Why the algorithm works

We choose m such that, for all a, b there exists c such that

$$p \mid (10a + b) + c(a + mb).$$

There are, of course, multiple possible choices for c and m. For instance, for division by 7, you could take $c = 4$ and $m = 5$:

$$(10a + b) + 4(a + 5b) = 14a + 21b$$

which is clearly divisible by 7. We could replace 5 by –2, as $5 \equiv -2 \pmod 7$; this would give smaller, and hence more convenient, numbers.

Now

$$p \mid (10 + c)a + (1 + mc)b \text{ for all } a, b \Rightarrow p \mid 10 + c \text{ and } p \mid 1 + mc.$$

As $p > 5$, p is coprime to 10, so the congruency equation $kp \equiv 9 \pmod{10}$ can always be solved for any prime p. Also, for this value of k, $\frac{1}{10}(kp + 1)$ is an integer; let this be m.

So the integer m satisfies $10m = 1 + kp$. Then

$$1 + mc = \frac{10 + c(1 + kp)}{10} = \frac{(10 + c) + kcp}{10}.$$

As 10 and p have no common factor,

$$p \mid \frac{(10 + c) + kcp}{10} \Leftrightarrow p \mid (10 + c) + kcp \Leftrightarrow p \mid 10 + c.$$

Hence $p \mid 1 + mc \Leftrightarrow p \mid 10 + c$,

So we can choose any c for which $c \equiv -10 \pmod p$; and with this choice of c and m, p divides $(10a + b) + c(a + mb)$. Thus $10a + b$ is divisible by p if, and only if, $a + mb$ is divisible by p. It is nice to observe that you don't need to use the value of c, or even to find it.

This proof will appear in my book written for the OCR option, forthcoming if I can find a publisher!

Reference

1. John Sykes, *A Level Further Mathematics for OCR A Additional Pure*, Cambridge University Press, 2021.

10.1017/mag.2023.76

On 107.03: Zoltan Retkes writes: The author gives an argument for the volume of an ungula (including a typo, where $z = 2y$ should be $z = 2x$). However, the argument can be replaced by this one-liner. If S is half the base circle lying in $x \geqslant 0$, then

$$\iint_S 2x\,dx\,dy = \int_{-a}^{a} dy \int_0^{\sqrt{a^2 - y^2}} 2x\,dx = \int_{-a}^{a} (a^2 - y^2)\,dy = \left[a^2 y - \frac{y^3}{3}\right]_{y=-a}^{y=a} = \frac{4a^3}{3}.$$

10.1017/mag.2023.77