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BIFURCATION AT INFINITY FOR EQUATIONS IN SPACES OF VECTOR-VALUED FUNCTIONS

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Abstract

New existence conditions, under which an index at infinity can be calculated, are given for bifurcations at infinity of asymptotically linear equations in spaces of vector-valued functions. The case where a bounded nonlinearity has discontinuous principal homogeneous part is considered. The results are applied to 2π-periodic problems for two–dimensional systems of ordinary differential equations and to a vector two-point boundary value problem.

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1. Introduction

The word bifurcation is very widely used in modern scientific language [10]. Essentially, it is concerned with the qualitative changes in the dynamical behavior of a system that may occur when parameters of the system are subjected to small changes. In a narrower sense, in mathematical bifurcation theory a particular parameter value in an equation with parameters is called a *bifurcation point* if new solutions arise or current solutions cease to exist, or both, for nearby parameter values. Over the past decade problems of *bifurcation at infinity*, that is, where arbitrarily large solutions arise at certain parameter values, have attracted considerable attention, for example [3, 4, 5, 9, 11, 12, 13]. This has been motivated, on the one hand, by a large number of applications and, on the other hand, by the need for a new mathematical perspective. The obvious idea of inverting a state variable and reducing the problem to a standard

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one of bifurcation at zero is not particular productive here because the associated superposition operators that are usually present cease to be superposition operators under inversion. Moreover, there is usually no convenient analytical way of representing solutions with very large norm. Consequently topological methods similar to the principle of changing index are especially important here. Such methods are the main topic of this paper.

Consider the equation $B(x, \lambda) = 0$ in a Banach space E for some operator $B(x, \lambda)$ which depends on a parameter $\lambda \in \Lambda = [a, b]$.

DEFINITION 1. A parameter value λ_0 is called a *bifurcation point at infinity* or, equivalently, an *asymptotic bifurcation point* if for every $\varepsilon > 0$ there exists a $\lambda_\varepsilon \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \bigcap \Lambda$ such that the equation $B(x, \lambda_\varepsilon) = 0$ has at least one solution x_ε satisfying $\|x_\varepsilon\| > \varepsilon^{-1}$.

The notion of an asymptotic bifurcation point was introduced in the early 1950s by Mark Krasnosel'skii, who initiated their study by topological methods with the so-called principle of changing index [8, 1]. This principle is applicable for equations $B(x, \lambda) = 0$ of the type $x = T(x, \lambda)$ where the operator $T(x, \lambda)$ is completely continuous (that is, compact and continuous) in the both of its variables. An operator $\Phi x = x - Tx$ is called a *vector field* and is said to be *completely continuous* when the operator T is completely continuous.

DEFINITION 2. Let a completely continuous vector field Φx be defined and non-degenerate for $\|x\| \geq r_0$. Then the *rotation* (see [8]) of this field on the boundary of every ball $B(r, 0) = \{x \in E;\ \|x\| \leq r\}$ is defined and has a common value for all $r > r_0$ which is called the *index at infinity* of the field Φx and is denoted by $\operatorname{ind}_\infty \Phi$.

If the value of the index at infinity of a field $\Phi_\lambda x = x - T(x, \lambda)$ is not defined for some parameter value $\lambda = \lambda_0$, then this value λ_0 is an asymptotic bifurcation point of the equation $x - T(x, \lambda) = 0$.

PROPOSITION 1 (Principle of changing index [8]). *Consider an equation $x = T(x, \lambda)$ in the Banach space E where the operator $T(x, \lambda)$ is completely continuous in both variables $x \in E$ and $\lambda \in [a, b]$. Suppose that the indices at infinity of the field $\Phi_\lambda x = x - T(x, \lambda)$ are defined for two different parameter values λ_1 and λ_2 and satisfy*

$$\operatorname{ind}_\infty \Phi_{\lambda_1} \neq \operatorname{ind}_\infty \Phi_{\lambda_2}. \tag{1}$$

Then there exists at least one asymptotic bifurcation point for the equation $x = T(x, \lambda)$ in the interval $[\lambda_1, \lambda_2]$.

This statement and its various reformulations are called the *principle of changing index*. The reformulations are mainly related to the problem of how to find two such values of the parameter with different indices. The following variation is the most widely used.

Let the index at infinity of the field Φ_λ be defined for every λ from some neighborhood of λ_0. Suppose that this index is constant for $\lambda < \lambda_0$ with value $\operatorname{ind}_\infty \Phi_{\lambda_0-0}$ and that it is also constant for $\lambda > \lambda_0$ with value $\operatorname{ind}_\infty \Phi_{\lambda_0+0}$.

PROPOSITION 2 ([8]). *Suppose that at least two numbers of the three numbers*

$$\operatorname{ind}_\infty \Phi_{\lambda_0}, \qquad \operatorname{ind}_\infty \Phi_{\lambda_0-0}, \qquad \operatorname{ind}_\infty \Phi_{\lambda_0+0}$$

are defined and different. Then λ_0 is an asymptotic bifurcation point for the equation $x = T(x, \lambda)$.

If the index $\operatorname{ind}_\infty \Phi_{\lambda_0}$ is defined and differs from zero, then multiplicity results are valid for parameter values close to λ_0 under assumptions of Proposition 2. For example, if $\operatorname{ind}_\infty \Phi_{\lambda_0} = 1$ and $\operatorname{ind}_\infty \Phi_{\lambda_0+0} = -1$, then, generally speaking, for parameter values $\lambda > \lambda_0$ close to λ_0 the equation $x = T(x, \lambda)$ has three solutions: one with index 1 which is bounded for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$, and two branches of solutions with index -1 which tend to infinity as $\lambda \to \lambda_0$.

The typical situation in applications is where nothing is known about the index $\operatorname{ind}_\infty \Phi_{\lambda_0}$, but the other two numbers, $\operatorname{ind}_\infty \Phi_{\lambda_0-0}$ and $\operatorname{ind}_\infty \Phi_{\lambda_0+0}$, are known and are different.

Most theorems on asymptotic bifurcation points concern asymptotically linear equations, which will be defined in the next section. In this case, if some value of the parameter is an asymptotic bifurcation point, then the kernel of the principal at infinity linear part $x - A_\lambda x$ of the field $x - T(x, \lambda)$ is non-trivial. If the parameter is included in the principal linear part as a multiplier, that is $A_\lambda x = \lambda A x$, then any eigenvalue μ of the linear operator A of odd multiplicity (for example, a simple eigenvalue) generates an asymptotic bifurcation point $\lambda = \mu^{-1}$.

This paper is organized as follows. In the next section we define an asymptotically linear vector field and state a theorem on the calculation of the index for a non-degenerate asymptotically linear completely continuous vector field. Asymptotically homogeneous nonlinearities are introduced in Section 3 and a theorem on the calculation of the index at infinity is stated for vector fields in abstract Banach spaces with a degenerate linear part and a continuous non-degenerate asymptotically homogeneous nonlinearity. Section 4 contains a new theorem on the asymptotic homogeneity of superposition operators, which is proved in Section 5. The corresponding theorem on the index calculation for fields in spaces of vector-valued functions which have degenerate linear part and a discontinuous non-degenerate asymptotically homogeneous nonlinearity is presented in Section 6. Such types of nonlinearities appear naturally in applications, examples of which are considered in Sections 7 and 8, while some concluding remarks are given in Section 9.

2. Index at infinity of non-degenerate asymptotically linear vector fields

A vector field $\Phi(x)$ is called *linear* if it can be represented as $\Phi x = x - Ax$, where A is a linear operator. A linear vector field is always zero at $x = 0$ and except this singular point a linear vector field either has no other singular points at all (if 1 does not belong to the spectrum of the operator A) or it degenerates on a non-trivial subspace (if 1 belongs to the spectrum of A).

If 1 is a regular value for a completely continuous linear operator A, then 0 is an isolated (in fact, the unique) singular point of the vector field $\Phi x = x - Ax$ and its index coincides with the index of the vector field Φx at infinity. The rotation of this vector field on the boundary of a given domain $\mathscr{D}$ is either equal to zero if $0 \notin \mathscr{D}$ or coincides with the index of zero if $0 \in \mathscr{D}$.

PROPOSITION 3. *Let β denote the sum of multiplicities of all real eigenvalues of A which are greater than* 1. *Then*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta. \tag{2}$$

For a proof of this assertion see, for instance, [8].

DEFINITION 3. An operator T and vector field $\Phi x = x - Tx$ are called *asymptotically linear* if T admits the representation $Tx = Ax + Fx$ where A is a linear operator and F is an operator which satisfies $\lim_{\|x\|\to\infty} \|Fx\|/\|x\| = 0$. The operator A is called the *asymptotic derivative* of the asymptotically linear operator T, or the *derivative of T at infinity*, while the linear vector field $x - Ax$ is called the *main linear part* of the vector field $x - Tx$. The main linear part is said to be *non-degenerate* if 1 does not belong to the spectrum of the operator A, otherwise *degenerate*.

Asymptotic derivatives of completely continuous operators are always completely continuous [8].

The following theorem of Leray and Schauder follows from theorems on calculating of rotation of a vector field in terms of its main part.

PROPOSITION 4 ([8, 1]). *Let a vector field $\Phi x = x - Tx$ be asymptotically linear with the non-degenerate main linear part $x - Ax$. Then the index of the vector field Φ at infinity is defined and is given by* $\operatorname{ind}_\infty \Phi = (-1)^\beta$, *where β denotes the sum of multiplicities of real eigenvalues of A which are greater than* 1.

The main part of the present paper is devoted to the calculation of the index at infinity of asymptotically linear vector fields with a degenerate main linear part. In the pioneering papers [9, 11], conditions of the form

$$\lim_{x\to+\infty} f(x) = f^+, \qquad \lim_{x\to-\infty} f(x) = f^-$$

allow the index at infinity to be calculated for Hammerstein type vector fields $\Phi x = x - A(x + f(x))$ with bounded scalar nonlinearities $f(x) : \Omega \to \mathbb{R}^1$. An extensive literature is devoted to investigations of concrete boundary value problems with such nonlinearities; see [2] and references therein.

A generalization of such results to vector-valued functions have been carried out in [3] to provide results on the calculation of the index at infinity of vector fields with a degenerate main linear part and with a non-degenerate next order term. They depend heavily on the continuity of the next order (after linear) non-degenerate term of the vector field.

3. Asymptotically homogeneous vector fields

The results of this section on vector fields in Banach spaces were announced in [4] and proved in [5]. They will be extended in subsequent sections of this paper.

DEFINITION 4. A nonlinear operator Q in the Banach space E is said to be *homogeneous of degree* 0, or just *homogeneous*, if

$$Q(x) = Q(\lambda x), \qquad \forall\lambda > 0, \quad x \in E.$$

Any constant vector field is homogeneous by definition and linear combinations of homogeneous vector fields are also homogeneous. Only functions of the form

$$q(x) = \begin{cases} q^-, & x < 0, \\ q^0, & x = 0, \\ q^+, & x > 0, \end{cases} \tag{3}$$

are homogeneous for a one-dimensional space E. In general, a homogeneous nonlinearity is determined by its values on the unit sphere and at the coordinate origin.

If A is a linear operator and Q is a homogeneous operator, then the operator QA is homogeneous. In fact, FQ is homogeneous for an arbitrary operator F.

In spaces of, for instance, scalar-valued functions defined on a given set $\Omega \subset \mathbb{R}^m$, a superposition operator $x(t) \mapsto q(t, x(t))$ is homogeneous if it is generated by a

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homogeneous function $q(t, x)$ which admits a representation (3) at each t, that is with

$$q(t,x) = \begin{cases} q^-(t), & x < 0, \\ q^0(t), & x = 0, \\ q^+(t), & x > 0. \end{cases} \tag{4}$$

In spaces of vector functions $\Omega \to \mathbb{R}^n$ examples of homogeneous nonlinearities can be given by the superposition operators $x(t) \mapsto f(t, x(t))$ generated by functions $f(t, x) = C(t)x/|x|$ where $C(t)$ is a $n \times n$ matrix and $|\cdot|$ is a given norm in $\mathbb{R}^n$. Functions $f(t, x)$ of the form $f(t, \operatorname{sign} x_1, \dots, \operatorname{sign} x_n)$ also generate homogeneous superposition operators.

If a homogeneous operator is not constant, then it must be discontinuous at zero. Such operators can also have other discontinuity points. A natural example of a discontinuous homogeneous operator on the plane $\mathbb{R}^2$ is given by the superposition operator $(x_1, x_2) \mapsto (\operatorname{sign} x_1, 0)$ which is discontinuous not only at zero, but also everywhere on the straight line $x_1 = 0$.

The superposition operator $Qx(t) = q(t, x(t))$ generated by the function (4) is also discontinuous in function spaces. If, for instance, $q(t, x) = q(x)$ and $q^- \neq q^+$, then the discontinuity points of Q are dense in the L^p spaces. The totality of such points is also dense in the space C outside of the sets of strictly positive or strictly negative functions. Nevertheless, this operator has a sufficient quantity of continuity points for it to still be useful in many applications. Criteria for the continuity of a superposition operator with discontinuous characteristics at a given point in spaces of integrable functions can be found in [7].

In the spaces L^∞ or C the operator Q can be discontinuous even at 'very nice' functions $x_0(t)$ which are equal to zero only at a single point. For instance, the operator $x(t) \mapsto \operatorname{sign} x(t)$ for $t \in [0, 1]$ is discontinuous at the function $x^*(t) = t - 1/2$ as an operator from C to L^∞. Fortunately, superposition operators Q are often combined with linear integral operators A which possess some substantial improvability properties. The operator $x(t) \mapsto \operatorname{sign} x(t)$ is continuous at the function $x^*(t)$ as an operator from the space L^∞ to the space L^2 provided that the function $x(t)$ vanishes only at the set of zero Lebesgue measure, while a linear operator A is often continuous as an operator acting from the space L^2 back to the space L^∞. As a result the operator AQ is continuous in L^∞ at all points $x_0 = x_0(t)$ satisfying the condition

$$\operatorname{mes}\{t \in \Omega : x_0(t) = 0\} = 0.$$

Let E_1 be a finite dimensional subspace of a Banach space E and P_1 a fixed projector on this subspace, so $P_1 E = E_1$ and $P_1^2 = P_1$.

DEFINITION 5 ([4]). An operator F is said to be *asymptotically homogeneous* in the space E (with respect to the subspace E_1 and the projector P_1) if it can be represented as the sum $F = Q + B$ where the operator Q is homogeneous and the operator B satisfies the following 'vanishing at infinity' condition:

$$(5)\qquad \lim_{R\to+\infty} \sup_{e_1\in E_1,\ \|e_1\|=1,\ h\in E,\ \|h\|<c} \|P_1B(Re_1+h)\| = 0$$

for each $c > 0$.

The main example of an asymptotically homogeneous operator in a function space is given by the superposition operator $f(t,x) = q(t,x) + \psi(t,x)$, where the function $q(t,x)$ is homogeneous and $\psi(t,x)$ satisfies the condition

$$(6)\qquad \lim_{|x|\to\infty} \sup_{t\in\Omega} |\psi(t,x)| = 0.$$

If q is continuous on the unit sphere S, then the equality

$$\lim_{R\to+\infty} \sup_{e_1\in E_1,\ \|e_1\|=1,\ h\in E,\ \|h\|_{L^1}<c} \|\psi(Re_1+h)\|_{L^1} = 0,$$

which is stronger than (5), can often be established for such operators. The corresponding operator is then asymptotically homogeneous with respect to an arbitrary projector on a subspace E_1 of L^1. On the other hand, if $f(t,x)$ satisfies a Caratheodory condition (that is, is continuous in x and measurable in t) and $q(t,x)$ is discontinuous in x at some points of S, then (6) is never valid.

Let us return now to the calculation of the index of a completely continuous asymptotically linear vector field $\Phi x = x - Ax - Fx$ with a degenerate linear part $x - Ax$. Let $E_1 = \operatorname{Ker}(I - A)$ and suppose that $1 \in \sigma(A)$ has no generalized eigenvector. That is, $E_1 = \{e(t) : Ae = e\}$. Then there exists a projection $P_1 : E \to E_1$ which commutes with A. For example, an explicit construction is given for such a projector in Section 6 for the case $E = L^2$.

THEOREM 1 ([5]). *Let the operator $F = Q + B$ be asymptotically homogeneous, where Q is homogeneous and B satisfies condition* (5) *with a finite dimensional subspace E_1 and projector P_1 defined by the linear operator A. Suppose that the finite dimensional vector field P_1Qe on the sphere $U = \{e \in E_1,\ \|e\| = 1\}$ is nondegenerate, that is $P_1Qe \neq 0$ for all $e \in U$, and that the operator $P_1Q : E \to E_1$ is continuous at each point of U. Then the index* $\operatorname{ind}_\infty \Phi$ *is defined and given by*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \gamma(P_1Q, U),$$

where $\gamma(P_1Q, U)$ denotes the rotation of the vector field P_1Q on the sphere U in the finite dimensional subspace E_1.

In applications the subspace E_1 is often one or two dimensional, so the rotation $\gamma(P_1Q, U)$ can be calculated explicitly.

4. Asymptotic homogeneity of superposition operators in spaces of vector functions

Let Ω be a closed bounded domain in a finite dimensional space such as $[0, 1]$ in $\mathbb{R}^1$ or a square or a circle in $\mathbb{R}^2$. We will consider operators, vector fields and equations in spaces E of functions $x(t) : \Omega \to \mathbb{R}^n$. Denote by $\langle \cdot, \cdot \rangle$ the scalar product in the space $\mathbb{R}^n$ and by $|\cdot|$ the corresponding norm.

Consider an arbitrary finite dimensional subspace $E_1 \subset E$ of vector-valued functions which are continuous on Ω. Denote $U = \{e(t) : e(t) \in E_1, \|e\| = 1\}$ and suppose that each non-zero function $e(t) \in E_1$ satisfies (see [2] and the references therein)

$$\text{mes}\,\{t \in \Omega : e(t) = 0\} = 0. \tag{7}$$

Let us fix a closed set $\Delta \subset S$ on the unit sphere $S = \{x \in \mathbb{R}^n : |x| = 1\} \subset \mathbb{R}^n$. Generally, in applications, this set is 'small', often having codimension 2. Denote by $\rho(u, \Delta)$ the distance between a point u in the sphere and the set Δ and for each function $e(t) \in E_1$ write

$$\chi(\delta, \Delta, e) = \text{mes}\left\{t \in \Omega : \rho\left(\frac{e(t)}{|e(t)|}, \Delta\right) \le \delta\right\}.$$

The main assumptions in the theorem formulated below on asymptotic homogeneity of the superposition operator $x(t) \mapsto f(t, x(t))$ are the following: there exist a set Δ such that

(1) The limit

$$\lim_{R\to+\infty} f(t, Ru) = q(t, u) \tag{8}$$

exists for each $u \in S \setminus \Delta$. The limit in (8) is supposed to be uniform in $t \in \Omega$ and in u belonging to any given closed subset of S which is disjoint with Δ, and the limit function $q(t, u)$ satisfies a Caratheodory condition for $u \notin \Delta$.

(2) The equality

$$\chi(0, \Delta, e) = 0. \tag{9}$$

holds for each function $e(t) \in E_1$.

Assumption 1 can be reformulated as follows:

(1*) The equality

$$\lim_{R\to+\infty} \sup_{t\in\Omega,\ u\in\Delta_*} |f(t, Ru) - q(t, u)| = 0 \tag{10}$$

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holds for each $\Delta_* \subset S$ such that $\overline{\Delta}_* \bigcap \Delta = \emptyset$.

Equality (7) together with the main assumption guarantees that the operator

$$Qx(t) = \begin{cases} q\,(t, x(t)/|x(t)|), & x(t) \neq 0, \\ 0, & x(t) = 0 \end{cases} \tag{11}$$

is continuous as an operator in L^1 (and in other L^p spaces for $p < \infty$) at every point of U (see [7]). The compactness of U guarantees the uniform continuity of this operator on U.

Let us suppose also that the functions $f(t, x)$ and $q(t, u)$ are both uniformly bounded.

THEOREM 2. *The operator $x(t) \mapsto f(t, x(t))$ is asymptotically homogeneous in the space $E = L^2 = L^2(\Omega, \mathbb{R}^n)$ under the assumptions listed above.*

This theorem was proved in [3] with other terminology for the case $\Delta = \emptyset$. The closure G of the totality of discontinuity points of the function $q(t, u)$ can be taken as the set Δ. Theorem 2 can be generalized to the case when the set G varies with t. Note also that all that is said in this section is interesting only for vector-valued functions, since for scalar functions the sphere S consists only of two points and the question of the continuity of the corresponding functions does not appear, that is, $q(t, u)$ is automatically continuous at both points of the sphere in one–dimensional space.

An example where condition (9) does not hold will be given in Section 9. Note also that with the use of Theorem 2 the solvability results in [12] can be generalized.

5. Proof of Theorem 2

We first prove a lemma.

LEMMA 1. *Let*

$$\chi(\delta, \Delta, E_1) \stackrel{\text{def}}{=} \sup_{e(t) \in U} \chi(\delta, \Delta, e). \tag{12}$$

Then

$$\lim_{\delta \to 0} \chi(\delta, \Delta, E_1) = 0. \tag{13}$$

PROOF. Suppose the contrary. Then there exists a number $\varepsilon > 0$ and a sequence of functions $e_n(t) \in U$ satisfying the inequalities $\chi\,(1/n, \Delta, e_n) > \varepsilon$, or, what is the same,

$$\operatorname{mes}\left\{t \in \Omega : \rho\left(\frac{e_n(t)}{|e_n(t)|}, \Delta\right) \leq \frac{1}{n}\right\} > \varepsilon.$$

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Since E_1 is finite dimensional, and hence all the norms in E_1 are equivalent, we can suppose without loss of generality that the sequence $e_n(t)$ converges uniformly to a function $e^*(t) \in U$. The continuity of measure, condition (7) and the finite dimensionality of the subspace E_1 together imply that

$$\lim_{\delta\to 0} \sup_{e(t)\in U} \operatorname{mes}\{t \in \Omega : |e(t)| \le \delta\} = 0. \tag{14}$$

Therefore

$$\operatorname{mes}\left\{t \in \Omega : \rho\left(\frac{e_n(t)}{|e_n(t)|}, \Delta\right) \le \frac{1}{n},\ |e_n(t)|,\ |e^*(t)| > \delta_0\right\} > \frac{\varepsilon}{2} \tag{15}$$

for all sufficiently large n and some fixed δ_0. However, inequality (15) contradicts (9) with $e = e^*$ because

$$\delta_n = \sup_{|e_n(t)|, |e^*(t)| > \delta_0} \left| \frac{e^*(t)}{|e^*(t)|} - \frac{e_n(t)}{|e_n(t)|} \right|$$

tends to zero as $n \to \infty$ and

$$\begin{aligned} &\operatorname{mes}\left\{t \in \Omega : \rho\left(\frac{e_n(t)}{|e_n(t)|}, \Delta\right) \le \frac{1}{n},\ |e_n(t)|,\ |e^*(t)| > \delta_0\right\} \\ &\quad \le \operatorname{mes}\left\{t \in \Omega : \rho\left(\frac{e^*(t)}{|e^*(t)|}, \Delta\right) \le \frac{1}{n} + \delta_n\right\} = \chi\left(\frac{1}{n} + \delta_n, \Delta, e^*\right) \to 0. \end{aligned}$$

This proves the lemma.

Let us now complete the proof of Theorem 2. To this end we will prove the equality

$$\lim_{R\to+\infty} \sup_{e\in U,\ \|h\|_{L_1}\le c} \left\| f(t, Re + h) - q\left(t, \frac{e(t)}{|e(t)|}\right) \right\|_{L^p} = 0, \tag{16}$$

for $p \in [1, \infty)$, which is stronger than (5) with $E = L^2$. This equality for $p > 1$ follows from the same equality for $p = 1$ by virtue of the uniform boundedness of the functions $f(t, x)$ and $q(t, u)$.

To estimate the value of

$$J = \int_\Omega \left| f(t, Re(t) + h(t)) - q\left(t, \frac{e(t)}{|e(t)|}\right) \right| dt$$

we choose an arbitrary $\varepsilon > 0$ and will show that $J < \varepsilon$ holds for all sufficiently large R. By the Chebyshev inequality,

$$\operatorname{mes}\{t \in \Omega : |h(t)| > \mu\} \le \|h\|_{L^1}/\mu$$

and by the boundedness of the functions $f(t,x)$ and $q(t,x)$ the inequality

$$\int_{\{t:\,|h(t)|>\mu\}} \left| f(t, Re(t)+h(t)) - q\left(t, \frac{e(t)}{|e(t)|}\right)\right| dt \le \frac{\varepsilon}{5}$$

is valid for some sufficiently large μ and all R. By (14) and the boundedness of the functions $f(t,x)$ and $q(t,x)$, the analogous inequality

$$\int_{\{t:\,|e(t)|\le\delta\}} \left| f(t, Re(t)+h(t)) - q\left(t, \frac{e(t)}{|e(t)|}\right)\right| dt \le \frac{\varepsilon}{5}$$

is also valid for sufficiently small δ for all R. Let us surround the set Δ on the sphere S with a sufficiently small neighbourhood $N = \{u : \rho(u,\Delta) < \eta\}$. By Lemma 1 the point $e(t)/|e(t)|$ belongs to this neighbourhood N for t from a set $G(e,\eta)$ which has arbitrarily small measure uniformly with respect to all $e(t) \in U$. Let us fix now a neighbourhood N satisfying the inequality

$$\int_{G(e,\eta)} \left| f(t, Re(t)+h(t)) - q\left(t, \frac{e(t)}{|e(t)|}\right)\right| dt < \frac{\varepsilon}{5}.$$

In what follows the values μ, δ and the set $G = G(e,\eta)$ are supposed to be fixed. Denote

$$\Omega^* \stackrel{\text{def}}{=} \{t \in \Omega : |h(t)| \le \mu,\ |e(t)| > \delta,\ t \notin G(e,\eta)\}.$$

The inequality $J < \varepsilon$ for large R will be true if we can show that

$$J_1 = \int_{\Omega^*} \left| q\left(t, \frac{Re(t)+h(t)}{|Re(t)+h(t)|}\right) - q\left(t, \frac{e(t)}{|e(t)|}\right)\right| dt$$

and

$$J_2 = \int_{\Omega^*} \left| f(t, Re(t)+h(t)) - q\left(t, \frac{Re(t)+h(t)}{|Re(t)+h(t)|}\right)\right| dt$$

satisfy the estimates $J_1, J_2 \le \varepsilon/5$ for all sufficiently large R. For this, note that for large R and for $t \in \Omega^*$ the value of

$$\left| \frac{e(t)}{|e(t)|} - \frac{Re(t)+h(t)}{|Re(t)+h(t)|} \right|$$

can be made arbitrarily small uniformly with respect to e, h and t. Therefore we can suppose without loss of generality that the both of

$$\frac{e(t)}{|e(t)|} \quad \text{and} \quad \frac{Re(t)+h(t)}{|Re(t)+h(t)|}$$

are uniformly separated from the set Δ for all sufficiently large R for $t \in \Omega^*$. Hence J_2 tends to zero as $R \to \infty$ by the assumption (8) and the fact that $|Re(t)+h(t)| \to \infty$ uniformly, while J_1 tends to zero by the uniform continuity of superposition operator (11).

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6. An index theorem

In this section we will again use the space $L^2 = L^2(\Omega, \mathbb{R}^n)$ of square–integrable functions $x(t) : \Omega \to \mathbb{R}^n$ with the usual norm $\|\cdot\|_{L^2}$ generated by the scalar product in $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$, that is,

$$\|\cdot\|_{L^2} = \sqrt{(\cdot,\cdot)}, \qquad (x, y) = \int_\Omega \langle x(t), y(t)\rangle\, dt.$$

Let $A : L^2 \to L^2$ be a completely continuous linear operator and suppose that $f(t, x) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded function satisfying a Caratheodory condition. Consider in L^2 the completely continuous vector field

$$\Phi x = x - A(x + f(t, x)), \tag{17}$$

which is asymptotically linear with asymptotic derivative $I - A$. If $1 \notin \sigma(A)$, where $\sigma(A)$ is the spectrum of the operator A, then $\operatorname{ind}_\infty \Phi = (-1)^\beta$, where β is the sum of multiplicities of all real eigenvalues of the operator A greater than 1.

On the other hand, if $1 \in \sigma(A)$ then the asymptotic derivative $I - A$ is degenerate and some properties of the nonlinearity $f(t, x)$ must be used to compute the index. Let $E_1 = \operatorname{Ker}(I - A)$ and suppose that $E_1 = \{e(t) : \ Ae = e\}$ holds (this means that the eigenvalue 1 of A does not have a generalized eigenvector). Denote by P_1 a projector onto E_1 which commutes with A, which can be constructed as follows. Let $e_1, \dots, e_m$, where $m = \dim E_1$, be a basis of E_1 and let $g_1, \dots, g_m$ be a basis in $E_1^* = \operatorname{Ker}(I - A^*) \subset L^2$ which satisfy

$$\int_\Omega \langle e_i(t), g_j(t)\rangle\, dt = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol. The projector P_1 can then be defined as

$$P_1 x(\cdot) = \sum_{i=1}^m e_i(\cdot) \int_\Omega \langle g_i(t), x(t)\rangle\, dt.$$

THEOREM 3. *Suppose that the bounded nonlinearity $f(t, x)$ satisfies the conditions of Theorem 2 for some set Δ and function $q(t, u)$ and that vector field $\Psi e = P_1 q\,(t, e(t)/|e(t)|)$ is non-degenerate on U. Then*

$$\operatorname{ind}_\infty \Phi = (-1)^\beta \gamma(\Psi, U).$$

Theorem 3 follows immediately from Theorems 1 and 2. Analogues of Theorems 2 and 3 can be formulated for the space L^p with $p \neq 2$ and used to study nonlinear degenerate elliptic PDE.

7. Example 1

Consider the 2-dimensional system

$$\begin{cases} x_1' = x_2 + \arctan(x_1) + b_1(t,\lambda), \\ x_2' = -x_1 + \arctan(x_2) + b_2(t,\lambda), \end{cases} \tag{18}$$

which can be rewritten vectorially as

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t,\lambda)$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \arctan(x_1) \\ \arctan(x_2) \end{pmatrix}, \quad \mathbf{b}(t,\lambda) = \begin{pmatrix} b_1(t,\lambda) \\ b_2(t,\lambda) \end{pmatrix},$$

where the functions $b_j(t,\lambda)$ are 2π-periodic in t and continuous in both variables.

We are interested in the existence of 2π-periodic solutions of this system and its asymptotic bifurcation points. The difficulty here is that the linear part $\mathbf{x}' - A\mathbf{x}$ is degenerate for the 2π-periodic problem, specifically the equation $\mathbf{x}' = A\mathbf{x}$ has a two–dimensional subspace E_1 of 2π-periodic solutions with an orthonormed basis

$$\mathbf{e}_1(t) = \frac{1}{\sqrt{2\pi}}\{\sin t, \cos t\}, \qquad \mathbf{e}_2(t) = \frac{1}{\sqrt{2\pi}}\{\cos t, -\sin t\}.$$

Consider the function

$$\varphi(\lambda) = \left| \int_0^{2\pi} (b_2(t,\lambda) + i b_1(t,\lambda)) e^{-it}\, dt \right| - 8$$

where $|\cdot|$ is the complex modulus.

THEOREM 4. *If $\varphi(\lambda) < 0$ then, for this value of λ, system* (18) *has at least one 2π-periodic solution.*

THEOREM 5. *Let $\varphi(\lambda_0) = 0$ and let the function $\varphi(\lambda)$ take values of both signs in any neighbourhood of the point λ_0. Then λ_0 is an asymptotic bifurcation point for system* (18).

Let L^2 be the space of the vector functions $\mathbf{x}(t) : [0, 2\pi] \to \mathbb{R}^2$ with the usual scalar product denoted as $(\cdot,\cdot)$ and consider the operator $\mathbf{y} = \mathbf{A}\mathbf{x}$ which puts into correspondence to any $\mathbf{x} \in L^2$ the 2π-periodic solution $\mathbf{y}(t)$ of the linear differential equation $\mathbf{y}' - A\mathbf{y} + \mathbf{y} = \mathbf{x}$. In other words, the operator $\mathbf{A}$ is the inverse operator for differential operator $\mathbf{y} \mapsto \mathbf{y}' - A\mathbf{y} + \mathbf{y}$ with 2π-periodic boundary conditions. This

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operator $\mathbf{A}$ exists since the spectrum of the differential operator is separated from zero, while 1 belongs to the spectrum $\sigma(\mathbf{A})$. Let the subspace E_1 corresponds to the eigenvalue 1. The operator $\mathbf{A}$ is completely continuous in the space L^2 and the operator equation

$$\mathbf{x} = \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda))$$

is equivalent in a natural sense to 2π-periodic problem for system (18).

Consider in L^2 the completely continuous vector field

$$\Phi_\lambda \mathbf{x} = \mathbf{x} - \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda)).$$

We want to show that the index at infinity of this vector field is defined if $\varphi(\lambda) \neq 0$ and that $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$ if $\varphi(\lambda) < 0$ with $\operatorname{ind}_\infty \Phi_\lambda = 0$ if $\varphi(\lambda) > 0$ (Here $^{\cdots}$ is an integer power, the precise value of which is not important just now). This will prove both Theorems 4 and 5.

To calculate the index for the cases considered we use Theorem 2. Put $\mathbf{q}(t, \mathbf{x}, \lambda) = (\operatorname{sign} x_1, \operatorname{sign} x_2)^\top + \mathbf{b}(t, \lambda)$ and let Δ consists of 4 points $u_1 = 0$, $u_2 = \pm 1$ and $u_1 = \pm 1$, $u_2 = 0$. All of the conditions of Theorem 2 are fulfilled, so to calculate $\operatorname{ind}_\infty \Phi_\lambda$ it is necessary only to calculate the rotation $\gamma(\lambda)$ of the field $P_1\mathbf{q}(t, \mathbf{x}, \lambda)$ on U. The operator P_1 has the form $P_1\mathbf{x} = (\mathbf{e}_1, \mathbf{x})\mathbf{e}_1(t) + (\mathbf{e}_2, \mathbf{x})\mathbf{e}_2(t)$. Let us parameterize the circle $U \in E_1$ as $U = \{\mathbf{e}_\psi(t) = \cos\psi \mathbf{e}_1(t) + \sin\psi \mathbf{e}_2(t),\ \psi \in [0, 2\pi]\}$ and calculate $\Psi_\lambda(\mathbf{e}_\psi) = P_1\mathbf{q}(t, \mathbf{e}_\psi(t), \lambda)$. After rather simple, but cumbersome, computations we obtain

$$\Psi_\lambda(\mathbf{e}_\psi) = \frac{8}{\sqrt{2\pi}}\mathbf{e}_\psi(t) + P_1\mathbf{b}(t, \lambda),$$

so Ψ_λ is a one-to-one mapping of the circle U into the circle U_λ of radius $8/\sqrt{2\pi}$ which is centered at the point $P_1\mathbf{b}(t, \lambda)$. If $\varphi(\lambda) < 0$, then the origin is surrounded by U_λ and $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$, while if $\varphi(\lambda) > 0$ then the origin is not surrounded by U_λ, hence $\operatorname{ind}_\infty \Phi_\lambda = 0$.

For more details on the computation of the rotation of planar vector fields see [6].

8. Example 2

We now consider the two–point boundary value problem

$$\begin{cases} x_1'' - 4x_1 + 5x_2 = \arctan(x_1 + 2x_2) + b_1(t, \lambda), \\ x_2'' - 2x_1 + 3x_2 = \arctan(2x_1 - x_2) + b_2(t, \lambda), \\ x_1(0) = x_2(0) = x_1(\pi) = x_2(\pi) = 0, \end{cases} \tag{19}$$

or equivalently

$$\mathbf{x}'' = A\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda), \qquad \mathbf{x}(0) = \mathbf{x}(\pi) = 0.$$

Here

$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} \arctan(x_1 + 2x_2) \\ \arctan(2x_1 - x_2) \end{pmatrix}$$

and the function

$$\mathbf{b}(t, \lambda) = \begin{pmatrix} b_1(t, \lambda) \\ b_2(t, \lambda) \end{pmatrix}$$

is continuous in both variables.

THEOREM 6. *Suppose that the function*

$$\varphi(\lambda) = \left| \int_0^\pi (b_1(t, \lambda) + b_2(t, \lambda)) \sin t \, dt \right| - 4 \tag{20}$$

is strictly negative for some λ. *Then system* (19) *has at least one solution for this* λ.

THEOREM 7. *Suppose that the function* (20) *is equal to zero for some* λ *and that this function takes values of both signs in any neighborhood of* λ_0. *Then* λ_0 *is an asymptotic bifurcation point for system* (19).

The differential operator $\mathbf{x}'' - \mathbf{A}\mathbf{x}$ has a non-trivial one-dimensional kernel

$$E_1 = \{\alpha \mathbf{e}(t),\ \alpha \in \mathbb{R}^1\}, \qquad \mathbf{e}(t) = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin t.$$

Let $\mathbf{A} = (\mathbf{x}'' - A\mathbf{x} + \mathbf{x})^{-1}$ with boundary conditions $\mathbf{x}(0) = \mathbf{x}(\pi) = 0$. Then the system (5) is equivalent to the operator equation $\mathbf{x} = \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda))$. Consider the vector field $\Phi_\lambda \mathbf{x} = \mathbf{x} - \mathbf{A}(\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t, \lambda))$ where

$$\mathbf{q}(t, \mathbf{x}, \lambda) = \begin{pmatrix} \operatorname{sign}(x_1 + 2x_2) \\ \operatorname{sign}(2x_1 - x_2) \end{pmatrix} + \mathbf{b}(t, \lambda)$$

and define $\Delta = S \setminus (S_1 \cup S_2)$ where

$$S_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S,\ \left(x_1 - \frac{\sqrt{2}}{2} \right)^2 + \left(x_2 - \frac{\sqrt{2}}{2} \right)^2 < \varepsilon \right\}$$

and

$$S_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in S, \left(x_1 + \frac{\sqrt{2}}{2} \right)^2 + \left(x_2 + \frac{\sqrt{2}}{2} \right)^2 < \varepsilon \right\}$$

for some sufficiently small positive ε. After some easy computations we have

$$P_1\mathbf{x} = (\mathbf{e}, \mathbf{x})\mathbf{e}, \qquad \Psi_\lambda(\pm\mathbf{e}) = P_1\mathbf{q}(t, \pm\mathbf{e}, \lambda) = s_\pm^\lambda \mathbf{e},$$

$$\begin{aligned} s_+^\lambda &= \frac{1}{\sqrt{\pi}} \int_0^\pi \sin t \big(\operatorname{sign}(3\sin t) + \operatorname{sign}(\sin t) + b_1(t,\lambda) + b_2(t,\lambda)\big)\, dt \\ &= \frac{1}{\sqrt{\pi}} \int_0^\pi \sin t \big(2 + b_1(t,\lambda) + b_2(t,\lambda)\big)\, dt \\ &= \frac{1}{\sqrt{\pi}} \left(\int_0^\pi \sin t \big(b_1(t,\lambda) + b_2(t,\lambda)\big)\, dt + 4 \right) \end{aligned}$$

and

$$s_-^\lambda = \frac{1}{\sqrt{\pi}} \left(\int_0^\pi \sin t \left(b_1(t,\lambda) + b_2(t,\lambda)\right)\, dt - 4 \right).$$

Hence $s_+^\lambda \cdot s_-^\lambda > 0$ and $\operatorname{ind}_\infty \Phi_\lambda = 0$ if $\varphi(\lambda) > 0$, and $s_+^\lambda \cdot s_-^\lambda < 0$ and $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$ if $\varphi(\lambda) < 0$. This proves both of the theorems of this section.

Note that Theorem 2 is inapplicable for the problem

$$\begin{cases} x_1'' - 4x_1 + 5x_2 = \arctan(x_1 + 2x_2) + b_1(t,\lambda), \\ x_2'' - 2x_1 + 3x_2 = \arctan(x_1 - x_2) + b_2(t,\lambda), \\ x_1(0) = x_2(0) = x_1(\pi) = x_2(\pi) = 0, \end{cases}$$

because $\mathbf{q}(t, \mathbf{x}, \lambda)$ contains the term $\operatorname{sign}(x_1 - x_2)$ which is not continuous at the point $\mathbf{e}(t)$.

9. Concluding remarks

(1) In the proofs of Theorems 4 – 7 we did not calculate the exponent in the formula $\operatorname{ind}_\infty \Phi_\lambda = (-1)^{\cdots}$ for $\varphi(\lambda) < 0$. This exponent depends on the spectrum $\sigma(\mathbf{A})$ and can be easily calculated if required.

(2) Theorems 4 – 7 can be rewritten without any changes for nonlinearities $\mathbf{f}(\mathbf{x}) + \theta(t, \mathbf{x}, \lambda)$ with an arbitrary Caratheodory function $\theta(t, \mathbf{x}, \lambda)$ satisfying

$$\lim_{|\mathbf{x}| \to \infty} \sup_{t \in \Omega,\, \lambda \in \Lambda} |\theta(t, \mathbf{x}, \lambda)| = 0.$$

(3) The function $\mathbf{b}(t, \lambda)$ need be only integrable in t, not continuous, but continuity in λ is essential.

(4) We used the function $q(t, x)$ defined on $\Omega \times S$, but we could have supposed it defined on $\Omega \times \mathbb{R}^n$ by $q(t, x) = q\,(t, x/|x|)$ if $x \neq 0$ with $q(t, x) = 0$ if $x = 0$.

(5) The closure of the set of discontinuity points for the function $q(t, u)$ can naturally be chosen as the set Δ, but the situation can arise where the essential part of the sphere S is not covered by the points $u = e(t)/|e(t)|$ for $e \in E_1$ and $t \in \Omega$ for one-dimensional sets Ω and E_1 (of course $n > 2$ here), in which case the set of points u is a one-dimensional submanifold of the sphere S which is a manifold of the dimension $n - 1 > 1$. Then there is no need to assume that condition (8) holds 'almost everywhere' on the sphere S; it suffices to assume that it holds in a neighbourhood of the corresponding one-dimensional submanifold.

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