Desargues' involution theorem: from history to applications

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Desargues' Involution Theorem is a powerful problem solving tool to anyone interested in projective geometry and its contemporary applications. To give a better understanding of this fundamental result, we present the history of the idea and we illustrate several direct applications.

1. *History*

The origin of Desargues' inquiries in the realm of projective geometry dates back to the remote time when most of Europe was embroiled in the turmoil of the Thirty Years' War.

Girard Desargues (1591-1661) used the concept of Involution for the first time in his most important work *Rough Draft on Conics* (1639). Its content can be found in [1], where he first studied sets of three pairs of points on a line (B, H); (C, G); (D, F) and then defined that they form a '*tree*' if there is a point A on the same line such that

$$AB \cdot AH = AC \cdot AG = AD \cdot AF. \tag{1}$$

Furthermore, he defined that those three pairs of points are in involution if

$$\frac{GD}{CD} \cdot \frac{GF}{CF} = \frac{GB}{CB} \cdot \frac{GH}{CH}. \tag{2}$$

As relation (2) suggests, Desargues was aware of earlier results known by Greek mathematicians. Indeed, he had available Commandino's Latin edition of Euclid's *Elements*, the first four books on conics by Apollonius and two editions of Pappus' *Collection* [1]. He exploits the invariant of the *ratio of the ratios in which the straight line joining two points is divided by another two points*, known but not emphasised, in Greek literature. He is mostly ignored, if not ridiculed, by his contemporaries due to the language he used. His manuscript, *Rough Draft on Conics* was forgotten, but discovered by Michel Chasles (1793-1880) in 1845 and published in 1864 together with an explanation of the terms used by Desargues. Despite being neglected, his ideas were preserved by many disciples, among them Bosse, Pascal and de la Hire. The recognition of his work came from Charles Julien Brianchon (1783-1864) in his *Mémoire sur les lignes du second ordre* (1817) and Jean-Victor Poncelet (1788-1867) in *Traité des proprietes projective* (1822) [2]. The latter contains fundamental ideas such as the cross-ratio, perspective and involution, which boosted the development of projective geometry during the nineteenth century, with Poncelet making his own contribution.

Desargues' approach to Involution was not projective; he defined it as a relation of distances between different pairs of points (see relations (1) and (2)).

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An equivalent definition more closely related to the current concept of Involution was given by Karl von Staudt (1798-1867): *An involution is a projectivity of period two, that is, a projectivity which interchanges pairs of points* [3].

Poncelet brought the '*principle of duality*' to light, claiming that it is his own discovery. However, its nature was more clearly understood by Joseph Diaz Gergonne (1771-1859). This principle, used in projective geometry, asserts that *every definition remains significant, and every theorem remains true, when we interchange the words point and line* [3]. The dual principle will be used subsequently in the Article.

At the end of this brief historical introduction, it is important to note the reason behind the expansion of Projective Geometry during the Renaissance period, which is art-related, as artists were trying to achieve a sense of depth and three-dimensional view in their plane paintings.

2. *Development*

For a brief introduction to the basic terminology used in projective geometry see [4, p. 242].

We now state von Staudt's definition of involution, mentioned in the previous section, and we put it into perspective with respect to the other forms of the concept of involution:

Definition 1: Involution (von Staudt). An involution is a projectivity of period two, that is, a projectivity which interchanges pairs of points.

The above definition suggests that we may look at involution as a function which intercharges pairs of points on a line in the projective plane. Recalling that projectivity preserves the cross-ratio of four points on a line (see [5, Chapter 9]), denoted in this Article by $(A, B; C, D) = \frac{AC}{AD} \div \frac{BC}{BD}$, the following definition will be more intuitive:

Definition 2: Involution (as function). Denote by $\mathcal{S}$ the set of all points on a line in the projective plane (thus including the point at infinity on the line). Then if an involution is a function $f : \mathcal{S} \to \mathcal{S}$ such that $f(f(X)) = X$ for all $X \in \mathcal{S}$, and for any four points $(A, B, C, D) \in \mathcal{S}$

$$(A, B; C, D) = (f(A), f(B); f(C), f(D)).$$

The pair of points $(X, f(X))$ is called a *point-pair*.

Definition 3: Involution (on a pencil). Denote by $\mathcal{L}$ the set of all lines in the projective plane containing a point P. Then an involution is a function $f : \mathcal{L} \to \mathcal{L}$ for which $f(f(l)) = l$, for all $l \in \mathcal{L}$ and for any four lines $(l_1, l_2, l_3, l_4) \in \mathcal{L}$ we have $(l_1, l_2; l_3, l_4) = (f(l_1), f(l_2); f(l_3), f(l_4))$. Similarly, pairs of lines $(l, f(l_1))$ are called line-pairs.

In order to unify Desargues' definition of Involution with the projective definition, we will write his condition again for three pairs of points in involution; with some manipulation (2) becomes

$$\frac{GD}{CD} \cdot \frac{GF}{CF} = \frac{GB}{CB} \cdot \frac{GH}{CH} \Leftrightarrow \frac{GD}{CD} \div \frac{GB}{CB} = \frac{GH}{CH} \div \frac{GF}{CF}$$

$$\Leftrightarrow (G, C; D, B) = (G, C; H, F).$$

So, renaming the previous pairs conveniently with $(A, A_1); (B, B_1); (C, C_1)$, the equation above becomes $(A, A_1; B, C) = (A, A_1; C_1, B_1)$.

3. *Main theorem and its variants*

A very special characteristic of Desargues' Involution Theorem is the fact that it has many variants, generated both by duality and by degenerate cases of quadrilaterals/quadrangles. In this section, we state these variants.

Let us start with a preliminary result which is part of Desargues' Involution Theorem, but which was first presented by Pappus in his book VII, prop. 130; see ([6, pp. 106-108}. Pappus' statement of this result was expressed in terms of collinearity of points having a particular metric relation. Desargues proved this part using Menelaus' theorem.

Proofs for all these results may be found in [7].

Theorem 1: The three pairs of opposite sides of a complete quadrilateral are cut by any line (not through a vertex) in three pairs of conjugate points on an involution.

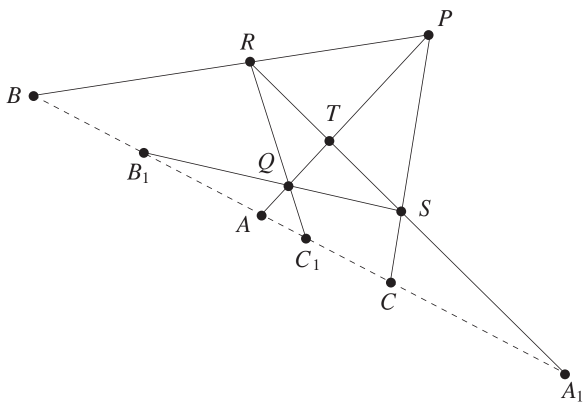

FIGURE 1

Theorem 2: (Desargues' Involution Theorem [7]). If a quadrangle is inscribed in a conic and if a line l not passing through a vertex of the quadrangle intersects the conic at two points, these points are a point-pair of the involution on l determined by the pairs of opposite sides of the quadrangle, i.e. the involution described in the previous theorem.

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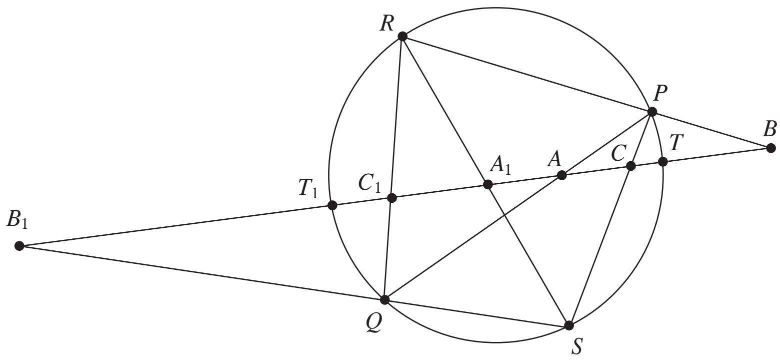

FIGURE 2

Theorem 3: (Dual of Desargues' Involution Theorem [7]). If we consider a quadrilateral circumscribed to a conic and if, from a point P not on a side of quadrilateral, we draw the two tangents to the conic, these tangents constitute a pair of the involution of the pencil from P determined by the pairs of opposite vertices of the quadrilateral.

Both forms of Desargues' Involution Theorem can be stated for degenerate cases of a quadrangle/quadrilateral, when some vertices coincide. We present the theorem for three points. Statements for the two-points cases are obtained analogously.

Theorem 4: (Desargues Involution Theorem - three points). Let ABC be a triangle inscribed in a conic. A line l intersects sides AB, AC and BC of the triangle at X_1, X_2 and Y_2 respectively. The same line intersects the tangent to the conic at A in Y_1 and the conic in two points Z_1, Z_2. Then the pairs of points (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) are pairs of some involution on l.

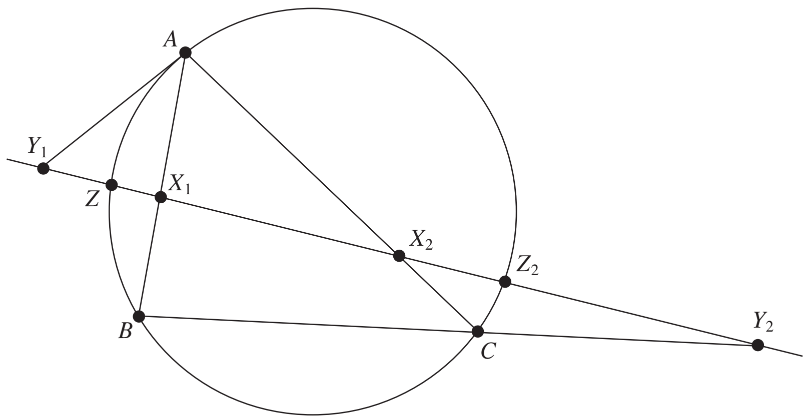

FIGURE 3

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Theorem 5: (Dual of Desargues Involution Theorem - three points). Let ABC be a triangle and let P be a point in the plane. If a conic inscribed in triangle ABC touches side BC at D, and PX_1, PX_2 are the tangents from P to that conic, then (PA, PD), (PB, PC) and (PX_1, PX_2) are pairs of some involution of the pencil of lines passing through P.

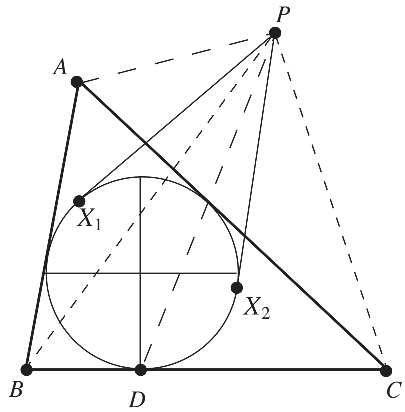

FIGURE 4

4. *Examples*

Desargues' Involution Theorem is one of the most productive tools in elementary geometry. Here we only scratch the surface of its wide applications, but we believe the examples selected are very instructive and demonstrate the utility of this beautiful theorem.

Example 1: (Papelier - [8] page 129). Let ABC be a triangle and let $\mathscr{C}(O, R)$ be its circumscribed circle. If diameter d of circle $\mathscr{C}$ intersects the sides of the triangle at A_1, B_1, C_1 and if A', B', C' are the reflections of A_1, B_1, C_1 in O, prove that AA', BB',CC' are concurrent at a point of the circle $\mathscr{C}$.

Proof: Let D_1, D' be the intersection points of d with $\mathscr{C}$ and let M be the second intersection of AA' and $\mathscr{C}$. Let B'', C'' be the intersections of MB and MC with d, respectively. By Desargues' Involution Theorem applied to the quadrangle $ABCM$ and line d we get an involution with pairs (A_1, A'), (B_1, B'), (C_1C'), (D_1, D'). However, pairs (A_1, A'),(B_1, B''), (C_1C''), (D_1, D') also form an involution, namely the one determined by reflection about point O. Since the last two involutions share the pairs (A_1, A'), (D_1, D'), they are in fact the same involution, meaning that $B' = B''$ and $C' = C''$, implying the desired concurrency on $\mathscr{C}$.

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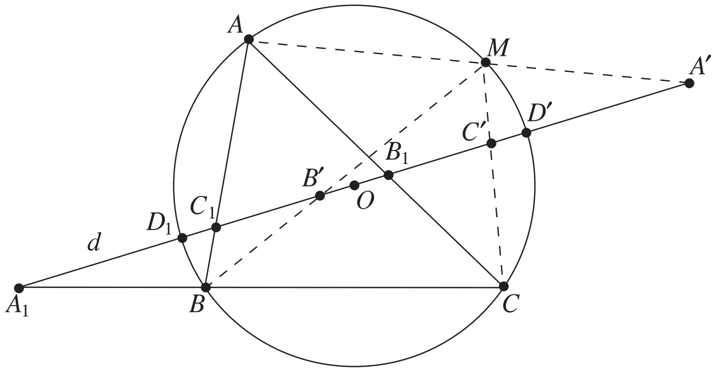

FIGURE 5

Example 2: (Butterfly Theorem). Let M be the midpoint of a chord XY of a circle, through which two other chords AB and CD are drawn. AD and BC intersect chord XY at Q and P respectively. Then M is the midpoint of QP.

Proof: By Desargues' Involution Theorem, pairs (XY), (PQ), (MM) will be in involution, call it f. Let ∞ denote the point at infinity on the support line. Because M, the midpoint of XY, is a fixed point under the involution, the other fixed point under the involution must be at infinity, as

$$(X, Y; \infty, M) = -1 = (X, Y; f(\infty), M)$$

by the involution condition, so that $f(\infty) = \infty$. Now, since (MM), $(\infty\,\infty)$ are pairs, it means that the involution is in fact the reflection about M, meaning that M is the midpoint of PQ.

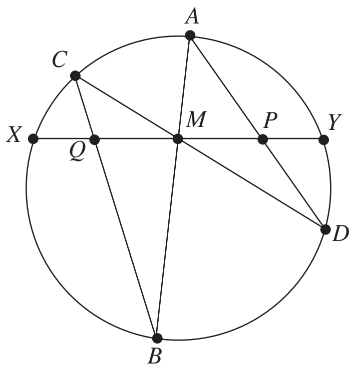

FIGURE 6

Example 3: (Newton-Gauss). The midpoints of diagonals of a complete quadrilateral are collinear.

Proof: Let (AA_1), (BB_1), (CC_1) be the pairs of opposite vertices in the quadrilateral and let S be one of the intersection points of the circles with diameters AA_1 and CC_1. Suppose S is a real point. If we consider the pencil of perpendicular rays (SC, SC_1), (SA, SA_1), by the dual of Desargues' Involution Theorem it follows that (SB, SB_1) is also a pair in this involution. However, each ray is perpendicular to its conjugate in this involution (see Lemma 1 below). Hence S lies on the circle with diameter BB_1. As a result, we can deduce that circles with diameters AA_1, BB_1 and CC_1 are coaxial, meaning that their centres are collinear, and the conclusion follows. We still have to establish the following result.

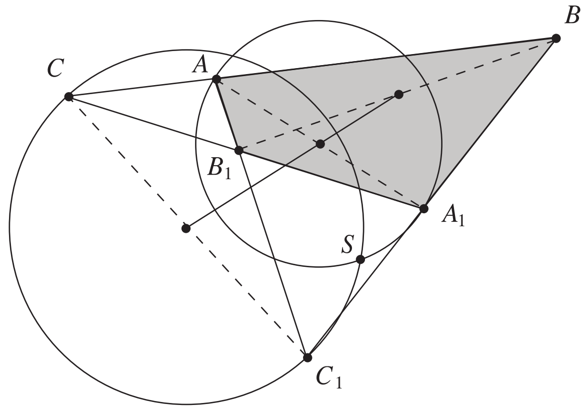

FIGURE 7

Lemma 1: (See [9, p. 33]). If in a pencil in involution, there are two pairs of conjugate rays which are perpendicular to each other, then every ray in this involution is perpendicular to its corresponding ray.

Proof: Let a, a' and b, b' be two pairs of perpendicular lines from a pencil in involution through S. Consider a circle through S and let it meet a, a', b, b' again at A, A', B, B' respectively. Then AA' and BB' are diameters of the circle, meeting at its centre O. So, for any other pair c, c' of the involution (which is well-determined) meeting the circle again at C and C', CC' passes through O, meaning that c, c' are orthogonal.

Example 4: (USAMO 2012). Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A_1, B_1, C_1 be the points where the reflections of lines PA, PB, PC with respect to γ intersect lines BC, AC, AB respectively. Prove that A_1, B_1, C_1 are collinear.

Proof: Let $C'' = A_1B_1 \cap AB$. Considering the quadrilateral $CAC''A_1$, by the dual of Desargues' Involution Theorem, the line-pairs (PA, PA_1), (PB, PB_1) and (PC, PC'') belong to an involution on the pencil with vertex P. But the reflection in γ also determines an involution on the same pencil, with line-pairs (PA, PA_1), (PB, PB_1) and (PC, PC_1). Since the two involutions share two pairs, they coincide, whence $C'' = C_1$.

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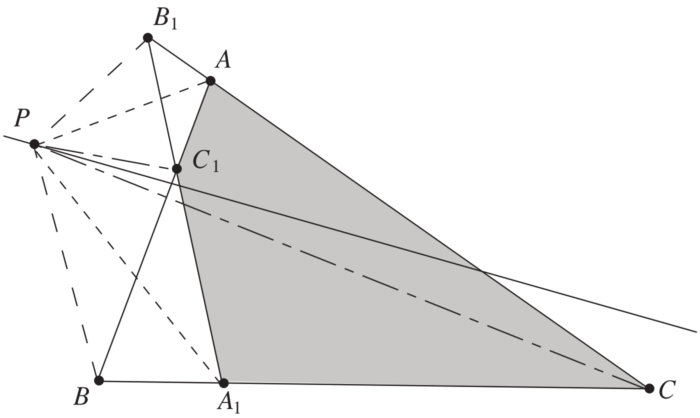

FIGURE 8

Example 5: (IMO Shortlist 2005). Let ABC be a triangle, and let M be the midpoint of side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again at two points X and Y. If the lines AX and AY intersect BC at points P and Q, prove that $BP = CQ$.

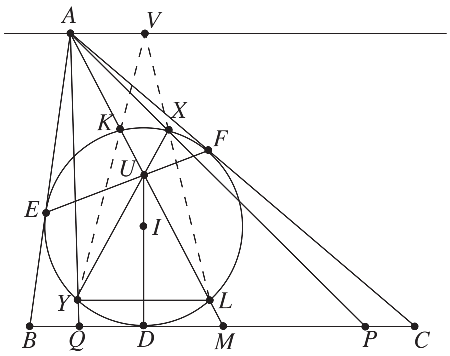

FIGURE 9

Proof: Define P_∞ the point at infinity on BC. Let D, E, F be the contact points of γ with sides BC, AB and AC respectively, and let $U = KL \cap XY$, $V = LX \cap KY$. It is easy to prove that AM, EF, DI are concurrent (see Lemma 2 below). Obviously, their intersection point is U ($KXLY$ is an isosceles trapezium). It follows that U belongs to the polar of A, so by La Hire's Theorem, the polar of U passes through A. On the other hand, U lies on the polar of V by construction, meaning that V lies on the polar of U. Thus AV is the polar of U. But the polar of U is a line parallel to BC because $UI \perp BC$. Hence $AV \parallel BC$.

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By the dual of Desargues' Involution Theorem in quadrilateral $KXLY$, the pairs of lines (AK, AL); (AX, AY) and (AV, AP_∞) are in involution. Projecting this pencil on BC we have an involution of pairs of points (M, M); (P, Q); (P_∞, P_∞), so that M is the midpoint of PQ.

Lemma 2: Let ABC be a triangle, and let D, E, F be the points of contacr of the incircle with BC, AB, AC. Consider M the midpoint of BC and I the incentre of ABC. Then, AM, DI, EF are concurrent.

Proof: Denote by Γ the incircle and by the ∞ the point at infinity on BC. Consider l the line through A parallel to BC, so that l and BC meet at ∞. Let $U = ID \cap EF$, $M' = AU \cap BC$. As EF is the polar of A with respect to the incircle of ABC and U lies on it, by La Hire's theorem, A lies on the polar of U with respect to Γ. Since UI is perpendicular to BC, UI is perpendicular to l, meaning that l is the polar of U with respect to Γ. The last observation implies that $(AE, AF; AU, A\infty) = -1$ and projecting onto BC we obtain $(B, C; M', \infty) = -1$, meaning that $M' = M$ and we are done.

Example 6: (Taiwan Team Selection Test for the IMO). Let M be any point on the circumcircle of triangle ABC. Suppose that the tangents from M to the incircle meet BC at two points X_1 and X_2. Prove that the circumcircle of triangle MX_1X_2 intersects the circumcircle of ABC again at the A-mixtilinear point of contact. (The A-mixtilinear incircle of a triangle ABC is defined as the unique circle tangent internally to the circumcircle of ABC and to the sides AB and AC, respectively. The A-mixtilinear point of contact represents the tangency point between the A-mixtilinear incircle and the circumcircle of triangle ABC.)

Proof: Suppose without loss of generality that M lies on the same side as A with respect to BC. Let γ be the incircle of ABC and let $\gamma \cap BC = D$. The tangency point of the A-mixtilinear incircle and the circumcircle will be denoted by T. By the dual of Desargues' Involution Theorem (the three points variant), there is an involution swapping $(MX_1; MX_2)$; $(MB; MC)$; $(MA; MD)$. If $MA \cap BC = N$, projecting the involution on BC, we get another involution (D, N); (B, C); (X_1, X_2). This is an elliptic involution because M and T will lie on opposite sides of BC, and circles passing through pairs of points in the involution and M will be coaxial. Hence all such circles pass through another common point. We want to show that this point is in fact T.

To this end, it is enough to show that points M, D, T, N lie on the same circle. If $TD \cap \odot ABC = A_1$, then $AA_1 \,//\, BC$ (see Lemma 3 below), so $\angle MTD = \angle MTA_1 = \angle MAA_1 = \angle MND$ and the conclusion follows.

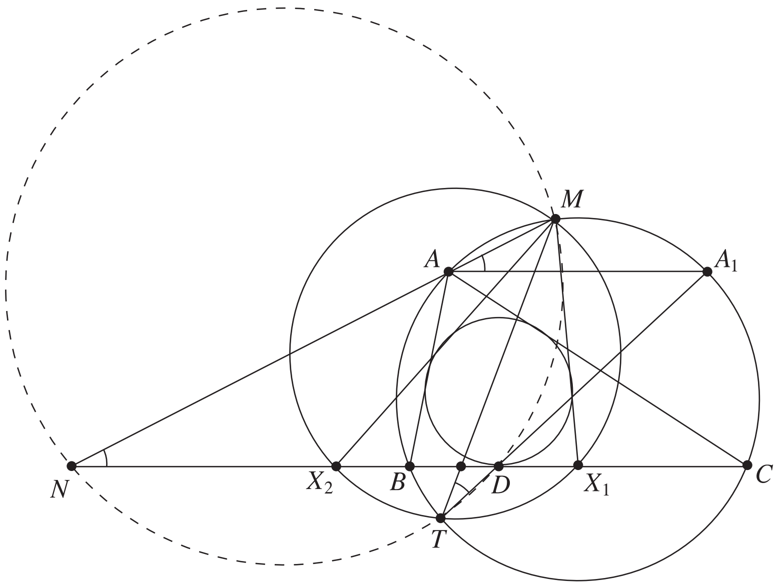

FIGURE 10

Lemma 3: (See [5, p. 69]) Let ABC be a triangle. Let T be the A-mixtilinear point of contact and let D be the tangency point of the incircle and BC. Then, if A_1 is the second intersection of TD and the circumcircle of ABC, AA_1 is parallel to BC.

Example 7: (Author) Consider a triangle ABC and let A_1, B_1, C_1 be the tangency points of the A-excircle with sides BC, AB and AC respectively. Let D be the tangency point of the incircle of triangle ABC with side BC. Let lines A_1C_1 and A_1B_1 intersect the altitude from A of the triangle ABC at M and N respectively. If AD intersects the A-excircle of ABC at R, prove that line A_1R bisects segment $[MN]$.

Proof: Denote by Z the foot of the altitude from A on BC, and by X and Y the intersection points of this altitude with the A-excircle.

By Desargues' Involution theorem (the two points variant) for line AZ and points A_1, C_1 on the circle, it follows that pairs (A, Z), (N, N), (X, Y) are in involution on the line AZ. Similarly, for points A_1 and B_1 on the circle, it follows that pairs (A, Z), (M, M), (X, Y) are in involution on the same line.

Thus there is an involution interchanging the pairs of points (A, Z), (M, M), (N, N) and (X, Y). But any involution on a line is an inversion of some non-zero power (see the lemma below), so this involution is an inversion centred at the midpoint of segment $[M, N]$, because these points are their own mates in this involution. Moreover, the centre of the involution, say P, lies on the radical axis of the A-excircle, $\odot A_1XY$ and $\odot A_1AZ$.

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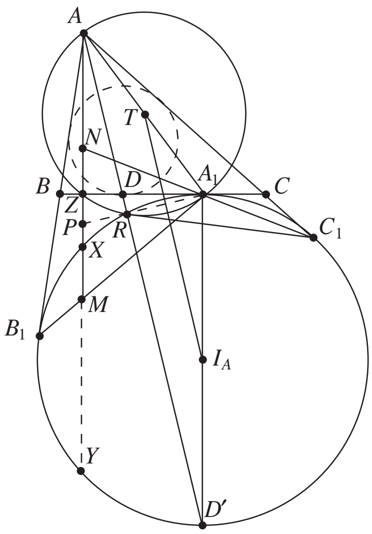

FIGURE 11

Let A_1R meet AZ in P, and let D be diametrically opposite to A_1 on the excircle. By homothety, A, D, D' and R are collinear. Then $\angle D'RA_1 = \frac{\pi}{2}$, so $\angle ARA_1 = \frac{\pi}{2}$ as well. Since $\angle AZA_1 = \frac{\pi}{2}$, the point R lies on $\odot AZA_1$. Thus A_1R is the radical axis of $\odot AZA_1$ and $\odot XYA_1$, and it follows that (P, ∞) is another pair of the involution on AZ. So $(M, N; P, \infty) = -1$, whence the result.

Lemma 4: Any involution on a line is an inversion of some non-zero (possibly negative) power.

Proof: Solution by AoPS user Tina Sprout. Let P, X, X_1, Y, Y_1 be some points in involution on a line l such that X, X_1 and Y, Y_1 are reciprocal pairs. If we choose P such that its reciprocal point is P_∞, the cross-ratio of the previous points will be

$$(P, P_\infty; X, Y) = (P_\infty, P; X_1, Y_1)$$

$$\Rightarrow \frac{PX}{PY} = \frac{PY_1}{PX_1} \Rightarrow PX \cdot PX_1 = PY \cdot PY_1$$

which means that this involution is an inversion around P.

Acknowledgement

The author wishes to thank the referee of *The Mathematical Gazette* for their review of the article and their valuable advice. Their solutions to the Example problems were partially integrated in the final form of the article, in order to clarify the exposition.

References

1. J. V. Field, J. Gray, *The geometrical work of Girard Desargues*, Springer-Verlag (1987).
2. C. Taylor, *An introduction to the ancient and modern geometry of conics*, Deighon Bell (1881).
3. H. S. M. Coxeter, *Projective geometry* (2nd edn.), Springer-Verlag (1987).
4. H. S. M. Coxeter, *Introduction to geometry* (2nd edn.), John Wiley and Sons (1969).
5. E. Chen, *Euclidean geometry in mathematical olympiads*, Mathematical Association of America (2016).
6. L. Cremona, *Elements of projective geometry* (2nd edn.), Clarendon Press (1885).
7. F. Ayres, Jr., *Theory and problems of projective geometry*, Schaum Publishing Co. (1967).
8. L. Nicolescu, W. G. Boskoff, *Probleme practice de geometrie* (Practical problems in geometry) Romanian language, Bucharest: Ed. Tehnica (1990).
9. A. G. Pickford, *Elementary projective geometry*, Cambridge University Press (1909).

10.1017/mag.2023.6

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