Morley's Trisection Theorem: an Extension and its Relation to the Circles of Apollonius.

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INTRODUCTORY PART.

1. In Morley's trisection theorem there are three triads of parallel lines which by their intersection with each other form equilateral triangles. The three lines EF, $E_{11}F_{11}$, $E_{22}F_{22}$ (*v.* Taylor and Marr) form one of these triads, and the equations are :—

$$(EF). \quad x\sin\alpha_2 + y\sin\beta_1 + z\sin\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots(1)$$

$$(E_{11}F_{11}). \quad x\sin\alpha + y\sin\beta_2 + z\sin\gamma_2 = 0 \quad \ldots\ldots\ldots\ldots(2)$$

$$(E_{22}F_{22}). \quad x\sin\alpha_1 + y\sin\beta + z\sin\gamma = 0 \quad \ldots\ldots\ldots\ldots(3)$$

where $\qquad \alpha = \frac{1}{3}A, \;\; \alpha_1 = \frac{1}{3}A + \frac{2\pi}{3}, \;\; \alpha_2 = \frac{1}{3}A + \frac{4\pi}{3}$

and β, γ, β_1, γ_1, etc., are similarly defined.

Subtract (2) from (3) and

$$x\cos\alpha_2 + y\cos\beta_1 + z\cos\gamma_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Multiply (1) by $\cos\epsilon$ and (4) by $\sin\epsilon$, add, and

$$x\sin(\alpha_2+\epsilon) + y\sin(\beta_1+\epsilon) + z\sin(\gamma_1+\epsilon) = 0 \quad \ldots\ldots\ldots(5)$$

(5) is a line parallel to (1), (2), or (3).

ϵ is arbitrarily chosen and temporarily fixed.

(5) will be named $\overline{211}_\epsilon$ or simply $\overline{211}$.

2. The line

$$x\sin(\alpha_2+\epsilon) + y\sin(\beta_1+\epsilon) + z\sin(\gamma_1+\epsilon) = 0$$

or $\overline{211}_\epsilon$ is the polar of the point

$$[\sin(\alpha_2+\epsilon), \;\sin(\beta_1+\epsilon), \;\sin(\gamma_1+\epsilon)]$$

or $(211)_\epsilon$ with respect to the imaginary conic $x^2+y^2+z^2=0$.

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If the point (211) be given, the *corresponding* line $\overline{211}$ can be obtained by the following construction :—

(1) Find the isogonal conjugate of (211) and call it Q.

(2) Join AQ to meet BC in X′
BQ to meet CA in Y′, etc.

(3) Find X the harmonic conjugate of X′ with respect to BC.
So for Y and Z

The line XYZ is $\overline{211}$.

Similarly $\overline{121}$ is obtained from (121)
and $\overline{112}$ „ „ (112).

3. The point (211) lies on the line

$$x\sin(\beta-\gamma)+y\sin\left(\gamma-\alpha+\frac{4\pi}{3}\right)+z\sin\left(\alpha-\beta+\frac{2\pi}{3}\right)=0$$

or $$x\sin p+y\sin q_2+z\sin r_1=0$$

or briefly the line $\overline{pq_2r_1}$, where

$$p=\beta-\gamma,\ q=\gamma-\alpha,\ \text{etc.},\ p_1=p+\frac{2\pi}{3},\ \text{etc.}$$

(121) lies on the line $\overline{p_1qr_2}$,
(112) on $\overline{p_2q_1r}$.

The three lines $\overline{pq_2r_1}$, etc., were first discovered by Mr F. G. Taylor in another connexion and shown to meet in the symmedian point.

PART I.—*The First Group of Equilateral Triangles.*

4. To form an equilateral triangle three lines are required, viz., $\overline{211}_\epsilon$, $\overline{121}_\epsilon$, $\overline{112}_\epsilon$. This triangle would correspond in Morley's diagram, where $\epsilon=0$, to DEF.

In the same diagram

$$E_{11}F_{11} \text{ is } \overline{022}_{\epsilon=0}$$
$$E_{22}F_{22} \text{ is } \overline{100}_{\epsilon=0}.$$

5. The problem is to find $\overline{211_\epsilon}$, $\overline{121_\epsilon}$, $\overline{112_\epsilon}$. To find these lines a point $(111)_\epsilon$ is required. Call this point R. (*v.* § 6).

AR meets $\overline{pq_2r_1}$ in (211). [*v.* § (3).]
BR meets $\overline{p_1qr_2}$ in (121).
CR meets $\overline{p_2q_1r}$ in (112).

Hence by means of $(111)_\epsilon$ the three points $(211)_\epsilon$, etc., are found from which the equilateral triangle $\overline{211}$, $\overline{121}$, $\overline{112}$ can be derived. This triangle may be called ${}_\epsilon$DEF, and the point (111) the "primary point" from which it is derived. For ${}_\epsilon$DEF we may write simply DEF.

6. The point $(111)_\epsilon$ or, at length,

$$[\sin(\alpha_1+\epsilon), \sin(\beta_1+\epsilon), \sin(\gamma_1+\epsilon)]$$

lies on the line

$$x\sin p + y\sin q + z\sin r = 0,$$

or, briefly, $\overline{pqr}$, one of a group of three important lines $\overline{pqr}$, $\overline{p_1q_1r_1}$, $\overline{p_2q_2r_2}$ discovered by Mr Taylor and shown to meet in the point

$$\left[\sin\left(A+\frac{2\pi}{3}\right), \sin\left(B+\frac{2\pi}{3}\right), \sin\left(C+\frac{2\pi}{3}\right)\right],$$

or, briefly, $(\sin A_1 \sin B_1 \sin C_1)$ one of the points of intersection of the Apollonius circles. This point will be called H_1.

7. It will be assumed that ${}_\epsilon$DEF and ABC are in perspective. The axis of perspective is :—

$$\begin{vmatrix} x & y & z \\ 0 & \operatorname{cosec}(\beta_1+\epsilon) & -\operatorname{cosec}(\gamma_1+\epsilon) \\ -\operatorname{cosec}(\alpha_1+\epsilon) & 0 & \operatorname{cosec}(\gamma_1+\epsilon) \end{vmatrix} = 0$$

for ${}_\epsilon$EF meets BC in $[0, \operatorname{cosec}(\beta_1+\epsilon), -\operatorname{cosec}(\gamma_1+\epsilon)]$ and so for ${}_\epsilon$FD.

This equation becomes $\Sigma x\sin(\alpha_1+\epsilon)=0$, the equation of the the line $\overline{111}$, which *corresponds* to the primary point (111) from which the triangle DEF is derived. It will be assumed, without further proof, that the axis of perspective of any equilateral triangle, with ABC, is the line *corresponding* to the primary point from which that equilateral triangle is derived.

The line $\overline{111}_\epsilon$ passes through (pqr) for any value of ϵ since $\Sigma\sin(\alpha_1+\epsilon)\sin(\beta-\gamma)\equiv 0$. The characteristic property of ${}_\epsilon$DEF is that, for any value of ϵ, the axis of perspective with ABC passes through (pqr). In Morley's diagram, the axes of perspective of DEF, $D_{11}E_{11}F_{11}$, $D_{22}E_{22}F_{22}$ with ABC pass through (pqr).

The point (pqr) may be called the *source of perspective*, or, briefly, the *source* of ${}_\epsilon$DEF.

8. By operating from three primary points (000), (111), (222) on $\overline{pqr}$, we could reach the three triads of lines [$\overline{100}$, $\overline{010}$, $\overline{001}$], [$\overline{211}$, etc.], [$\overline{022}$, etc.], which, for a given value of ϵ, form a system of equilateral triangles corresponding to Morley's particular system where $\epsilon = 0$.

It will be better, however, to be content with (111) on $\overline{pqr}$ and proceed to $\overline{p_1 q_1 r_1}$ any point of which has coordinates of the type $\sin(\alpha_2 + \epsilon)$, $\sin(\beta_1 + \epsilon)$, $\sin(\gamma + \epsilon)$, or, briefly, $(210)_\epsilon$.

9. This point (210) will serve as the primary point of another type of equilateral triangle. Call (210) T.

Join AT to cut $\overline{p_1 q r_2}$ in (010) for the coordinates of a point on $\overline{p_1 q r_2}$ are at discretion (121), (202), or (010). (*v.* § 3).

Join BT to cut $\overline{p_2 q_1 r}$ in (220).

Join CT to cut $\overline{p q_2 r_1}$ in (211).

The three points (010), (220), (211) lead to the three *corresponding* lines from which an equilateral triangle of the type $D_{02}E_{21}F_{10}$ is formed.

It is interesting to note (i) that $\overline{211}$, a line which is related to A, is obtained from CT, that $\overline{010}$, which is related to B, is obtained from AT, and so for $\overline{200}$, (ii) that $D_{02}E_{21}$ is inclined at 120° to DE. We shall find later that $\overline{pqr}$ and $\overline{p_1 q_1 r_1}$, from primary points on which the triangles ${}_\epsilon DEF$, ${}_\epsilon D_{02}E_{21}F_{10}$ are derived, are also inclined at 120°.

10. The primary point being (210), the axis of perspective of $D_{02}E_{21}F_{10}$ with ABC is $\overline{210}$.

$\overline{210_\epsilon}$ passes through $(p_1 q_1 r_1)$ for any value of ϵ.

The characteristic of ${}_\epsilon D_{02}E_{21}F_{10}$ is that its axis of perspective with ABC passes through $(p_1 q_1 r_1)$, which may be called the *source* of $D_{02}E_{21}F_{10}$. In Morley's particular system the axes of perspective of $D_{02}E_{21}F_{10}$, $D_{10}E_{02}F_{21}$, $D_{21}E_{10}F_{02}$ with ABC pass through $(p_1 q_1 r_1)$.

11. Proceed to $\overline{p_2 q_2 r_2}$, and take a point on it as primary point, viz. (012). From (012) we can reach the three lines $\overline{112}$, $\overline{022}$, $\overline{010}$ which form an equilateral triangle of the type $D_{01}E_{12}F_{20}$.

The axis of perspective is $\overline{012}$.

The *source* of ${}_\epsilon D_{01}E_{12}F_{20}$ is $(p_2 q_2 r_2)$.

12. *Summary.*

By operating from primary points (111), (210), (012), we reach three types of triangles DEF, $D_{02}E_{21}F_{10}$, $D_{01}E_{12}F_{20}$, each with a characteristic *source*. These sources (pqr), $(p_1q_1r_1)$, $(p_2q_2r_2)$ are collinear on $\Sigma x \sin A_1 = 0$, which is interpreted as follows. The point $(\sin A_1, \sin B_1, \sin C_1)$ lies on $\overline{pqr}$, $\overline{p_1q_1r_1}$, $\overline{p_2q_2r_2}$, and may be regarded as $(111)_{\epsilon'}$, $(210)_{\epsilon''}$, or $(012)_{\epsilon'''}$. As (111) it will be the primary point of a triangle $_{\epsilon'}$DEF with source $\overline{pqr}$, as (210) it will be the primary point of another triangle $_{\epsilon''}D_{02}E_{21}F_{10}$ inclined at 120° to DEF with source $(p_1q_1r_1)$, as (012) it will be the primary point of $_{\epsilon'''}D_{01}E_{12}F_{20}$ with source $(p_2q_2r_2)$. But, as we are operating from $(\sin A_1, \sin B_1, \sin C_1)$, the axis of perspective of any of these three triangles with ABC is $\Sigma x \sin A_1 = 0$, on which, therefore, the three sources must lie. (*v.* § 7).

Part II.—*The Second Group of Equilateral Triangles.*

13. The primary points, from which the equilateral triangles can be derived, may be exhibited in three triads:—

(000), (111), (222) on $\overline{pqr}$
(210), (021), (102) on $\overline{p_1q_1r_1}$
(012), (120), (201) on $\overline{p_2q_2r_2}$.

The secondary points, which *correspond* to the sides of the equilateral triangles, are:—

(211), (022), (100) on $\overline{pq_2r_1}$
(121), (202), (010) on $\overline{p_1qr_2}$
(112), (220), (001) on $\overline{p_2q_1r}$.

The number of groups is 27, and there remain, therefore, three triads:—

(200), (011), (122) on $\overline{pq_1r_2}$
(020), (101), (212) on $\overline{p_2qr_1}$
(002), (110), (221) on $\overline{p_1q_2r}$.

These lines $\overline{pq_1r_2}$, etc., have been shewn by Mr Taylor to pass through the point $\left[\sin\left(A + \frac{4\pi}{3}\right), \text{etc.}\right]$, or, briefly, $(\sin A_2, \sin B_2, \sin C_2)$, the other point of intersection of the circles of Apollonius. This point will be called H_2.

It will be shewn later that, like the $\overline{pqr}$ lines, these $\overline{pq_1r_2}$ lines are inclined to each other at 120°, and that this property is reflected in the inclination to each other of the equilateral triangles to which these $\overline{pq_1r_2}$ lines are related.

14. As primary point take (200) on $\overline{pq_1r_2}$ and call it V.

AV meets $\overline{pq_2r_1}$ in (100).

BV meets $\overline{p_2q_1r}$ in (220).

CV meets $\overline{p_1qr_2}$ in (202).

The sides of the equilateral triangles derived from (200) are thus $\overline{100}$, $\overline{220}$, $\overline{202}$. This is the triangle ${}_\epsilon D_{11}E_{10}F_{01}$. The triangle has to be read in the clockwise rotation. The transposition of E and F will be understood if it is noticed that AV meets its proper line $\overline{pq_2r_1}$, whereas BV meets $\overline{p_2q_1r}$, a line related to C and CV meets $\overline{p_1qr_2}$, a line related to B.

The primary point being (200), the axis of perspective with ABC is $\overline{200}$.

The *source* of ${}_\epsilon D_{11}E_{10}F_{01}$ is (pq_1r_2).

In Morley's particular system the axes of perspective of $D_{11}E_{10}F_{01}$, $D_{22}E_{21}F_{12}$, and $D_{00}E_{02}F_{20}$ with ABC pass through (pq_1r_2).

15. Operating from (020) on $\overline{p_2qr_1}$ as primary point, we reach the triangle ${}_\epsilon D_{01}E_{11}F_{10}$. The sides of this triangle are inclined at 120° to those of the previous, and $\overline{p_2qr_1}$, $\overline{pq_1r_2}$ are inclined at the same angle, as will be proved later.

The axis of perspective is $\overline{020}$, and the *source* is (p_2qr_1).

16. From (002) on $\overline{p_1q_2r}$ we reach ${}_\epsilon D_{10}E_{01}F_{11}$. The *source* is (p_1q_2r).

17. The vertices of the nine equilateral triangles, if we take Morley's particular system, in this second group are distinct, but each vertex has already appeared as a vertex in the first group. The *sources* of the three types of triangles in the second group are collinear on $\Sigma x \sin A_2 = 0$, which is interpreted thus:

$(\sin A_2 \sin B_2 \sin C_2)$ lies on $\overline{pq_1r_2}$, $\overline{p_2qr_1}$, $\overline{p_1q_2r}$, and may be regarded as $(200)_{\epsilon'}$ or $(020)_{\epsilon''}$ or $(002)_{\epsilon'''}$. It will thus give rise to three different equilateral triangles inclined to each other at 120°. As the axis of perspective with ABC of any of these is $\Sigma x \sin A_2 = 0$, the line which *corresponds* to the primary point ($\sin A_2$, etc.), it follows that the three *sources* must lie on $\Sigma x \sin A_2 = 0$ since each *source* lies on the axis of perspective.

Part III.—*The Geometry of* $\overline{100}_\epsilon$ *and the nine pqr lines.*

18. The line $\overline{100}_\epsilon$ or ${}_\epsilon E_{22}F_{22}$ is

$$x\sin(\alpha_1+\epsilon)+y\sin(\beta+\epsilon)+z\sin(\gamma+\epsilon)=0.$$

If $\overline{100}_\epsilon$ meet BC in K, the coordinates of K are 0, $-\sin(\gamma+\epsilon)$, $\sin(\beta+\epsilon)$. If K be found, $\overline{100}_\epsilon$ is a line through K inclined at $(\gamma-\beta)$ to CB. It will serve us better, however, to find X, where X is on BC and AX is the line isogonal to AK.

Construction for X. (Fig. 1).

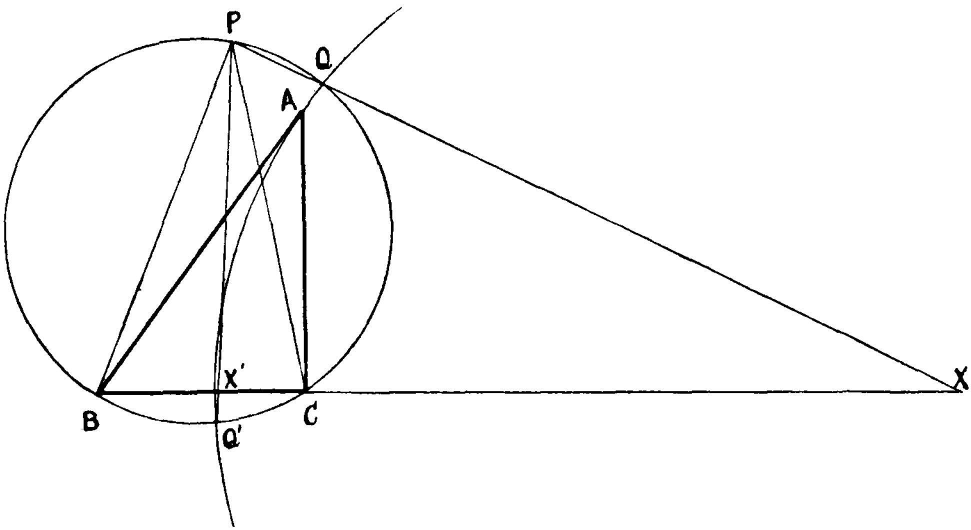

Fig. 1.

Draw the Apollonius circle through A. Make the angle $CBP=\beta+\epsilon$ and angle $BCP=\gamma+\epsilon$. Q and Q′ are the intersections of the circle BPC and the Apollonius circle.

PQ meets BC in X. PQ′ meets BC in X′.

$$\frac{BX}{XC}=\frac{BX/BQ}{XC/CQ}\cdot\frac{BQ}{CQ}=\frac{\sin BQX}{\sin CQX}\cdot\frac{\sin C}{\sin B}$$

$$=\frac{\sin(\gamma+\epsilon)}{\sin(\beta+\epsilon)}\cdot\frac{\sin C}{\sin B}$$

(for $BQX=\pi-BQP=\pi-BCP=\pi-(\gamma+\epsilon)$

$CQX=PBC=\beta+\epsilon$,

and Q lies on the Apollonius circle,

$\therefore\ BQ/CQ=\sin C/\sin B$

$\therefore\ BX\sin B/XC\sin C=\sin(\gamma+\epsilon)/\sin(\beta+\epsilon)$,

or if 0, y, z be the co-ordinates of X,

$$z/y=-\sin(\gamma+\epsilon)/\sin(\beta+\epsilon).$$

Similarly, if X′ be 0, y', z',

$$z'/y'=\sin(\gamma+\epsilon)/\sin(\beta+\epsilon),$$

and, therefore, (AB,XX′) is harmonic.

Find AK, the line isogonal to AX with K on BC. As the tangent at P is inclined at $(\gamma-\beta)$ to CB, draw through K the line parallel to this tangent, and $\overline{100_\epsilon}$ is obtained.

19. The line $\overline{p\,q\,r}$ is $\Sigma x\sin(\beta-\gamma)=0$. This line, as we have seen, is the locus of the primary points of the triangle ${}_\epsilon$DEF for any value of ϵ. In Morley's particular system Mr Taylor has shown that it is the "line of poles" of DEF, $D_{11}E_{11}F_{11}$, $D_{22}E_{22}F_{22}$. In the previous construction for $\overline{100_\epsilon}$ let $\epsilon=-\alpha$. (Fig. 2).

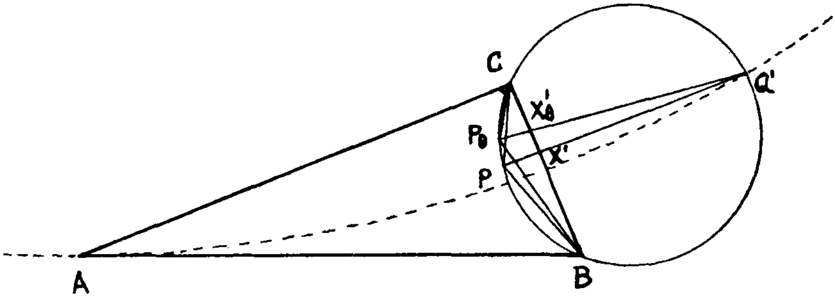

Fig. 2.

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The angle $CBP = \beta - \alpha$ and $BCP = \gamma - \alpha$.

The angle $BQ'C = (\beta - \alpha + \gamma - \alpha) = \frac{\pi}{3} - A$.

Now Q′ lies on the Apollonius circle through A, and since $BQ'C = \frac{\pi}{3} - A$, it is (H_1) one of the two points of intersection of the three Apollonius circles, and its coordinates are $\sin A_1 \sin B_1 \sin C_1$. (*v.* § 6).

The coordinates of X′, the meet of PQ′ and BC, (*v.* § 18) are $0, \sin(\beta - \alpha), \sin(\gamma - \alpha)$ or $0, -\sin(\alpha - \beta), \sin(\gamma - \alpha)$, and hence X′ is a point on $\Sigma x \sin(\beta - \gamma) = 0$ or $\overline{pqr}$. But Q′ (or H_1) is also a point on $\overline{pqr}$.

$$\therefore \overline{pqr} \text{ is the line } Q'X'.$$

20. The line $\overline{(pqr)}_\theta$ is $\Sigma x \sin(\beta - \gamma + \theta) = 0$. Rotate the CP of Fig. 2 to the right through an angle θ to the position CP_θ .

The angle $CBP_\theta = \beta - \alpha - \theta$,

and the angle $BCP_\theta = \gamma - \alpha + \theta$.

P_θ lies on the circle BPC which intersects the Apollonius circle in Q′ or H_1.

Join $P_\theta Q'$ to meet BC in X'_θ.

The coordinates of X'_θ (*v.* § 18) are

$$0, \ \sin(\beta - \alpha - \theta), \ \sin(\gamma - \alpha + \theta),$$

or

$$0, \ -\sin(\alpha - \beta + \theta), \ \sin(\gamma - \alpha + \theta),$$

and, therefore, X'_θ is a point on $\overline{(pqr)}_\theta$.

Since $\Sigma \sin A_1 \sin(\beta - \gamma + \theta) \equiv 0$, Q′ (or H_1) is also a point on $\overline{(pqr)}_\theta$ and hence $Q'X'_\theta$ is the line $\overline{(pqr)}_\theta$.

21. The angle between Q′X′ and $Q'X'_\theta$ is $PQ'P_\theta = PCP_\theta = \theta$, Thus $\overline{(pqr)}_\theta$ is inclined at an angle θ to $\overline{pqr}$. When $\theta = \frac{2\pi}{3}$, $\overline{(pqr)}_\theta$ becomes $\overline{p_1q_1r_1}$, and hence $\overline{p_1q_1r_1}$ and $\overline{pqr}$ are inclined to each other at 120°, and so for $\overline{p_2q_2r_2}$ and $\overline{p_1q_1r_1}$.

22. Any line through H_1 is of the form $\Sigma x \sin(\beta - \gamma + \theta) = 0$. When $\theta = \gamma - \beta$ this equation becomes

$$y\sin(2\gamma - \alpha - \beta) + z\sin(\gamma + \alpha - 2\beta) = 0,$$

or
$$y\sin\left(C - \frac{\pi}{3}\right) + z\sin\left(\frac{\pi}{3} - B\right) = 0,$$

or
$$y\sin C_1 - z\sin B_1 = 0.$$

This is the line AH_1, and thus AH_1 is inclined at $(\gamma - \beta)$ to $\overline{pqr}$.

23. The line $\overline{pq_1r_2}$ is

$$x\sin(\beta - \gamma) + y\sin\left(\gamma - \alpha + \frac{2\pi}{3}\right) + z\sin\left(\alpha - \beta + \frac{4\pi}{3}\right) = 0.$$

This line is the locus of the primary points of the triangle $_\epsilon D_{11}E_{10}F_{01}$. In Morley's particular system Mr Taylor has shown that it is the "line of poles" of $D_{11}E_{10}F_{01}$, $D_{22}E_{21}F_{12}$, $DE_{02}F_{20}$.

In the construction for $\overline{100}_\epsilon$ (*v.* § 18) let

$$\epsilon = \frac{2\pi}{3} - \alpha \quad \text{(Fig. 3)}.$$

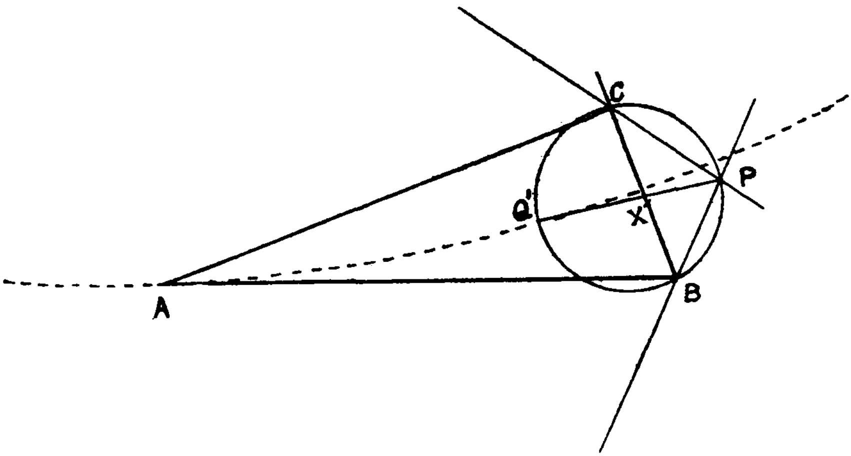

Fig. 3.

The angle

$$BPC = \left(\frac{2\pi}{3} + \beta - \alpha\right) + \left(\frac{2\pi}{3} + \gamma - \alpha\right) - \pi = \frac{2\pi}{3} - A.$$

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The circle BPC cuts the Apollonius circle in Q′ on the side of BC remote from P.

The angle

$$BQ'C = \pi - \left(\frac{2\pi}{3} - A\right) = \frac{\pi}{3} + A.$$

Hence Q′ is the second point of intersection of the three Apollonius circles, viz. H_2, and its coordinates are $\sin A_2$, $\sin B_2$, $\sin C_2$. (*v.* § 13).

If X′ is the meet of PQ′ and BC, QX′ is the line $\overline{(p\,q_1r_2)}$. (*cf.* § 19).

24. The line $\overline{(p\,q_1r_2)_\theta}$ is

$$x\sin(\beta - \gamma + \theta) + y\sin\left(\gamma - \alpha + \frac{2\pi}{3} + \theta\right) + z\sin\left(\alpha - \beta + \frac{4\pi}{3} + \theta\right) = 0.$$

By a proof similar to that of § 21 it can be shown that $\overline{(p\,q_1r_2)_\theta}$ and $\overline{p\,q_1r_2}$ are inclined at an angle θ.

Hence (*cf.* § 21) $\overline{pq_1r_2}$, $\overline{p_1q_2r}$, $\overline{p_2qr_1}$ are inclined to each other at 120°.

25. AH_2 is inclined to $\overline{p\,q_1r_2}$ at $(\gamma - \beta)$ (*cf.* § 22).

26. As AH_1 is inclined at $(\gamma - \beta)$ to $\overline{p\,q\,r}$ and AH_2 has the same inclination to $\overline{p\,q_1r_2}$, it follows that $\overline{p\,q\,r}$ and $\overline{p\,q_1r_2}$ make with each other an angle equal to H_1AH_2.

Hence $\overline{p\,q\,r}$ and $\overline{p\,q_1r_2}$ intersect on the circumference of the Apollonius circle through A.

$\overline{pqr}$ and $\overline{pq_1r_2}$ meet on the Apollonius circle (A).
$\overline{p_1q_1r_1}$ and $\overline{p_1q_2r}$ „ „ „ „
$\overline{p_2q_2r_2}$ and $\overline{p_2qr_1}$ „ „ „ „
$\overline{pqr}$ and $\overline{p_2qr_1}$, etc., meet on the Apollonius circle (B).
$\overline{pqr}$ and $\overline{p_1q_2r}$, „ „ „ „ „ (C).

(*v.* Fig. 5.)

27. The line $\overline{pq_2r_1}$ is

$$x\sin(\beta - \gamma) + y\sin\left(\gamma - \alpha + \frac{4\pi}{3}\right) + z\sin\left(\alpha - \beta + \frac{2\pi}{3}\right) = 0.$$

This line is the locus of the points $(211)_\epsilon$ which *correspond* to the lines $\overline{211_\epsilon}$ or ${}_\epsilon EF$. In Morley's particular system Mr Taylor has shown that it is the "line of poles" of a certain triad of non-equilateral triangles.

In the construction for $\overline{100_\epsilon}$ (*v.* § 18) let

$$\epsilon = \frac{4\pi}{3} - \alpha \quad \text{(Fig. 4)}.$$

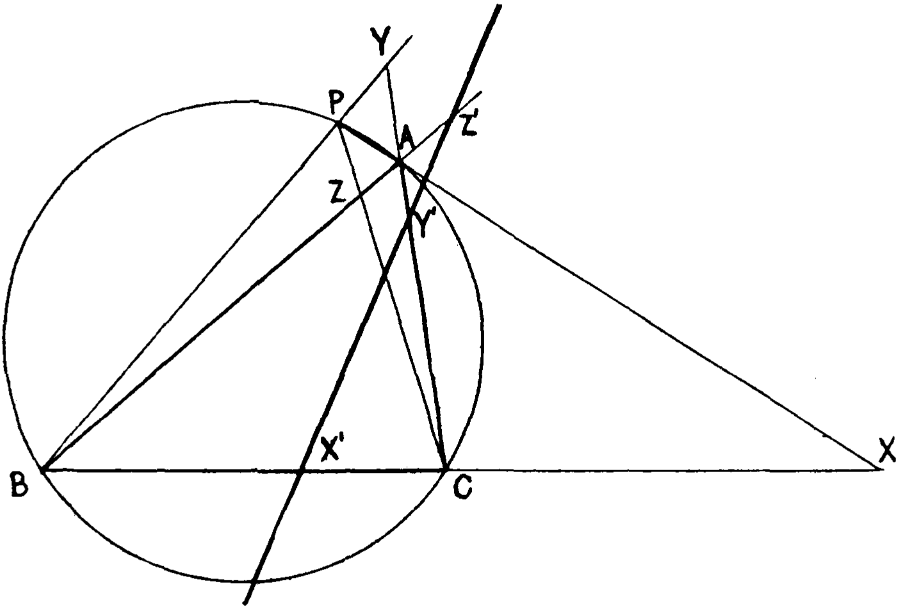

Fig. 4.

The angle

$$BPC = \pi - \left[\left(\frac{4\pi}{3} + \beta - \alpha\right) + \left(\frac{4\pi}{3} + \gamma - \alpha\right) - 2\pi\right] = A.$$

Hence P lies on the circumcircle of ABC and the circle BPC or ABC meets the Apollonius circle in A, and the tangent at P is parallel to EF.

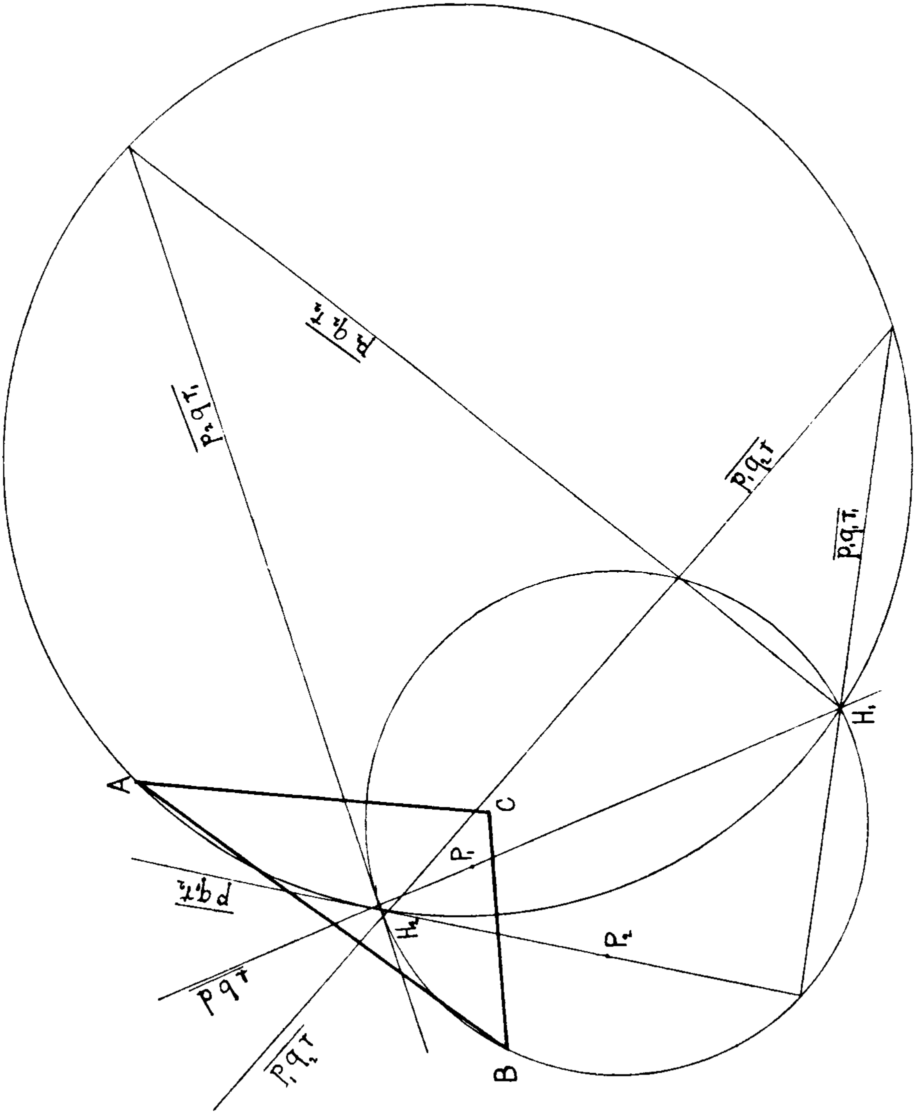
A
B
C
H_1
H_2
P_1
P_2
$\overline{pqr}$
$\overline{p_1q_2r}$
$\overline{p_1q_1r_1}$
$\overline{p_2qr_1}$
$\overline{p_2q_2r_2}$
$\overline{pq_1r_2}$

Fig. 5.

A now takes the place of Q in the construction for $\overline{100_\epsilon}$ and the coordinates of X, the point of intersection of PA and BC, are (*v.* § 18)

$$0, \ -\sin\left(\frac{4\pi}{3}+\beta-\alpha\right), \ \sin\left(\frac{4\pi}{3}+\gamma-\alpha\right),$$

or

$$0, \ \sin\left(\alpha-\beta+\frac{2\pi}{3}\right), \ \sin\left(\gamma-\alpha+\frac{4\pi}{3}\right).$$

X, therefore, is the harmonic conjugate of the meet of $\overline{p\,q_2r_1}$ and BC.

Similarly Y is the harmonic conjugate of the meet of $\overline{p\,q_2r_1}$ and CA. So for Z.

The construction for $\overline{p\,q_2r_1}$ is, then, as follows :— (*v.* Fig. 4).

Find P the point of contact of the upper tangent to the circle ABC, which is parallel to EF. Let PA meet BC in X, etc. Let X′ be the harmonic conjugate of X with regard to BC, etc.

The line X′Y′Z′ is $\overline{pq_2r_1}$.

The three lines $\overline{p\,q_2r_1}$, $\overline{p_1q\,r_2}$, $\overline{p_2q_1r}$ are obtained in the same manner from three points P, which lie on the circumference of ABC, and are the vertices of an equilateral triangle with sides parallel to those of DEF.

28. Let the coordinates of P on the circumcircle be $1/\alpha$, $1/\beta$, $1/\gamma$. Since the circumcircle is the isogonal conjugate of the line at infinity, (α, β, γ) is a point on the line at infinity, and, therefore, $\alpha\sin A+\beta\sin B+\gamma\sin C=0$. The line X′Y′Z′ can easily be proved to have the equation $x\alpha+y\beta+z\gamma=0$, and therefore must pass through the symmedian point (sinA, sinB, sinC) since

$$x:y:z=\sin A:\sin B:\sin C$$

satisfies $\Sigma x\alpha=0$.

Take any point on X′Y′Z′, the coordinates of which are $(211)_\epsilon$, but which may now be named $x_0y_0z_0$. Just as the point (211) *corresponds* to the line $\overline{211}$, so the point $(x_0y_0z_0)$ corresponds to

$$xx_0+yy_0+zz_0=0.$$

But since $(x_0y_0z_0)$ is on $\Sigma x\alpha=0$,

$$x_0\alpha+y_0\beta+z_0\gamma=0,$$

and therefore the equation $\Sigma xx_0=0$ is satisfied by

$$x:y:z=\alpha:\beta:\gamma.$$

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This means that $\Sigma xx_0 = 0$ always passes through $(\alpha\beta\gamma)$ for any position of $x_0y_0z_0$ on X′Y′Z′. Since $(\alpha\beta\gamma)$ is a point at infinity, it follows that $\Sigma xx_0 = 0$ remains parallel for different positions of $x_0y_0z_0$. It has already been shewn that, for any value of ϵ, $\overline{211_\epsilon}$ is parallel to EF, but this proof, which is independent of the particular position of P on the circumcircle, shows that a series of parallel lines related to P can be obtained for any position of P on the circumcircle.

29. If the coordinates of P be $\operatorname{cosec}(B - C)$, $\operatorname{cosec}(C - A)$, $\operatorname{cosec}(A - B)$, then the X′Y′Z′ corresponding to this P is $\Sigma x \sin(B - C) = 0$. This line is the Hessian axis, and contains the two points $H_1(\sin A_1$, etc.) and $H_2(\sin A_2$, etc.). Hence (§ 28) $\Sigma x \sin A_1 = 0$ and $\Sigma x \sin A_2 = 0$ are parallel, *i.e.* the two lines which contain respectively the *sources* of the triangles of the first and second groups are parallel.

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