{(2)} \not\subset X'$ (above it was shown that $C \cup Y^{(2)}$ is contained in a unique complete intersection $X = S \cap T$ of cubic surfaces), so $Y^{(2)} \not\subset \Gamma'$. This implies that the component of Γ' supported on Y is superficial (see [1]). Let V be the union of the components of Γ' which are not supported on Y. The degree of V is at most three, so the only way that V could fail to be superficial is if V is a triple line $L^{(2)}$. In this case V is not a local complete intersection, hence neither is $C \cup \Gamma'$, a contradiction. It follows that Γ' is superficial, proving part (b).

Remark 3.3. Applying theorem 2.1 with $Y \subset \mathbf{P}^3$ a straight line, $d = 2$ and $3 \leqslant s \leqslant t$, we obtain a multiplicity four structure Γ on the line Y which contains $Y^{(2)}$ and satisfies $p_a(\Gamma) = 5 - s - t$. In particular, we obtain one for each arithmetic genus $\leqslant -1$. On the other hand, any locally Cohen–Macaulay multiplicity four structure Γ on Y as above gives an exact sequence (see [1])

$$0 \to \mathcal{I}_\Gamma \to \mathcal{I}_{Y^{(2)}} \to \mathcal{O}_Y(a) \to 0.$$

Since $\mathcal{I}_{Y^{(2)}}$ is generated in degree 2, it follows that $a \geqslant -2$ and hence $p_a(\Gamma) \leqslant -1$. It follows that any multiplicity four structure Γ which contains $Y^{(2)}$ deforms flatly to Γ' which links to a smooth connected curve.

Remark 3.4. All the smooth connected curves C produced in this paper have the property that there is a *unique* complete intersection $S \cap T$ which links it to a curve which is not superficial. It would be interesting to know if there exist any such C which are contained in a moving family of complete intersections (of minimal degree) which link C to curves which are not superficial.

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