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# The Real Effects of Financing and Trading Frictions

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## Abstract

I develop a model revealing the interplay between a stock's liquidity and the policies and value of the issuing firm. The model shows that bid-ask spreads increase not only the firm's cost of capital but also the opportunity cost of cash, then lowering cash reserves, increasing liquidation risk, and reducing firm value. These outcomes are stronger when internalized by liquidity providers, simultaneously leading to a wider bid-ask spread. A two-way relation between the firm and the liquidity of its stock arises, implying that shocks arising within the firm or in the stock market have more complex implications than previously understood.

## I. Introduction

Corporate financial constraints and investors' trading frictions appear to go hand in hand in the cross section of firms. Large firms enjoy an easy access to external financing, and their stocks are very liquid. At the other side of the spectrum, small firms are largely financially constrained, and their stocks are relatively illiquid. Indeed, small firms typically face delays and costs when raising fresh funds, an issue that has spurred the creation of an ad hoc committee within the U.S. Securities and Exchange Commission (SEC).[1] Moreover, their stocks are characterized by non-negligible bid-ask spreads, low trading volume, and other microstructure frictions (e.g., Novy-Marx and Velikov (2016), Chung and Zhang

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[1]Small firms are typically less known and more vulnerable to capital market imperfections. In contrast, large, established firms are more likely to have continued relations with financial institution. The U.S. SEC Advisory Committee in Small and Emerging Companies pointed out that small firms often struggle to attract capital (see https://www.sec.gov/spotlight/advisory-committee-on-small-and-emerging-companies.shtml).

(2014)). As small firms represent more than 80% of U.S. firms over the past 40 years (Hou, Xue, and Zhang (2020)), understanding the frictions affecting their performance is of utmost importance.

This paper develops a dynamic model showing that trading frictions and financial constraints are deeply related and investigates their real effects. The model studies a firm that has assets in place (generating a stochastic flow of revenues) and a growth option. The firm faces uncertainty in its ability to raise external financing, as small firms do in the real world. Crucially, the firm's shareholders face frictions when trading the stocks, as is typically the case for small-capitalization firms. The model shows that the bid-ask spread associated with trading the firm's stocks affects corporate policies and value. In turn, corporate policies and value feed back into the bid-ask spread. Thus, the model highlights a two-way relation between the policies and value of the firm and the liquidity of its stock. Through this relation, shocks arising within the firm or in financial markets have more nuanced implications than previously understood.

To understand the strengths at play, consider first how a positive (exogenous) bid-ask spread affects the optimal policies and value of the issuing firm. By imposing a cost on investors when trading, a bid-ask spread leads shareholders to require a larger return to invest in the stock. That is, in the spirit of Amihud and Mendelson (1986), the cost of capital increases. As a novel implication, this paper illustrates that the greater cost of capital also implies an increase in the opportunity cost of keeping cash inside the firm. The model demonstrates that firms whose stocks are traded at larger bid-ask spreads are more financially constrained. These firms are less likely to raise external financing and keep smaller precautionary cash reserves; thus, they are more exposed to forced liquidations. Moreover, these firms face an underinvestment problem, as the additional return required by the investors erodes the profitability of investment opportunities. Overall, a positive bid-ask spread leads to a decrease in firm value.

When allowing the bid-ask spread to be endogenous, a feedback effect arises. In this richer setup, the illiquidity-driven decrease in firm value leads to an increase in the endogenous bid-ask spread, as liquidity providers with outside opportunities then extract larger rents from shocked shareholders as a proportion of the value of their claim. As a result, the detrimental effects of stock illiquidity on corporate policies and value strengthen; in particular, the firm holds less cash, financial constraints tighten, the probability of liquidation increases, and firm value declines. The ensuing two-way relation between the bid-ask spread and the policies and value of the issuing firm implies that shocks affecting liquidity provision also impact corporate outcomes. Similarly, shocks affecting firm operations also impact the liquidity of its stock. A rich set of novel implications follows.

First, the interplay between the firm and the liquidity of its stock implies that shocks to firm attributes bear indirect effects, which are novel to the existing literature. The analysis reveals that a tightening in the firm's access to external financing or an increase in its cash flow volatility—which the literature recognizes as prime drivers of corporate cash reserves (see, e.g., Opler, Pinkowitz, Stulz, and Williamson (1999), Bates, Kahle, and Stulz (2009))—should affect a firm's cash hoarding behavior not only by increasing its precautionary demand (as highlighted by previous works) but, as these shocks increase the bid-ask spread on the firm's stock, by also impacting its opportunity cost. This channel should weaken the firm's ability to accumulate cash exactly when its demand for cash increases (i.e., when the firm needs it the most).[2] Thus, revisiting empirical tests to account for stock liquidity would shed new light on the quantitative impact of such standard cash determinants.

Second, and relatedly, this novel channel exacerbates the impact of adverse shocks on the firm's probability of liquidation. Indeed, a deterioration in the firm's access to external financing and an increase in cash flow volatility both raise the probability of liquidation of constrained firms. Yet, as such shocks affect stock liquidity too (as just described), they also harm the firm's ability to keep cash. This additional effect makes firms less financially resilient. That is, firms are less able to withstand prolonged periods of losses and, after a given cumulative loss, they exhibit a higher probability of forced liquidation. Overall, the analysis then indicates that measures of financial constraints could fruitfully account for stock illiquidity to improve their predictive power.

Third, the two-way relation between stock illiquidity and corporate policies also implies that shocks arising in the market for the stock (e.g., shocks making liquidity provision more costly or weakening the bargaining position of shocked shareholders) have indirect effects too. That is, as a direct effect, such shocks naturally make the stock less liquid. Furthermore, as illustrated, the ensuing greater costs of trading borne by the firm's shareholders affect corporate policies, tighten financial constraints, and decrease firm value, worsening the illiquidity of the firm's stock further. Through this mechanism, shocks originating in the market of the firm's stock propagate to firm policies and outcomes, eventually bearing an amplified impact on the liquidity of the firm stock.

On top of providing novel testable predictions, the paper finds empirical validation in the documented impact of stock illiquidity on corporate outcomes. The model predicts that these firms should face severe financial constraints because of their larger costs of external *and* internal equity, which reduce the probability of external financing and the size of the firm's precautionary cash reserves, consistent with Nyborg and Wang (2021). As a result, these firms face higher liquidation risk, consistent with Brogaard, Li, and Xia (2017). Notwithstanding these constraints, these firms should exhibit larger payouts in the cross section to compensate investors for trading frictions, as documented by Banerjee, Gatchev, and Spindt (2007). These firms should also suffer from underinvestment, consistent with Campello, Ribas, and Wang (2014) and Amihud and Levi (2023). Not only this paper proposes a model that rationalizes all of these empirical findings but also shows that these effects are amplified when liquidity providers internalize the negative effect of bid-ask spreads on firm value.

## A. Related Literature

This paper contributes to the strand of dynamic corporate finance models with financing frictions, including Bolton, Chen, and Wang (BCW) (2011), Décamps, Mariotti, Rochet, and Villeneuve (DMRV) (2011), Hugonnier, Malamud, and

[2]That is, when external financing tightens or when cash flows are more volatile.

Morellec (HMM) (2015), Malamud and Zucchi (2019), or Della Seta, Morellec, and Zucchi (2020).[3] These papers show that financing frictions, such as costs or uncertainty in raising external funds, should increase a firm's propensity to keep precautionary reserves. Whereas these extant papers impose an exogenous cost of holding cash, the current model shows that this cost can arise endogenously when accounting for trading frictions faced by firm shareholders. Importantly, the analysis illustrates that the interplay between stock illiquidity and corporate policies makes firms less financially resilient.

The paper is also related to the theoretical literature modeling endogenous feedback effects. Brunnermeier and Pedersen (2009) show that there is a two-way link between an asset's market liquidity and traders' funding liquidity. Traders provide market liquidity, which in turn depends on their funding ability. Because of margin requirements, traders' funding depends on the asset's market liquidity. The current paper instead focuses on the relation between the funding liquidity of a given firm and the market liquidity of its stocks, unraveling a novel two-way relation between a stock's bid-ask spread and the policies and value of the issuing firm. Another related paper is He and Milbradt (2014), who endogenize *bond* illiquidity into a Leland-type model of endogenous default, in which shareholders can inject fresh equity at no cost. He and Milbradt provide a decomposition of credit spreads into a default and a liquidity component, then matching several cross-sectional patterns of bid-ask spreads and credit spreads. Conversely, the present paper focuses on *stock* illiquidity and builds on the strand of dynamic corporate finance models with financing frictions, in which shareholders face costs or uncertainty in their ability to raise additional financing. This novel feedback effect illustrates that shocks arising within the firm or in financial markets have more nuanced effects than previously understood.

Finally, the paper relates to the literature showing that stock illiquidity impacts corporate policies and outcomes. Fang, Noe, and Tice (2009) show that firms with liquid stocks are more valuable. Campello et al. (2014) find that stock liquidity improves corporate investment and value. Amihud and Levi (2023) show that illiquidity lowers corporate investment, R&D, and inventory. Nyborg and Wang (2021) show that stock liquidity increases a firm's propensity to hold cash. Banerjee et al. (2007) reveal that firms with illiquid stocks pay out more dividends. Brogaard et al. (2017) find that stock liquidity reduces firms' bankruptcy risk. Notably, the paper provides a unified framework that supports the existing empirical evidence.[4]

The paper proceeds as follows: Section II describes the model. Section III describes the model solution. Section IV derives the model predictions, starting with the special case in which liquidity provision is exogenous and then moving to the case with endogenous liquidity provision. Section V concludes. Proofs are gathered in Appendices A–E.

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[3]Other papers in this strand are Bolton, Chen, and Wang (2013), Hugonnier and Morellec (2017), Dai, Giroud, Jiang, and Wang (2024), or Bruegem, Marfè, and Zucchi (2023).

[4]Because the two-way relation between stock illiquidity and corporate policies and value amplifies the effect of illiquidity on firm policies, the current paper confirms the directional effect suggested by existing empirical work.

# II. The Model

Time is continuous, and uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$. Agents are risk-neutral, and discount cash flows at rate $\rho > 0$.

## A. The Firm

I consider a small firm operating a set of assets in place, which generate continuous and stochastic cash flows. This flow is modeled as an arithmetic Brownian motion, $(Y_t)_{t \geq 0}$, whose dynamics evolve as

$$dY_t = \mu dt + \sigma dZ_t. \tag{1}$$

The parameters $\mu$ and $\sigma$ are strictly positive and represent the mean and volatility of corporate cash flows, and $(Z_t)_{t \geq 0}$ is a standard Brownian motion. The firm has access to a growth option that has the potential to increase its income stream from $dY_t$ to $dY_t^+ = dY_t + (\mu_+ - \mu)dt, \mu_+ > \mu$, by paying a lump-sum cost $I > 0$. That is, the drift can assume two values $\mu_i = \{\mu, \mu_+\}$. Investment is assumed to be irreversible.

The process in equation (1) implies that the firm can make operating losses. If capital supply was perfectly elastic, such losses could be covered by raising outside financing immediately and at no cost. In practice, small firms face financing frictions, such as uncertainty or costs in raising funds. I model this uncertainty by assuming that the firm raises new funds at the jump times of a Poisson process, $(N_t)_{t \geq 0}$, with intensity $\lambda$, as in HMM. That is, if the firm decides to raise outside funds, the expected financing lag is $1/\lambda$ periods. If $\lambda \to 0$, the firm cannot raise external funds at all (equivalently, it takes an infinite waiting period to raise fresh funds upon searching) and relies on cash reserves to cover operating losses. If $\lambda \to \infty$, the waiting time upon searching for external funds is zero (i.e., the firm has access to outside financing at no delays).[5] Notably, as shown in the paper, the discount on newly issued equity is related to trading frictions faced by firms' investors.

Because capital supply is uncertain, the firm has incentives to retain earnings in cash reserves. I denote by $(C_t)_{t \geq 0}$ the firm's cash reserves at any $t$. Cash reserves earn a constant rate, $r \leq \rho$. Whenever $r < \rho$, keeping cash entails an opportunity cost.[6] In contrast with extant cash holdings models, in which the strict inequality $r < \rho$ is needed to depart from the corner solution featuring firms piling infinite cash reserves, I allow for the $r = \rho$ case. The cash reserves process satisfies[7]

[5]Section SA.2 of the Supplementary Material analyzes the corner case $\lambda \to \infty$, in which the firm faces no financing frictions as it immediately finds external financing upon searching. Absent financing frictions, financial policies become trivial. Thus, the model with financing frictions proves to be a more comprehensive analysis of the impact of stock illiquidity on firm value, which is particularly relevant in light of the empirical observation that firms with illiquid stocks are typically financially constrained, as discussed in the introduction. In the main body of the paper, I then focus on the case $\lambda < \infty$.

[6]This cost can be interpreted as a free cash flow problem (Jensen (1986)) or as tax disadvantages (Graham (2000)).

[7]Upon investing (i.e., the cash flow drift increases from $\mu$ to $\mu_+$), the cost $I$ is financed with either cash or external financing. Because the paper focuses on the decision of whether or not to invest (rather than on the investment timing), I do not explicitly spell out the outflow $I$ in the dynamics of cash reserves.

$$dC_t = rC_t dt + \mu_i dt + \sigma dZ_t - dD_t + f_t dN_t. \tag{2}$$

$dD_t \geq 0$ represents the instantaneous flow of payouts at time $t$. $f_t \geq 0$ denotes the instantaneous inflow of funds when financing opportunities arise, in which case management stores the proceeds in the cash reserves. This assumption is consistent with the strong, positive correlation between equity issues and cash accumulation documented by McLean (2011) or Eisfeldt and Muir (2016). Notably, $D$ and $f$ are endogenous. Equation (2) implies that the firm's cash reserves increase with external financing, retained earnings, and the interest earned on cash, whereas they decrease with payouts and operating losses.

As in previous cash management models (see, e.g., HMM, BCW, or DMRV), the cash reserves of the firm need to always remain nonnegative as an operating constraint. Subject to this constraint, management can distribute cash and liquidate the firm's assets at any time. Yet, liquidation is inefficient, as the recovery value of assets is smaller than the firm's first best, $\mu_i/\rho$, due to liquidation costs. These costs erode a fraction, $1-\phi \in (0,1]$, of the firm's first best, so the liquidation value is $\ell_i = \phi\mu_i/\rho$.[8] I denote by $\tau$ the endogenous time of liquidation.

## B. Transacting the Firm Stocks

The key departure from previous dynamic corporate finance models with financing frictions is the explicit consideration of stock transactions and the costs thereof. There are two types of traders: investors (who may buy, hold, and eventually sell the stock) and trading firms (or liquidity providers, which ease investors' trading).

Investors are ex ante identical and infinitely lived. Each of them has measure zero and cannot short sell. Investors can be hit by liquidity shocks. As in previous contributions (e.g., Duffie, Garleanu, and Pedersen (2005), among others), liquidity shocks trigger a sudden need for liquidity that reduces the subjective valuation of the asset by a fraction $\chi$. Thus, $\chi$ can be interpreted as a holding cost, that is, as the opportunity cost of being locked into an undesired asset position, because of take-it-or-leave-it investments, unpredictable financing needs, or unpredictable changes in hedging needs, for example. The liquidity shock vanishes once the shocked investor either sells his stock or bears the loss $\chi$. Liquidity shocks are idiosyncratic, independent across investors, and occur at the jump times of a Poisson process $(M_t)_{t \geq 0}$ with intensity $\delta > 0$. In turn, non-liquidity-shocked shareholders have no immediate need to trade and, thus, are indifferent between keeping the stock or selling it at its fundamental value.[9]

Trading firms are agents who provide liquidity in the market for the stock, then helping liquidity-shocked shareholders unload their holdings. Throughout the paper, trading firms and liquidity providers will be used interchangeably. Trading firms have no intrinsic demand for the firm's stock and should be interpreted as pass-through intermediaries. Trading firms are active on both sides of transactions. In the spirit of Stoll (2000), trading firms bear a proportional order-flow cost $\nu$ on each round-trip

[8] Recall that $\mu_i = \{\mu, \mu_+\}$ depending on whether the firm has exercised the growth option. Thus, if the firm has cash flow drift $\mu$, the liquidation value is simply denoted by $\ell$.

[9] Following previous contributions (see, e.g., He and Milbradt (2014)), I assume that the mass of non-liquidity-shocked investors is larger than that of liquidity-shocked shareholders.

transaction. On the ask side, trading firms transact with non-liquidity-shocked investors. As non-liquidity-shocked investors have no immediate need to trade, the gain to trading firms on this side of transactions is null. On the bid side, trading firms transact with liquidity-shocked shareholders. Because shocked shareholders value the asset at a discount $\chi$, trading firms can extract surplus from this side of transactions.

When shocked shareholders and trading firms meet, they need to bargain over the terms of the trade. I denote by $\theta$ the bargaining power of trading firms, and by $1-\theta$ the bargaining power of shocked shareholders. Furthermore, I assume that trading firms have an outside option denoted by $\omega$. Whereas this modeling is stylized, it captures the key idea that trading firms may weigh their decision to provide liquidity in the market of the stock against opportunities arising in other markets, whose payoff is independent of the value of the firm's stock. In the following, $b_t$ represents the trading firms' bid price (i.e., the price at which shocked shareholders sell the stock to trading firms) and $\eta_t$ represents the associated bid-ask spread, which are endogenously derived.

## C. The Firm's Problem with Endogenous Bid-Ask Spread

Firm management maximizes equity value. Namely, cash retention and payouts ($D$), financing ($f$), liquidation ($\tau$), and investment ($I$) are set to maximize

$$V(c;\eta) = \sup_{(D,f,\tau,I)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}(dD_t - f_t dN_t - \Phi(\eta_t,\chi)\,dM_t) + e^{-\rho\tau}\ell\right]. \tag{3}$$

As in previous cash management models (DMRV, BCW, or HMM), the first term in the expectation operator is the discounted value of net payouts to shareholders, whereas the second term is the discounted liquidation value. Differently from previous works, net payouts to shareholders are not just the difference between the expected present value of all future dividends ($dD_t$) and the expected present value of all future gross issuance proceeds ($f_t dN_t$, which is akin to a negative payout), because frictions incurred in trading the firm stocks further drain the flow of net payouts to shareholders (represented by the term $\Phi(\eta_t,\chi)dM_t$). Namely, liquidity shocks, whose arrival is marked by the Poisson process $dM_t$, lead to the loss $\Phi(\eta_t,\chi)$, which depends on the holding cost upon keeping the stock ($\chi$) and the bid-ask spread incurred upon selling it ($\eta_t$).[10] In turn, the equilibrium bid-ask spread $\eta_t$ (equivalently, the bid price $b_t$) is pinned down by Nash bargaining between shocked shareholders and trading firms. As the model solution illustrates, the endogenous bid-ask spread not only affects (as clear from equation (3)) but also reflects firm policies and value.

## D. Discussion of the Assumptions

The model nests trading frictions faced by the firm's shareholders into a dynamic corporate finance model with financing frictions. That is, the paper contributes to previous dynamic corporate finance models in this strand by studying the impact of stock illiquidity. At the same time, the paper contributes to models of the effects of

[10]Because the loss $\Phi(\eta_t,\chi)$ is endogenous, it is derived in the model solution (see Section III).

liquidity demand/supply on asset valuations by explicitly focusing on the policies of the issuing firm. Whereas these models usually take the flow of dividends associated with a given stock as exogenous, the current paper endogenizes it. To keep the analysis simple, two assumptions are made. First, the trading costs borne by shocked (selling) investors are positive, whereas the costs borne by (buying) non-shocked investors are 0. This assumption is consistent with Brennan, Chordia, Subrahmanyam, and Tong (2012), who show that sell-order frictions are priced more strongly than buy-order ones.[11] Second, the model abstracts from asymmetric information about firm value. Indeed, recent evidence shows the importance of the non-information component of trading costs on asset prices (e.g., Chung and Huh (2016)).

Without loss of generality, the degree of stock illiquidity is derived by assuming Nash bargaining between shocked shareholders and trading firms.[12] While Nash bargaining has been used to model over-the-counter (OTC) markets, the applicability is broader as it represents a simple way to acknowledge that trading firms can capture some of the surplus from trade. Yet, the model results are robust to alternative ways of endogenizing the bid-ask spread. Namely, a previous version of the paper derived the bid-ask spread under the assumption of a competitive market for liquidity provision, where the bid-ask spread was pinned down by the zero-profit condition of trading firms. All the main model results continue to hold in this alternative version of the model.[13]

Given the paper's focus on firm policies (especially financial ones), it is crucial to assess that the results are not driven by some specific assumptions. The paper assumes that the firm faces uncertainty in its ability to raise fresh funds as HMM, an issue that is especially severe for small firms, as also pointed out by the U.S. SEC Advisory Committee in Small and Emerging Companies. As illustrated by the survey evidence in Lins, Servaes, and Tufano (2010), financing uncertainty is one of the top reasons behind corporate cash stockpiling. To assess the robustness to alternative modeling of the firm's financing frictions, I design two extensions. First, Section SA.1.1 of the Supplementary Material allows the firm to tap credit line availability. In fact, whereas small (or micro) firms often find it too costly (or unfeasible) to tap bond financing, they usually access debt by borrowing from banks. Second, Section SA.1.2 of the Supplementary Material assumes that the firm faces issuance costs whenever raising new equity, as in DMRV and BCW. The main model takeaways are preserved under these alternative assumptions.

# III. Model Solution

## A. Financing Frictions, the Bid-Ask Spread, and Firm Value

As in previous cash management models, the benefit of cash decreases with cash reserves. Its (opportunity) cost is the wedge between the return required by the investors and the return on cash. Thus, I conjecture that there is a target cash level,

[11]Brennan et al. (2012) show that the pricing of illiquidity emanates principally from the sell side. The underlying idea is that agents seldom face needs to buy stock urgently, but unexpected needs for cash may force them to suddenly sell stocks.

[12]I thank Thierry Foucault for suggesting this modeling approach.

[13]The results of this alternative version of the model are available in a previous working paper.

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$C_V$, at which the cost and benefit of cash are equalized. Above $C_V$, it is optimal to pay excess cash out to shareholders. Below $C_V$, shareholders retain earnings in cash reserves and search for financing.

Management can choose to liquidate when the firm holds positive cash reserves. However, because the firm is profitable in expectation and liquidation is costly, it is optimal to delay the liquidation time as much as possible.[14] In other words, subject to the operating constraint that cash reserves must be nonnegative, the firm is liquidated the first time that the cash reserves process hits $c=0$, in which case it cannot cover its losses. Thus, the endogenous time of liquidation is

$$\tau = \inf\{t \geq 0 : C_t \leq 0\}. \tag{4}$$

Assume that the firm does not have any growth option.[15] Using standard arguments, firm value satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{aligned}\rho V(c;\eta) = {} & (rc+\mu)V'(c;\eta) + \frac{\sigma^2}{2}V''(c;\eta) + \lambda \sup_f [V(c+f;\eta) - V(c;\eta) - f] \\ & + \delta[(1-\min[\eta;\chi])V(c;\eta) - V(c;\eta)].\end{aligned} \tag{5}$$

The left-hand side is the return required by the investors. The first two terms on the right-hand side represent the effect of cash retention and cash flow volatility on equity value. The third term represents the surplus from raising external financing, weighted by its likelihood. In Appendix A, the enterprise value $V(c)-c$ is shown to increase with $c$, so it is optimal to raise the cash buffer up to $C_V$ whenever financing opportunities arise.[16] Thus, the optimal refinancing amount is $f(c,\eta) = C_V - c$. As shown in the following, $C_V$ depends on the bid-ask spread and, thus, the optimal refinancing amount depends on the bid-ask spread too.

The novelty of equation (5) compared to previous cash management models is the last term on the right-hand side, which reflects the impact of liquidity shocks borne by the firm shareholders on firm value. Liquidity shocks are independent across investors, so a measure $\delta dt$ of shareholders is shocked on each time interval, and the ensuing loss is $\Phi(\eta,\chi) = \min[\eta;\chi]V(c;\eta)$. Next, the quantity $\eta$ is endogenously derived.

## B. Deriving Stock Liquidity

When providing liquidity in the market for the stock, trading firms are both on the bid and ask sides. On the ask side, trading firms sell stocks to non-liquidity-

[14]In fact, the drift in equation (1) is positive (meaning that the firm is viable in expectation), and cash flow shocks are transitory (i.e., the process in equation (1) follows an arithmetic Brownian motion). Thus, such shocks do not jeopardize the long-term prospects of the firm. Moreover, liquidation is costly as only a fraction of the present value of future cash flows is recovered, as per the definition of $\ell$.

[15]Following Décamps and Villeneuve (2007) and HMM, solving for firm value when there is no growth option is auxiliary to studying the optimal investment rule, which is studied in Section IV for both the cases with exogenous and endogenous bid-ask spread.

[16]The marginal value of cash satisfies $V'(c) \geq 1$ (see Appendix A for a proof). This implies that the first derivative of $V(c)-c$ is nonnegative. Clearly, it is not optimal to raise external financing to replenish the cash buffer beyond $C_V$; otherwise, the excess cash would be paid out as dividend.

shocked investors, so they extract no rents from this side of transactions.[17] On the bid side, they buy stocks from liquidity-shocked investors; because these investors value the stock at discount, trading firms can extract rents from this side of the transaction. The terms of the trade are pinned down by Nash bargaining. Namely, if shocked shareholders reject the intermediary offer, they incur a proportional holding cost $\chi$. In turn, trading firms have an outside option $\omega$. Standard arguments imply that the Nash bargaining solution for the bid price is

$$b(c;V) = \theta(1-\chi)V(c) + (1-\theta)[(1-\nu)V(c) - \omega], \tag{6}$$

where $\theta$ is the bargaining power of trading firms. The proportional bid-ask spread is then given by

$$\eta(c;V) = \frac{V(c) - b(c)}{V(c)} = \theta\chi + (1-\theta)\nu + \frac{(1-\theta)\omega}{V(c;\eta)}. \tag{7}$$

This expression illustrates that if the shareholders' holding cost is larger, it is relatively more costly for shocked shareholders to reject the offer of the liquidity provider, who can then extract more rents. Moreover, if the outside option of trading firms $\omega$ is larger, liquidity providers can extract more rents from shocked shareholders, all else equal. As equation (7) illustrates, $\eta$ depends on firm value; notably, the model can reproduce the negative relation between bid-ask spreads and market capitalization (see, e.g., Chung and Zhang (2014)). As equation (5) illustrates, firm value itself depends on $\eta$. That is, there is a two-way relation between $V$ and $\eta$. In the following, to ease the notation, I simply use $\eta(c)$ and $V(c)$.[18]

## C. Endogenous Liquidity and Firm Value

Plugging the endogenous bid-ask spread (equation (7)) into equation (5), we solve for firm value subject to the following boundary conditions. First, as explained above, the firm is liquidated when cash is exhausted and the firm cannot raise external funds. Thus,

$$V(0) = \ell \tag{8}$$

holds. Moreover, it is optimal to distribute all the cash exceeding $C_V$ as payouts. Firm value is thus linear for any $c \geq C_V$: $V(c) = V(C_V) + c - C_V$. Subtracting $V(c)$ from both sides of this equation, dividing by $c - C_V$, and taking the limit $c \to C_V$ gives

$$\lim_{c \uparrow C_V} V'(c) = 1. \tag{9}$$

That is, it is optimal to start paying out cash when the marginal value of one dollar inside the firm equals the value of a dollar paid out to shareholders. The target

[17] Recall that non-liquidity-shocked investors are indifferent between staying out of the market or buying the stock at its fundamental value.

[18] To guarantee that bargaining always goes through, the inequality $(\chi - \nu)\ell > \omega$ is assumed to hold. It ensures that trading firms can extract enough rents from shocked shareholders so that it is sufficiently attractive vis-à-vis their outside option (see Appendix A).

cash level that maximizes shareholder value is determined by the super-contact condition

$$\lim_{c \uparrow C_V} V''(c) = 0. \tag{10}$$

As in previous cash management models, firm value is increasing and concave in $c$ in the presence of financing frictions; that is, firm value increases with cash reserves, and the marginal value of cash is greater when the cash reserves are smaller (see Appendix A for a proof).

## IV. Model Analysis

### A. A Special Case: Exogenous Stock Illiquidity

To disentangle the strengths at play in the model, I start by investigating the special case in which the bid-ask spread is exogenous (i.e., the quantity $\eta$ in equation (5) is given and constant). If $\eta < \chi$, equation (5) boils down to

$$(\rho + \delta\eta)V(c) = (rc + \mu)V'(c) + \frac{\sigma^2}{2}V''(c) + \lambda[V(C_V) - V(c) - C_V + c]. \tag{11}$$

In line with the seminal work of Amihud and Mendelson (1986), the bid-ask spread leads shareholders to require a higher compensation to invest in the firm, as the left-hand side of this equation is increased by $\delta\eta$. This additional compensation is greater if the bid-ask spread is larger (i.e., higher $\eta$) or if shareholders need to trade more often (greater $\delta$).

To analyze the effects of stock illiquidity, I define the following quantities. First, the firm's payout probability satisfies

$$P^p(c, C_V) = E_c\left[e^{-\lambda \tau_d(C_V)}\right], \tag{12}$$

where $\tau_d(C_V)$ represents the first time that the cash reserves process, initially at a given $c < C_V$, reaches the target level $C_V$. Furthermore, the probability of liquidation while the firm is searching for external funds is given by

$$P^l(c, C_V) = E_c\left[e^{-\lambda \tau(C_V)}\right], \tag{13}$$

and, complementarily, the probability of external financing is $P^f(c, C_V) = E_c\left[1 - e^{-\lambda \tau(C_V)}\right]$, where $\tau(C_V)$ represents the first time that the cash process, reflected at $C_V$, is absorbed at 0 (see equation (4)). The next proposition summarizes the impact of an exogenous bid-ask spread on firm's decisions and outcomes (see Appendix B for a proof).

*Proposition 1.* A positive bid-ask spread leads to:

(1) a decrease in the target cash level—the greater the bid ask spread $\eta$, the smaller $C_V$);

(2) an increase in the firm's payout probability, i.e., $P^p(c, C_V)$ increases with $\eta$;

(3) an increase in the firm's probability of liquidation and a decrease in the probability of external financing, i.e., $P^l(c, C_V)$ increases with $\eta$ and $P^f(c, C_V)$ decreases with $\eta$;

(4) a decrease in the maximum amount that the firm is willing to pay to exercise the growth option compared to the case with perfect stock liquidity, as the zero-NPV cost is

$$I_V = \frac{\mu_+ - \mu}{\rho + \delta\eta} - (C_{V+} - C_V)\left[1 - \frac{r}{\rho + \delta\eta}\right] \tag{14}$$

with $C_{V+}$ denoting the post-investment target cash level;

(5) a decrease in firm value, i.e., $V(c)$ decreases with $\eta$.

Claim (1) of Proposition 1 explains how the bid-ask spread affects corporate cash hoarding choices. As in previous cash management models, the target cash level trades off the benefit of cash (stemming from providing financial flexibility to the firm facing financing frictions) against its opportunity cost. The current model shows that the bid-ask spread affects such opportunity cost. Indeed, as illustrated by equation (11), the bid-ask spread increases the return required by the investors. As a result, the cost of cash also increases, as the wedge between the return required by the investors and the return on cash widens.[19] Thus, the proposition shows analytically that a greater bid-ask spread leads to a smaller target cash level. That is, the bid-ask spread negatively affects the firm's ability to hold cash, consistent with the evidence in Nyborg and Wang (2021).

Cash retention and payout decisions are closely related. As illustrated by equation (12), the target cash level $C_V$ affects the probability with which the firm pays out dividends. Hence, by affecting $C_V$, the bid-ask spread also affects the probability of payout. Claim (2) of Proposition 1 suggests that a firm pays out more dividends if its stock is traded at a larger bid-ask spread, as the target cash level is hit more often. In so doing, the firm compensates shareholders for the frictions borne when trading the stock. This finding is in line with Banerjee et al. (2007), who suggest that investors view stock market liquidity and dividends as substitutes. When a firm's bid-ask spread is small, investors can create dividends to themselves by cashing out their investment. When the bid-ask spread is large, investors require the firm to pay out more dividends.

Because the bid-ask spread increases the cost of internal and external equity, the firm's financial resilience is also affected. As illustrated by equation (13), the choice of the target cash level affects the time $\tau(C_V)$ at which the firm is liquidated. Indeed, Claim (3) of Proposition 1 shows that the greater a firm's bid-ask spread, the higher the firm's probability of liquidation and the lower the firm's probability of external financing. Proposition 1 then suggests that bid-ask spreads exacerbate firms' financial constraints and increase a firm's threat of forced liquidations (a finding consistent with Brogaard et al. (2017)).

[19]This model then delivers finite target cash levels even when $r$ and $\rho$ coincide. In previous dynamic cash management models, differently, holding cash is not costly if $r = \rho$ and, thus, a financially constrained firm would pile infinite cash reserves in such case.

The bid-ask spread affects not only financial policies but also investment. Claim (4) of Proposition 1 suggests that a positive bid-ask spread leads to a decrease in the investment reservation price (i.e., it reduces the maximum amount that the firm is willing to pay to exercise the growth option). Consider the investment reservation price when the bid-ask spread is 0 and denote it by $I^*$.[20] If the investment cost lies in $[I_V, I^*]$, the growth option has negative NPV if the bid-ask spread is positive ($\eta > 0$), whereas it has positive NPV if the bid-ask spread is 0. Thus, the positive bid-ask spread leads to underinvestment, a result that is empirically consistent with Campello et al. (2014) and Amihud and Levi (2023).

To summarize, a positive bid-ask spread adversely impacts the financial and investment policies of the firm. Claim (5) of Proposition 1 then concludes that it also decreases firm value, consistent with the evidence by Fang et al. (2009). As I show next, such a drop in firm value has important implications when the bid-ask spread is endogenous.

## B. Endogenous Stock Illiquidity

As shown by Proposition 1 for the case with exogenous bid-ask spread, trading costs lead shareholders to require an additional compensation to invest in the firm. Such additional compensation constrains corporate policies (e.g., it makes it more costly to the firm to hold cash, which tightens the firm's financial constraints and increases the firm's probability of liquidation) and leads to a decrease in firm value. When allowing the bid-ask spread to be endogenous, such a drop in firm value is internalized in the terms of the trade between shocked shareholders and trading firms. Specifically, trading firms with outside opportunities can extract larger rents as a fraction of firm value—that is, the bid-ask spread widens. The greater bid-ask spread feeds back into firm value, then making the detrimental effects illustrated in the previous section stronger. This two-way relation between the firm and the liquidity of its stock gives rise to a rich set of novel implications, which I analyze next.

### 1. Baseline Parameterization

Before turning to the model implications (which are gauged quantitatively too), I describe the baseline parameterization reported in Table 1. The risk-free rate $\rho$ is set to 2%, and the return on cash is set to 1%. The resulting opportunity cost of cash is equal to 1%, as in BCW and DMRV. Because small firms tend to have lower sales and cash flows in the cross section (see, e.g., Fama and French (2008)), the drift $\mu = 0.05$ is set to be lower than the value used by DMRV and consistent with the bottom range of values in Whited and Wu (2006). Upon exercising the growth option, the drift is assumed to be 20% bigger (i.e., $\mu_+ = 0.06$). I set $\sigma = 0.12$, which is consistent with Graham, Leary, and Roberts (2015) and is higher than the value set by DMRV, as small firms have more volatile cash flows. I base liquidation costs on the estimates of Glover (2016) and set $\phi = 0.55$. The parameter $\lambda$ is set to 0.75, which is consistent with the frequency of equity issues by small firms reported by

[20]In this case, the zero-NPV cost would be $I^* = \frac{\mu_+ - \mu}{\rho} - \left(C_+^* - C^*\right)\left(1 - \frac{r}{\rho}\right)$, with $C^*$ (resp. $C_+^*$) denoting the pre-investment (post-investment) target cash level when the bid-ask spread is 0 ($\eta = 0$).

TABLE 1
**Benchmark Parameters**

Table 1 illustrates the benchmark parameters used in the quantitative analysis.

| Symbol | Description | Value |
|---|---|---|
| | *Firm* | |
| $\rho$ | Risk-free rate | 0.02 |
| $r$ | Return on cash | 0.01 |
| $\mu$ | Cash flow drift | 0.05 |
| $\mu_+$ | Post-investment cash flow drift | 0.06 |
| $\sigma$ | Cash flow volatility | 0.12 |
| $\phi$ | Recovery rate in liquidation | 0.55 |
| $\lambda$ | Arrival rate of financing opportunities | 0.75 |
| | *Stock transactions* | |
| $\delta$ | Arrival rate of liquidity shocks | 0.700 |
| $\chi$ | Shocked shareholders' holding cost | 0.012 |
| $v$ | Order-processing cost | 0.001 |
| $\omega$ | Trading firms' outside opportunities | 0.005 |
| $\theta$ | Trading firms' bargaining power | 0.30 |

Fama and French (2005). The intensity of the liquidity shock is set to $\delta = 0.7$, as in He and Milbradt (2014). The parameters related to trading firms' liquidity provision are varied extensively and chosen to convey realistic magnitudes of the bid-ask spreads for a small stock; under the baseline parameterization, the bid-ask spread ranges between 58 and 68 BPS.[21] While Nash bargaining is often used to model OTC markets, our baseline value of the bargaining power of trading firms reflects the idea that, in many stock markets, competition among liquidity providers limits their bargaining power. In the analysis that follows, such parameter is varied extensively.[22]

### 2. Liquidity Provision Affects Corporate Outcomes

As shown in Section IV.A, the exogenous bid-ask spread affects corporate policies and outcomes. When liquidity is endogenous, variables that directly affect the bid-ask spread impact corporate policies too, as formally shown in the next proposition.

*Proposition 2.* If liquidity providers face a larger order-processing cost $v$, more profitable outside opportunities $\omega$, or a greater bargaining power $\theta$, or if the shocked shareholders' holding cost $\chi$ is larger, the bid-ask spread widens. The endogenous bid-ask spread leads to: i) a lower target cash level; ii) a higher payout probability; iii) a higher probability of liquidation and a lower probability of external financing;

[21]Chung and Zhang (2014) report the median bid-ask spread for firms sorted by quintiles of market capitalization over the period of 1993–2009 (calculated using TAQ data). They report that the median bid-ask spread of smaller quintile firms is 0.0195 for NYSE/AMEX stocks and 0.0501 for NASDAQ stocks. Yet, they also note that the bid-ask spread has decreased over time (see also Hasbrouck (2009)): The median bid-ask spread for (all capitalization) NYSE/AMEX stocks went from 0.0094 in 1993 to 0.0034 in 2009, and from 0.0346 in 1993 to 0.0067 in 2009 for NASDAQ stocks. In the model parameterization, I take a conservative approach and take a relatively low value for the bid-ask spread. In so doing, I show that even small bid-ask spreads can bear substantial impact on corporate policies and value.

[22]Notably, by taking a smaller value of $\theta$, the bid-ask spread exhibits a more conservative value. The model results hold irrespective of the magnitude of this parameter.

iv) a decrease in the maximum amount that the firm is willing to pay to exercise the growth option compared to the case with perfect stock liquidity, as the zero-NPV costs is

$$(15)\qquad I_V = \frac{\mu_+ - \mu}{\rho + \delta(v + \theta(\chi - v))} - \left(1 - \frac{r}{\rho + \delta(v + \theta(\chi - v))}\right)(C_{V+} - C_V),$$

where $C_{V+}$ is the target cash level after growth option exercise; and v) a decrease in firm value.

Proposition 2 indicates that the costs of liquidity provision (being related to order-processing or foregone opportunities) are passed on to the firm's shareholders via a larger bid-ask spread and, through this channel, impact corporate policies and outcomes. Similarly, if shocked shareholders' holding cost is higher, or if their bargaining power is weaker (equivalently, trading firms' bargaining power is stronger), the bid-ask spread is wider, and so is its impact on corporate policies and value. Not surprisingly, the effects highlighted in Proposition 2 are similar to those described in Proposition 1—that is, the primitive variables affecting the bid-ask spread themselves impact corporate policies and outcomes. In the following, I analyze these effects quantitatively.

Table 2 quantifies these predictions by gauging how the primitive parameters affecting the bid-ask spread impact firm choices and outcomes compared to the case in which the stock is perfectly liquid (i.e., the bid-ask spread is 0). Under our baseline parameterization (in which case the bid-ask spread is 58 BPS for $c = C_V$), the target cash level decreases by 12% compared to the case with perfect stock liquidity. Moreover, the probability of payout increases by about 2.8% on average.[23] Overall, the firm is more financially constrained and faces a higher probability of liquidation; under the baseline parameterization, this probability increases on average by about 1.8%.

Table 3 further investigates the impact of the primitive parameters affecting the bid-ask spread on the firm's probability of liquidation at different levels of cash reserves. It shows that an increase in the parameters determining stock illiquidity raises the firm's probability of liquidation, and more so if the firm's cash reserves are smaller. Moreover, for a given cumulative shock, liquidation becomes relatively more likely if these parameters are larger (and, thus, the bid-ask spread is wider). Quantitatively, if the bid-ask spread was 0 (in which case the target cash level is denoted by $C^*$), a series of shocks reducing the cash buffer from $C^*/2$ to $C^*/4$ would increase the probability of liquidation from 1.97% to 14.11% (see the last row of Table 3). When instead the bid-ask spread is positive, a series of shocks reducing the cash buffer from $C_V/2$ to $C_V/4$ increase the probability of liquidation from 3.18% to 17.88% under our baseline parameterization. Moreover, the reduction from $C_V/2$ to $C_V/4$ would be caused by a cumulative loss that is 12% smaller. That is, firms traded at a larger bid-ask spread are less financially resilient.

[23] To calculate the probabilities in this table, I follow HMM and calculate them for a cross section of firms with cash reserves uniformly distributed between 0 and $C_V$.

TABLE 2

Endogenous Bid-Ask Spread and Corporate Outcomes

Table 2 reports the change in corporate policies and outcomes in the setup with endogenous bid–ask spread compared to a benchmark environment with zero bid–ask spread (and, thus, perfect stock liquidity). Namely, the table reports the change in the target cash level, in the probability of liquidation, in the probability of payout, in the zero-NPV investment cost, and in firm value (to fix ideas, calculated at the target cash level) as well as the bid–ask spread (also calculated at the target cash level) for different values of the order-processing cost $v$ (first panel), of the trading firms' outside opportunity $\omega$ (second panel), of the trading firms' bargaining power $\theta$ (third panel), and of the shocked shareholders' holding cost $\chi$ (fourth panel).

| | Target Cash Level | Liquidation Probability | Payout Probability | Zero-NPV Cost | Firm Value | Bid-Ask Spread (BPS) |
|---|---|---|---|---|---|---|
| $v=0.001$ | −12.01% | 1.77% | 2.79% | −13.88% | −17.95% | 58 |
| $v=0.003$ | −15.56% | 2.39% | 3.71% | −17.71% | −21.59% | 73 |
| $v=0.005$ | −19.36% | 3.13% | 4.75% | −21.34% | −24.97% | 88 |
| $v=0.007$ | −23.54% | 4.03% | 5.97% | −24.81% | −28.12% | 103 |
| $\omega=0.003$ | −10.91% | 1.59% | 2.51% | −13.69% | −16.32% | 52 |
| $\omega=0.006$ | −12.59% | 1.87% | 2.93% | −13.99% | −18.77% | 62 |
| $\omega=0.009$ | −14.50% | 2.20% | 3.43% | −14.36% | −21.23% | 72 |
| $\omega=0.012$ | −16.73% | 2.61% | 4.02% | −14.83% | −23.73% | 83 |
| $\theta=0.25$ | −10.85% | 1.58% | 2.50% | −12.34% | −16.73% | 54 |
| $\theta=0.45$ | −15.51% | 2.38% | 3.70% | −18.23% | −21.37% | 72 |
| $\theta=0.65$ | −20.36% | 3.33% | 5.04% | −23.51% | −25.47% | 90 |
| $\theta=0.85$ | −25.57% | 4.50% | 6.59% | −28.30% | −29.13% | 107 |
| $\chi=0.0075$ | −8.73% | 1.24% | 1.98% | −9.94% | −14.14% | 44 |
| $\chi=0.0100$ | −10.54% | 1.52% | 2.42% | −12.16% | −16.29% | 52 |
| $\chi=0.0125$ | −12.38% | 1.83% | 2.88% | −14.30% | −18.35% | 60 |
| $\chi=0.0150$ | −14.27% | 2.16% | 3.37% | −16.37% | −20.32% | 68 |

TABLE 3

The Firm's Probability of Liquidation with Stock Market Illiquidity

Table 3 reports the firm's probability of liquidation at different levels of cash reserves (i.e., at $C_V/2$, $C_V/4$, and $C_V/8$) when varying the order-processing cost $v$ (first panel), the trading firms' outside opportunity $\omega$ (second panel), the trading firms' bargaining power $\theta$ (third panel), and the shocked shareholders' holding cost $\chi$ (fourth panel). The bottom line reports the probability of liquidation when the firm's stock is perfectly liquid (i.e., when the bid–ask spread is 0).

| | $C_V/2$ | $C_V/4$ | $C_V/8$ |
|---|---|---|---|
| *Varying $v$* | | | |
| $v=0.001$ | 3.18% | 17.88% | 42.34% |
| $v=0.003$ | 3.67% | 19.17% | 43.84% |
| $v=0.005$ | 4.28% | 20.67% | 45.51% |
| $v=0.007$ | 5.08% | 22.45% | 47.41% |
| *Varying $\omega$* | | | |
| $\omega=0.003$ | 3.05% | 17.49% | 41.89% |
| $\omega=0.006$ | 3.26% | 18.08% | 42.58% |
| $\omega=0.009$ | 3.52% | 18.78% | 43.39% |
| $\omega=0.012$ | 3.85% | 19.62% | 44.35% |
| *Varying $\theta$* | | | |
| $\theta=0.25$ | 3.04% | 17.47% | 41.86% |
| $\theta=0.45$ | 3.67% | 19.15% | 43.82% |
| $\theta=0.65$ | 4.46% | 21.08% | 45.96% |
| $\theta=0.85$ | 5.52% | 23.37% | 48.37% |
| *Varying $\chi$* | | | |
| $\chi=0.0075$ | 2.79% | 16.76% | 41.00% |
| $\chi=0.0100$ | 3.00% | 17.37% | 41.74% |
| $\chi=0.0125$ | 3.23% | 18.01% | 42.50% |
| $\chi=0.0150$ | 3.49% | 18.69% | 43.29% |
| Perfect liquidity | 1.97% | 14.11% | 37.64% |

Table 2 also shows that the primitive parameters determining the bid-ask spread have a substantial impact on the firm's investment decisions, consistent with the evidence in Goldberg (2020). Namely, they lead to a sharp reduction in the maximum price that the firm is willing to pay to increase the cash flow drift from $\mu$

to $\mu_+$. Under the baseline parameterization, such a decrease in the maximum investment cost is 13.9%.[24] Overall, primitive parameters affecting the bid-ask spread have a substantial, detrimental effect on firm value, as quantified in the last column of Table 2; under the baseline parameterization, the decrease is about 18%.

### 3. Reassessing Standard Determinants of Cash Reserves and Liquidation Probability

The two-way relation between stock liquidity and firm value implies that shocks to firm characteristics bear indirect effects. Given the focus on cash reserves, I investigate the impact of shocks to major determinants of corporate cash holdings like the firm's access to external financing, cash flow volatility, and the return on cash (see Opler et al. (1999), Bates et al. (2009), and Azar, Kagy, and Schmalz (2016)).

Consider first a tightening in the firm's access to external financing; in the model, this shock is captured by a decrease in the parameter $\lambda$.[25] As a direct effect, the firm's precautionary demand for cash increases (and so does the target cash level) and firm value declines. Yet, when the bid-ask spread and firm value are jointly determined, such a direct effect simultaneously leads to an increase in the bid-ask spread which, as suggested by Proposition 2, should push the target cash level down. That is, this indirect effect should hamper the firm's ability to accumulate cash exactly when its demand for cash increases (precisely because access to financing is tighter).

Similar effects can be considered in the wake of an increases in the volatility of cash flows, $\sigma$. As a direct effect, an increase in $\sigma$ expands the firm's precautionary demand for cash (pushing the target cash level up) and decreases firm value.[26] When the bid-ask spread and firm value are jointly determined, yet, such a direct effect simultaneously leads to an increase in the bid-ask spread and should push the firm's target cash level down (a strength that goes in the opposite direction of the direct effect). We have the following result:

*Proposition 3.* The indirect effect of shocks to the access to external financing $\lambda$ or cash flow volatility $\sigma$ on the target cash level $C_V$ via stock illiquidity does not offset the direct effect of such shocks via the demand for cash. Thus, $C_V$ decreases with $\lambda$ and increases with $\sigma$.

This proposition illustrates that the indirect effect of shocks to $\lambda$ and $\sigma$ (channeled by the two-way relation between stock liquidity and corporate policies and value) does not offset the direct effect of shocks to $\lambda$ and $\sigma$ on the target cash level

[24] Equation (15) in Proposition 2 illustrates that $\omega$ affects $I_V$ through its impact on the target cash level only (whether the other parameters additionally enter this expression directly by affecting its denominator). In fact, as shown in Section SA.2 of the Supplementary Material, $\omega$ has no impact on the zero-NPV cost when there are no financing frictions, as the firm does not keep any cash.

[25] Indeed, a decrease in $\lambda$ implies that the firm's access to external financing is more uncertain (or, equivalently, it takes a longer time in expectation to secure financing upon searching).

[26] Because firm value is concave in cash reserves due to the presence of financing frictions (as discussed in Section III and proved in Appendix A), an increase in $\sigma$ decreases firm value (a standard result in cash management models).

TABLE 4

Cash Hoarding and Forced Liquidation with Endogenous Bid-Ask Spread

Table 4 reports the target cash level and the average liquidation probability in the benchmark environment in which stock liquidity is perfect (i.e., the bid–ask spread is 0, second and fifth columns), in the baseline parameterization with endogenous bid–ask spread (third and sixth columns), and the wedge between the two environments (fourth and seventh columns) when varying the firm's access to external financing ($\lambda$), cash flow volatility ($\sigma$), and the return on cash ($r$).

| | Target Cash Level | | | Liquidation Probability | | |
|---|---|---|---|---|---|---|
| | $C^*$ | $C_V$ | Wedge | $\bar{P}_l(C^*)$ | $\bar{P}_l(C_V)$ | Wedge |
| $\lambda = 0.25$ | 0.638 | 0.548 | −14.16% | 15.08% | 17.79% | 2.71% |
| $\lambda = 0.50$ | 0.584 | 0.509 | −12.84% | 13.60% | 15.66% | 2.06% |
| $\lambda = 0.75$ | 0.546 | 0.480 | −12.01% | 12.78% | 14.55% | 1.77% |
| $\lambda = 1.00$ | 0.516 | 0.458 | −11.41% | 12.22% | 13.81% | 1.59% |
| $\lambda = 1.25$ | 0.493 | 0.439 | −10.95% | 11.81% | 13.27% | 1.46% |
| $\sigma = 0.08$ | 0.347 | 0.312 | −9.83% | 11.48% | 12.75% | 1.27% |
| $\sigma = 0.10$ | 0.448 | 0.399 | −10.92% | 12.18% | 13.69% | 1.51% |
| $\sigma = 0.12$ | 0.546 | 0.480 | −12.01% | 12.78% | 14.55% | 1.77% |
| $\sigma = 0.14$ | 0.640 | 0.556 | −13.11% | 13.31% | 15.35% | 2.04% |
| $\sigma = 0.16$ | 0.731 | 0.626 | −14.26% | 13.80% | 16.13% | 2.33% |
| $r = 0.0050$ | 0.512 | 0.446 | −12.72% | 13.69% | 15.73% | 2.04% |
| $r = 0.0075$ | 0.527 | 0.463 | −12.28% | 13.25% | 15.14% | 1.89% |
| $r = 0.0100$ | 0.546 | 0.480 | −12.01% | 12.78% | 14.55% | 1.77% |
| $r = 0.0125$ | 0.568 | 0.500 | −11.99% | 12.25% | 13.94% | 1.69% |
| $r = 0.0150$ | 0.598 | 0.523 | −12.50% | 11.62% | 13.29% | 1.67% |

$C_V$. That is, when bid-ask spread and firm value and policies affect each other, a tightening in external financing or an increase in cash flow volatility leads the target cash level to increase but by less than if stock liquidity was perfect. Table 4 investigates these effects quantitatively. It shows that the wedge between $C_V$ and $C^*$ (i.e., the target cash level if the bid-ask spread is 0) widens as $\lambda$ decreases. That is, while $C^*$ increases as a direct effect of the tightening in capital supply, $C_V$ increases by less. The middle panel of the table focuses on cash flow volatility. It shows that the gap between $C_V$ and $C^*$ widens as $\sigma$ rises: Whereas $C^*$ increases as a direct effect of the increase in $\sigma$, $C_V$ increases by less.

Shocks to the firm's access to financing naturally impact its probability of liquidation. If access to financing tightens (i.e., $\lambda$ decreases), the firm's probability of liquidation increases as a direct effect. As just explained, such shock also leads the firm to keep smaller cash reserves compared to the case with perfect liquidity. This, in turn, pushes the probability of liquidation higher, beyond the impact solely driven by the aforementioned direct effect. Consistently, Table 4 lists that the probability of liquidation between the case with endogenous bid-ask spread and the case with perfect stock liquidity widens as $\lambda$ decreases. Similarly, while an increase in $\sigma$ directly inflates the probability of liquidation, the lower firm's reliance on cash reserves increases such probability further. Table 4 indeed confirms that the gap between the probability of liquidation with endogenous liquidity and the benchmark with perfect liquidity widens as $\sigma$ increases.

Consider now another standard determinant of corporate cash holdings: the return on cash $r$ which, in turn, affects the cost of keeping a precautionary cash buffer (see, e.g., Azar et al. (2016)). That is, whereas the analysis so far has examined determinants affecting the *benefit* of cash, I now focus on shocks to the opportunity cost of cash that are orthogonal to stock liquidity (i.e., shocks to $r$). When $r$ drops, keeping cash becomes more costly irrespectively of stock illiquidity,

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so the optimal level of cash reserves decreases. When stock liquidity is endogenous, however, a decrease in $r$ also leads to a drop in firm value, the bid-ask spread rises, and the target cash level should decrease further. As a result, when stock liquidity is endogenous, a negative shock to $r$ has a more detrimental effect on cash reserves and on the probability of liquidation compared to the benchmark case with perfect stock liquidity. The bottom panel of Table 4 illustrates these effects. As $r$ drops, the wedge between $C_V$ and $C^*$ widens, and the probability of liquidation increases.[27]

Overall, the analysis illustrates that the interplay between stock liquidity and corporate policies and value makes firms less financially resilient. Indeed, such interplay amplifies the impact of adverse shocks on the probability of forced liquidations.

### 4. Further on the Indirect Effects of Shocks

The analysis so far shows that the intertwined relation between corporate choices and stock liquidity implies that shocks to firm characteristics have indirect effect. Consider now the impact of operating shock. Negative (resp. positive) operating shocks deplete (replenish) the cash reserves and decrease (increase) firm value. The next result follows:

*Proposition 4.* For an operating shock of a given size, the resulting change in the bid price and in the bid-ask spread is greater if such shock is negative.

Proposition 4 shows that stock illiquidity reacts asymmetrically to negative or positive cash flow shocks. In fact, concavity of firm value in cash reserves, which stems from the presence of financing frictions as in previous cash management models (as discussed in Section III and proved in Appendix A), implies that, for an operating shock of a given size, firm value is more sensitive if the shock is negative (than if it is positive). Consistently, Hameed, Kang, and Viswanathan (2010) show that liquidity responds asymmetrically to shocks to asset values, deteriorating more sharply after negative ones. Figure 1 supports this pattern as the bid-ask spread is a steeper function of $c$ as the firm gets closer to the liquidation boundary (i.e., as $c$ gets closer to 0). Moreover, the more profitable the outside opportunity of trading firms $\omega$ are, the steeper the bid-ask spread is, and more so as $c$ gets closer to 0. The reason is that, as highlighted by equation (7), $\omega$ directly determines how much a decrease in firm value (e.g., driven by a cash flow shock) impacts the bid-ask spread too. That is, the parameter $\omega$ makes the bid-ask spread more sensitive to operating shocks and, thus, to changes in firm value.[28]

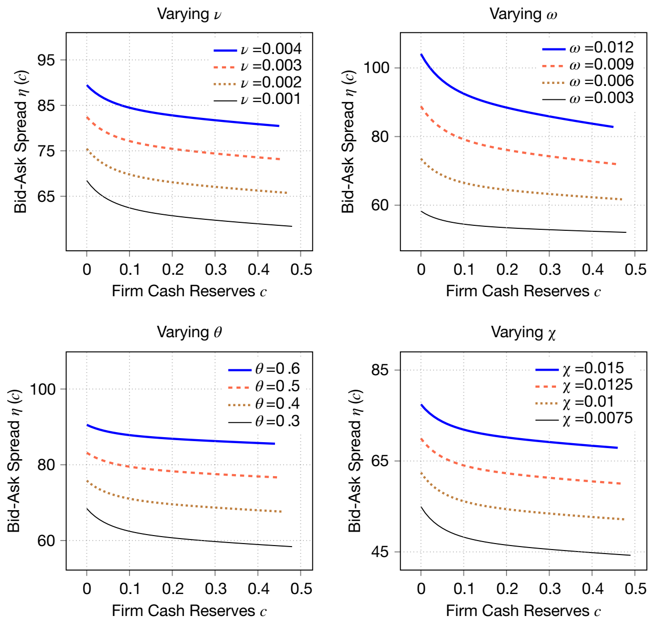
[27]It is worth noting that a non-monotonicity can arise in the wedge between $C_V$ and $C^*$ as $r$ gets closer to $\rho$, as shown in the table. The reason is that when stock liquidity is perfect, the opportunity cost of cash goes to 0 as $r$ approaches $\rho$, making it optimal to accumulate an unbounded cash reserves (i.e., $C^*$ goes to infinity). In turn, when the stock is illiquid, there continues to be an opportunity cost of holding cash even when $r \rightarrow \rho$, precisely driven by stock illiquidity. Thus, as $r$ approaches $\rho$, $C^*$ goes to infinity, whereas $C_V$ does not, then driving the observed non-monotonicity in the wedge $C_V/C^* - 1$.

[28]In turn, $\nu$ and $\chi$ do not enter the part of the bid-ask spread that is dependent on firm value. Conversely, the parameter $\theta$ affects both the part of the bid-ask spread that is independent of firm value (the first two terms in equation (7)) as well as the part that is dependent (the last term in equation (7)). Yet, because the greater $\theta$, the lower the weight of the term $\omega/V(c)$, the bid-ask spread becomes more responsive to cash flow shocks as $\theta$ decreases.

FIGURE 1

Endogenous Bid-Ask Spread

Figure 1 shows the endogenous bid–ask spread (in BPS) as a function of the firm cash reserves $c$ when varying the order-processing cost $\nu$, trading firms' outside opportunity $\omega$, trading firms' bargaining power $\theta$, and shocked shareholders' holding cost $\chi$.




Consider now shocks to the parameters that directly affect the liquidity of the firm's stock (i.e., shocks to $\nu$, $\omega$, $\theta$, or $\chi$). Through the two-way relation singled out by the model, such shocks not only impact stock liquidity but also corporate policies and value. That is, the model illustrates that such shocks affect the cost of stock illiquidity borne by shocked shareholders through a direct *and* an indirect channel. Indeed, an increase in these parameters leads to a direct decrease in the bid price at which shocked shareholders sell their stock (see equation (6))—equivalently, an increase in the bid-ask spread (equation (7)). This is the direct effect. Yet, as shown by Proposition 2, an increase in these parameters also affects corporate policies and reduces firm value, which, by equations (6) and (7), lead to a further decrease in the bid price and a further increase in the bid-ask spread. This is the indirect effect.

Table 5 analyzes the magnitude of these effects by considering increases (of different sizes) in these parameters with respect to the baseline parameterization. Namely, it reports the bid price and its percentage change following a shock to each of these parameters when ignoring (second and third columns) and when acknowledging (fourth and fifth columns) the two-way relation between stock liquidity and firm value. Across the different parameters and the magnitude of the increases, the table consistently shows that transactions become more costly for shocked shareholders (as the bid price decreases) when allowing for such two-way relation. For instance, an increase in the trading firms' bargaining power $\theta$ by 0.1 (over its baseline value of 0.3) leads to an about 0.1% decrease in the price at which shocked

TABLE 5

Shocks to Determinants of Stock Liquidity

Table 5 reports the bid price as well as the percentage change in the bid price following shocks increasing the order-processing cost $v$ (first panel, where I denote the magnitude of the increase by $\Delta v$), in the trading firms' outside opportunity $\omega$ (second panel), in the trading firms' bargaining power $\theta$ (third panel), and in the shocked shareholders' holding cost $\chi$ (fourth panel) compared to their baseline value. I focus on the direct effect in the second and third columns, whereas on the overall effect (including the indirect effect on corporate policies and value) in the fourth and fifth columns. To fix ideas, I calculate the bid–ask at the target cash level.

| | Direct Effect Only | | With Indirect Effect | |
|---|---|---|---|---|
| | Bid Price | Change | Bid Price | Change |
| *Baseline value:* $v = 0.001$ | | | | |
| $\Delta v = +0.001$ | 2.260 | −0.07% | 2.209 | −2.34% |
| $\Delta v = +0.002$ | 2.259 | −0.14% | 2.158 | −4.58% |
| $\Delta v = +0.003$ | 2.257 | −0.21% | 2.109 | −6.74% |
| $\Delta v = +0.004$ | 2.255 | −0.28% | 2.062 | −8.83% |
| *Baseline value:* $\omega = 0.005$ | | | | |
| $\Delta\omega = +0.001$ | 2.261 | −0.03% | 2.239 | −1.03% |
| $\Delta\omega = +0.003$ | 2.260 | −0.09% | 2.192 | −3.10% |
| $\Delta\omega = +0.005$ | 2.258 | −0.15% | 2.145 | −5.18% |
| $\Delta\omega = +0.007$ | 2.257. | −0.22% | 2.097 | −7.27% |
| *Baseline value:* $\theta = 0.30$ | | | | |
| $\Delta\theta = +0.1$ | 2.260 | −0.09% | 2.196 | −2.92% |
| $\Delta\theta = +0.2$ | 2.258 | −0.18% | 2.134 | −5.65% |
| $\Delta\theta = +0.3$ | 2.256 | −0.27% | 2.076 | −8.23% |
| $\Delta\theta = +0.4$ | 2.254 | −0.35% | 2.021 | −10.65% |
| *Baseline value:* $\chi = 0.012$ | | | | |
| $\Delta\chi = +0.002$ | 2.261 | −0.06% | 2.216 | −2.01% |
| $\Delta\chi = +0.004$ | 2.259 | −0.12% | 2.173 | −3.95% |
| $\Delta\chi = +0.006$ | 2.258 | −0.18% | 2.130 | −5.83% |
| $\Delta\chi = +0.008$ | 2.256 | −0.24% | 2.089 | −7.65% |

shareholders manage to sell their stock. However, when accounting for the indirect effect, the decrease is notably wider and equal to about 2.90%.

A natural question arises as for which firms the interplay between the firm and the liquidity of its stock is stronger. Table 6 points to smaller firms (in the model, exhibiting a lower $\mu$).[29] Notably, the table illustrates three important points. First, for lower $\mu$, the firm exhibits a larger bid-ask spread; this is consistent with the empirical observation that smaller firms are typically less liquid. Second, the relation between $\mu$ and the bid-ask spread is nonlinear; namely, a further decline in $\mu$ when it is lower leads to a relatively larger increase in the bid-ask spread. Third, this translates into a sharper drop in firm value compared to the counterfactual environment with perfect liquidity (see the last column of the table). That is, the ensuing deterioration in liquidity amplifies the impact of a decrease in $\mu$ on firm value. Lastly, the middle and bottom panels of this table report that such deterioration in liquidity and firm value is even bigger if $v$ or $\omega$ are larger.[30]

## C. Testable Predictions

This paper provides a theoretical framework that delivers a unified explanation for a set of empirical regularities relating stock market liquidity and corporate

[29]Empirically, firm size is typically gauged through firm sales, whose model counterpart is $\mu$.

[30]Whereas this table does not consider increases in the parameters $\theta$ and $\chi$ for the sake of brevity, such results are available from the authors.

TABLE 6

**Bid-Ask Spread and Firm Value in the Cross Section**

Table 6 reports the bid–ask spread (second column), the bid price (third column), and the firm value changes due to endogenous illiquidity (fourth column) when varying the parameter $\mu$, serving as a gauge for size. To fix ideas, these quantities are calculated when the firm holds its target cash level. The top panel focuses on the baseline parameterization (see Table 1), whereas the middle and bottom panels assume costlier liquidity provision (by assuming that $\omega = 0.008$ in the middle panel and $v = 0.005$ in the bottom panel).

| | Bid-Ask Spread | Bid Price | Value Drop (%) |
|---|---|---|---|
| *Baseline* | | | |
| $\mu = 0.03$ | 68 | 1.376 | −21.73% |
| $\mu = 0.04$ | 62 | 1.824 | −19.29% |
| $\mu = 0.05$ | 58 | 2.262 | −17.95% |
| $\mu = 0.06$ | 56 | 2.695 | −17.09% |
| $\mu = 0.07$ | 54 | 3.126 | −16.49% |
| $\mu = 0.08$ | 53 | 3.555 | −16.04% |
| *Higher* $\omega = 0.008$ | | | |
| $\mu = 0.03$ | 86 | 1.296 | −26.14% |
| $\mu = 0.04$ | 75 | 1.751 | −22.41% |
| $\mu = 0.05$ | 68 | 2.192 | −20.41% |
| $\mu = 0.06$ | 64 | 2.626 | −19.14% |
| $\mu = 0.07$ | 61 | 3.058 | −18.24% |
| $\mu = 0.08$ | 59 | 3.487 | −17.58% |
| *Higher* $v = 0.005$ | | | |
| $\mu = 0.03$ | 99 | 1.238 | −29.39% |
| $\mu = 0.04$ | 92 | 1.656 | −26.50% |
| $\mu = 0.05$ | 88 | 2.062 | −24.97% |
| $\mu = 0.06$ | 85 | 2.462 | −24.02% |
| $\mu = 0.07$ | 83 | 2.860 | −23.37% |
| $\mu = 0.08$ | 82 | 3.255 | −22.89% |

policies. As shown by Propositions 1 and 2, the paper shows that firms whose stocks are traded at a higher bid-ask spread hold less cash (Nyborg and Wang (2021)), pay out more dividends (Banerjee et al. (2007)), have a greater default probability (Brogaard et al. (2017)), invest less (Campello et al. (2014), Amihud and Levi (2023)), and are overall less valuable (Fang et al. (2009)). On top of these results, the analysis suggests novel testable predictions that exploit the interplay between bid-ask spread and the policies and value of the issuing firm.

First, the model suggests that empirical tests aimed at examining the determinants of corporate cash holdings should be revisited to account for their simultaneous impact on stock liquidity. The analysis indeed indicates that standard determinants such as the firm's access to external financing and cash flow volatility affect cash reserves not only through a benefit channel but, because they also impact the firm's bid-ask spread, they should also affect its opportunity cost. As discussed, the latter effect is novel to the literature and, importantly, should weaken the firm's ability to accumulate cash when its demand for cash increases. Thus, empirical tests could shed new light on the quantitative impact of such cash determinants by controlling for the simultaneous impact on stock liquidity.

Second, and related to the previous point, the model suggests that exogenous shocks to the firm's access to external financing or to cash flow volatility have an amplified impact on the firm's probability of liquidation when firm choices and stock liquidity are jointly determined. Thus, such two-way relation makes firms more fragile and more vulnerable to forced liquidations in the wake of such adverse shocks. Empirical work could then exploit exogenous shocks to the firm's access to external financing, which directly affect the firm's probability of liquidation, and investigate how illiquidity plays a role in amplifying such probability.[31] Cross-sectional heterogeneity could help gage the extent of the amplification, as the empiricist could identify firms that are insulated from the effect of illiquidity (and, thus, its impact on the probability of liquidation).[32]

Third, the analysis suggests that, to gauge the severity of a firm's financial constraints, empiricists should account for stock illiquidity. Indeed, the analysis in the paper indicates that stock liquidity affects the firm's probability of financing and of liquidation, then affecting its degree of financial constraints. While measures of financial constraints typically harness firm characteristics as useful predictors of financial constraints (such as the WW index suggested by Whited and Wu (2006) or the SA index of Hadlock and Pierce (2010)), this paper suggests that the liquidity of the firm's stock could improve the predictive power of such measures.

Fourth, the model suggests that shocks exacerbating the liquidity of a firm's stock bear an amplified impact when they also affect firm policies and value. A test of this prediction could then exploit exogenous (unexpected) shocks to the costs borne by liquidity providers or to their bargaining power, for instance. As liquidity providers are typically active in several stocks, such shocks would affect a pool of stocks, which would then help exploit a cross-sectional dimension that could validate the model mechanism. Indeed, the model suggests that the smaller stocks in the affected portfolio should be more exposed to the two-way relation between liquidity and firm value and, thus, should experience both the direct *and* the indirect effect described in Section IV.B. Following the shock, such stocks should experience a sharper change in their corporate policies. Namely, they would keep less cash and pay out dividends more often. Moreover, they would become less resilient to negative operating shocks, then exhibiting an increase in their liquidation probability. In addition, such firms would curtail their investment and become less valuable. Testing these effects would validate the mechanism at play in the model. Notably, the gap in the increase in the bid-ask spread between the firm exhibiting the largest deviations in corporate policies and those exhibiting no changes would help gage the extent of the amplification effect. In fact, the firms exhibiting no changes in corporate policies should be insulated by the two-way relation described in the paper.

## V. Concluding Remarks

This paper develops a model that sheds light on the two-way relation between a stock's illiquidity and the policies and value of the issuing firm. The model shows that bid-ask spreads increase the firms' cost of capital *and* the opportunity cost of cash. As such, they make firms more financially constrained, more exposed to forced liquidations, less prone to invest, and less valuable. The model shows that these outcomes

[31]For instance, Duchin, Ozbas, and Sensoy (2010) and Campello, Graham, and Harvey (2010) investigate the effects of shocks to the supply of financing using the 2007–2009 financial crisis as a laboratory.

[32]In the context of studying real effects of financial markets, Bond, Edmans, and Goldstein (2012) point to exploiting cross-sectional heterogeneity to identify firms that might be insulated from the effect of illiquidity and any potential feedback effect.

get reinforced when internalized by liquidity providers, leading to a wider bid-ask spread and lower firm value. This mechanism implies that frictions faced by liquidity providers are passed on to the firm's investors and, through this channel, have an impact on the policies, values, and survival rates of small firms. Overall, this two-way relation implies that shocks arising within the firm or in the market for its stock have more nuanced impacts than previously understood. More generally, the model suggests that the architecture of secondary market transactions has a prime effect on corporate decisions,[33] especially for firms that face severe financing frictions.

## Appendix A. Proof of the Results in Section III

The endogenous bid price and bid-ask spread stemming from Nash bargaining satisfy the expression reported in equations (6) and (7), using standard arguments and by straightforward calculations. Substituting equation (7) into equation (5) gives

$$(\text{A-1}) \qquad (\rho+\delta\nu(1-\theta)+\delta\chi\theta)V=(rc+\mu)V'+\frac{\sigma^2}{2}V''+\lambda[V(C_V)-C_V+c-V(c)] \\ -\delta\omega(1-\theta)$$

for any $c \le C_V$. Firm value is then solved subject to the boundary condition at the liquidation threshold and at $C_V$, as reported in the main text. To simplify the notation throughout, we define

$$(\text{A-2}) \qquad \Psi \equiv \delta\nu(1-\theta)+\delta\chi\theta.$$

It is possible to show that $V(c)$ is increasing and concave in $c$, as shown in the next lemma.

*Lemma A1.* $V'(c)>1$ and $V''(c)<0$ for any $c\in[0,C_V)$.

*Proof.* Simply differentiating equation (A-1) gives

$$(\rho+\lambda+\Psi-r)V'(c)=V''(c)(rc+\mu)+\frac{\sigma^2}{2}V'''(c)+\lambda.$$

By the conditions $V'(C_V)=1$ and $V''(C_V)=0$, it follows that $V'''(C_V)=\frac{2}{\sigma^2}(\rho+\Psi-r)>0$ as $r\le\rho$. Thus, there exists a left neighborhood of $C_V$ such that for any $c\in(C_V-\varepsilon,C_V)$, with $\varepsilon>0$, the inequalities $V'(c)>1$ and $V''(c)<0$ hold. Toward a contradiction, I assume that $V'(c)<1$ for some $c\in[0,C_V-\varepsilon]$. Then there exists a point $C_c\in[0,C_V-\varepsilon]$ such that $V'(C_c)=1$ and $V'(c)>1$ over $(C_c,C_V)$, so

$$(\text{A-3}) \qquad V(C_V)-V(c)>C_V-c$$

for any $c\in(C_c,C_V)$. For any $c\in(C_c,C_V)$, it must be also that

$$V''(c)=\frac{2}{\sigma^2}\{(\rho+\lambda+\Psi)V(c)-[rc+\mu]V'(c)-\lambda(V(C_V)+c-C_V)+\delta\omega(1-\theta)\}.$$

Using (A-3), jointly with $V(C_V)=\frac{rC_V+\mu-\delta\omega(1-\theta)}{\rho+\Psi}$, it follows that

[33]In this context, see Foucault, Pagano, and Roell ((2013), Chapter 10).

$$V''(c) < \frac{2}{\sigma^2}\{(\rho+\Psi)(V(C_V)+c-C_V)-rc-\mu+\delta\omega(1-\theta)\} = \frac{2}{\sigma^2}(c-C_V)(\rho+\Psi-r) < 0.$$

This means that $V'(c)$ is decreasing for any $c \in (C_c, C_V)$, which contradicts $V'(C_c) = V'(C_V) = 1$. It follows that $C_c$ cannot exist. So, $V'(c) > 1$ and $V''(c) < 0$ for any $c \in [0, C_V)$, and the claim follows. ■.

It is worth noting that bargaining goes through as long as the inequality $(\chi - \nu)V(c) > \omega$ holds[34]; given the monotonicity of firm value in $c$ proved in the above lemma, if such inequality holds at $c = 0$, it does hold for any $c$. Hence, $(\chi - \nu)\ell > \omega$ guarantees that the bid-ask spread satisfies an internal solution, which is assumed throughout our analysis.[35]

Equation (A-1) is similar to the HJB equation in HMM up to the effects of stock illiquidity. Notably, the bid-ask spread is not a direct policy of the firm, so the firm optimizes over a similar set of decisions. It is possible to exploit similar verification arguments.[36]

## Appendix B. Proof of Proposition 1

When the bid-ask spread is exogenous, equation (5) boils down to equation (11) in the main text. In this case too, it is optimal for the firm to raise funds up to the target cash level, as explained in Section III.

### B.1. Claim (1): Monotonicity of the Target Cash Threshold

I express the function $V(c)$ as a function of $X$, denoting the threshold satisfying $V'(X,X) - 1 = V''(X,X) = 0$. To prove the claim, I exploit the following auxiliary results

*Lemma B1.* The function $V(c,X)$ is decreasing in $X$.

*Proof.* To prove the claim, I take $X_1 < X_2$, and I define the auxiliary function $k(c) = V(c,X_1) - V(c,X_2)$, which satisfies

(B-1) $$(\rho+\delta\eta+\lambda)k(c) = (rc+\mu)k'(c) + 0.5\sigma^2 k''(c) + \lambda(X_1 - X_2)[r/(\rho+\delta\eta) - 1]$$

for any $c \in [0, X_1]$. By calculations, the function is positive at $X_2$ as $k(X_2) = (X_1 - X_2)[r/(\rho+\delta\eta) - 1] > 0$. By the definition of $X_1$ and $X_2$, the function $k(c)$ is decreasing and convex for $c \in [X_1, X_2)$. Therefore, $k(X_1) > 0$. Consider now the first derivative of the previously defined function, $k'(c)$, which satisfies

[34]From the perspective of shocked shareholders, this implies that $\eta(c)$ does not exceed $\chi$. From the perspective of trading firms, it means that the gain from trading (i.e., the bid-ask spread net of the order-processing cost) should exceed what trading firms can get from their outside option.

[35]Whereas this condition is imposed to simplify the analysis, it is possible to solve the model (and obtain similar implications) when relaxing it. When such inequality fails to hold, there is at most one threshold $\underline{C} \in [0, C_V]$ such that the proportional loss borne by liquidity-shocked investors is equal to $\eta(c)$ for any $c \in [\underline{C}, C_V]$, whereas it is equal to $\chi$ for any $c \in [0, \underline{C}]$.

[36]See also DMRV; in that setup, equity issuance is not stochastic but costly.

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$$\text{(B-2)} \qquad (\rho+\delta\eta+\lambda-r)k'(c)=(rc+\mu)k''(c)+0.5\sigma^2 k'''(c),$$

simply exploiting equation (B-1). Note that $k'(c)$ does not have a positive local maximum nor a negative local minimum; otherwise, equation (B-2) would not hold (respectively, $k'(c)>0=k''(c)>k'''(c)$ and $k'(c)<0=k''(c)<k'''(c)$ at a positive maximum and at a negative minimum). As $k$ is convex at $X_1$, this means that $k'$ is increasing at $X_1$, and therefore it must be negative for any $c\in[0,X_1]$. Jointly with $k(X_1)>0$, this means that $k(c)>0$ for any $c\in[0,X_2]$. The claim follows. ■.

*Lemma B2*. For a given payout threshold $X$ and two given $\eta_1>\eta_2$, $V(c,X,\eta_2)>V(c,X;\eta_1)$ holds for any $c\in[0,X]$.

*Proof.* I define the auxiliary function $h(c)=V(c,X;\eta_2)-V(c,X;\eta_1)$. I need to prove that, for a given payout threshold $X$, $h(c)>0$ for any $c\in[0,X]$. At $X$, the function is positive as

$$h(X)=(rX+\mu)\left(\frac{1}{\rho+\delta\eta_2}-\frac{1}{\rho+\delta\eta_1}\right)=(rX+\mu)\frac{\delta\eta_1-\delta\eta_2}{(\rho+\delta\eta_1)(\rho+\delta\eta_2)}>0,$$

as $h'(X)=h''(X)=0$. In addition, the function satisfies

$$[rc+\mu]h'(c)+\frac{\sigma^2}{2}h''(c)-(\rho+\lambda+\delta\eta_2)h(c)+\lambda h(X)=(\delta\eta_2-\delta\eta_1)V(c,X;\chi_1)$$

and the right-hand side is negative. Differentiating gives

$$[rc+\mu]h''(c)+\frac{\sigma^2}{2}h'''(c)-(\rho+\lambda+\delta\eta_2-r)h'(c)=(\delta\eta_2-\delta\eta_1)V'(c,X;\chi_1)$$

At $X$, I get $\frac{\sigma^2}{2}h'''(X)=\delta\eta_2-\delta\eta_1$, meaning that $h'''(X)<0$. This means that the second derivative is decreasing in a neighborhood of $X$, so one has $h''(c)>0$ in a left neighborhood of $X$ (recall that $h''(X)=0$). In turn, this means that $h'(c)$ is increasing in such a neighborhood of $X$, then implying that $h'(c)<0$ in a left neighborhood of $X$. Note that, by the ODE above, $h'(c)$ cannot have a negative local minimum. As $h'(X)=0$ and it is negative and increasing in a left neighborhood of $X$, this means that $h'(c)$ should be negative for any $c<X$, so $h(c)$ is always decreasing. As it is positive at $X$, it means that it should be always positive, so $h(c)>h(X)>0$; therefore, it is positive for any $c<X$. ■.

Exploiting the results above, I can prove the following lemma:

*Lemma B3*. For any $\eta_1>\eta_2$, $C_V(\eta_1)<C_V(\eta_2)$.

*Proof.* The payout thresholds $C_V(\eta_1)$ and $C_V(\eta_2)$ are the unique solution to the boundary conditions $V(0,C_V(\eta_2);\eta_2)-\ell=0=V(0,C_V(\eta_1);\eta_1)-\ell$. Exploiting Lemma B2, I now take, for instance, $X=C_V(\eta_1)$. It follows that $V(0,C_V(\eta_1);\eta_2)-\ell>0=V(0,C_V(\eta_1);\eta_1)-\ell$. As $V$ is decreasing in the payout threshold, this means that $C_V(\eta_1)<C_V(\eta_2)$ to get the equality $\ell-V(0,C_V(\eta_2);\eta_2)=0$. The claim follows. ■.

The next results stem from Lemma B3.

*Corollary B4.* When the bid-ask spread is positive, the target cash level is lower than in the benchmark case with no bid-ask spread (i.e., $C_V < C^*$).

### B.2. Claim (2): Probability of Payout

Using the insights from Dixit and Pindyck (1994), the dynamics of $P_p(c,X)$ are given by $P_p'(c)(rc+\mu)+\frac{\sigma^2}{2}P_p''(c)-\lambda P_p(c)=0$ subject to $P_p(0)=0$ and $P_p(X)=1$. The first boundary condition implies that when the controlled cash process is absorbed at 0, the firm liquidates and the payout probability is 0. The second boundary condition is obvious given that cash is paid out at $X$. The following lemma shows that greater bid-ask spreads are associated with larger payout probability:

*Lemma B5.* For any $\eta_1 > \eta_2$, $P_p(c,C_V(\eta_1)) \geq P_p(c,C_V(\eta_2))$.

*Proof.* By Lemma B3, $C_V(\eta_1) < C_V(\eta_2)$. To ease the notation throughout the proof, I define $X_1 \equiv C_V(\eta_1)$ and $X_2 \equiv C_V(\eta_2)$. Consider the function $h(c)=P_p(c,X_1)-P_p(c,X_2)$. Because of the boundary conditions at 0 and $X_1$, $h(0)=0$ and $h(X_1)=1-P_p(c,X_2)>0$. Note that $h(c)$ cannot have either a positive local maximum ($h(c)>0$, $h'(c)=0$, $h''(c)<0$) or a negative local minimum ($h(c)<0$, $h'(c)=0$, $h''(c)>0$) on $[0,X_1]$, as otherwise the equation $h''(c)\frac{\sigma^2}{2}+h'(c)[rc+\mu]-\lambda h(c)=0$ would not hold. Therefore, the function must be always positive and increasing over the relevant interval, and the claim follows. ■.

The result below is a straightforward consequence of Lemma B5 and the fact that, in the absence of trading costs, $\eta=0$.

*Corollary B6.* When trading the firm's stock is costly, the payout probability $P_p$ is larger than in the benchmark case with no trading costs (i.e., $P_p(c,C^*) < P_p(c,C_V)$).

### B.3. Claim (3): Probability of Liquidation

I derive the results regarding the probability of liquidation $P_l(c,X)$, as the probability of external financing is just $P_f(c,X)=1-P_l(c,X)$. Using standard methods (e.g., Dixit and Pindyck (1994)), the dynamics of $P_l(c,X)$ are given by

$$P_l'(c)(rc+\mu)+\frac{\sigma^2}{2}P_l''(c)-\lambda P_l(c)=0 \tag{B-3}$$

subject to $P_l(0)=1$ and $P_l'(X)=0$, where the first boundary condition is given by the definition of $P_l$, whereas the second boundary condition is due to reflection at the payout threshold. I prove that the probability of liquidation is higher when the firm's stocks are illiquid. In the following, I employ the generic function $P_l(c,X) \equiv P_l(c)$

*Lemma B7.* The probability $P_l(c,X)$ is decreasing and convex for any $c \in [0,X]$.

*Proof.* As $P_l'(X)=0$ and $P_l(X) \geq 0$, it must be that $P_l''(X)>0$ for equation (B-3) to hold. Then, there exists a left neighborhood of $X, [X-\varepsilon,X]$ with $\varepsilon>0$, over which $P_l'(c)<0$ and $P_l''(c)>0$. Toward a contradiction, suppose that there exists some $c \in [0,X-\varepsilon]$, where $P_l'(c)>0$. Then, there should be a $\bar{C}$ such that $P_l'(\bar{C})=0$, while

$P_l'(c) < 0$ for $c \in [\bar{C}, X]$. For any $c \in [\bar{C}, X]$, it must be that $P_l''(c) = \frac{2}{\sigma^2}[\lambda P_l(c) - P_l'(c)(rc+\mu)] > \frac{2}{\sigma^2}\lambda P_l(X) > 0$. Then, $P_l''(c) > 0$ for any $c \in [\bar{C}, X]$ means that $P_l'(c)$ is always increasing on $c \in [\bar{C}, X]$, contradicting $P_l'(\bar{C}) = P_l'(X) = 0$. The claim follows. ■.

Now I prove that $P_l(c, C_V) \geq P_l(c, C^*)$.

*Lemma B8.* For any $\eta_1 > \eta_2$, $P_l(c, C_V(\eta_1)) \geq P_l(c, C_V(\eta_2))$.

*Proof.* By Lemma B3, $C_V(\eta_1) < C_V(\eta_2)$. To ease the notation throughout the proof, I define $X_1 \equiv C_V(\eta_1)$ and $X_2 \equiv C_V(\eta_2)$. By Lemma B7, the functions $P_l(c, X_1)$ and $P_l(c, X_2)$ are positive, decreasing, and convex over the interval of definition. I define the auxiliary function $h(c) = P_l(c, X_1) - P_l(c, X_2)$. Note that $h(c)$ cannot have either a positive local maximum ($h(c) > 0$, $h'(c) = 0$, $h''(c) < 0$) or a negative local minimum ($h(c) < 0$, $h'(c) = 0$, $h''(c) > 0$) on $[0, X_1]$, as otherwise the equation $h''(c)\frac{\sigma^2}{2} + h'(c)[rc+\mu] - \lambda h(c) = 0$ would not hold. In addition, $h(0) = 0$, and $h'(X_1) = -P_l'(c, X_2) > 0$ because of the boundary conditions at 0 and at $X_1$. This means that the function is null at the origin, and increasing at $X_1$. Toward a contradiction, assume that $h$ is negative for some $c < X_1$. This would imply the existence of a negative local minimum, given that the function is null at 0 and it is increasing at $X_1$. This cannot be the case as argued above. Therefore, the function must be always positive, and the claim follows. ■.

The result below is a straightforward consequence of Lemma B8.

*Corollary B9.* When the bid-ask spread associated with the firm's stock is positive, the probability of liquidation $P_l$ is larger than in the case in which the bid-ask spread is 0 (i.e., $P_l(c, C^*) < P_l(c, C_V)$).

### *B.4. Claim (4): Zero-NPV Cost*

I exploit the dynamic programming result in Décamps and Villeneuve (2007) and HMM, establishing that the growth option has a non-positive NPV if and only if $V(c) > V_+(c - I)$ for any $c \geq 0$, where I denote by $V_+(c - I)$ the value of the firm after investment. To prove the claim, I rely on the following lemma:

*Lemma B10.* $V(c) \geq V_+(c - I)$ for any $c \geq I$ if and only if $I \geq I_V$, where $I_V$ satisfies the expression (14) reported in Proposition 1.

*Proof.* I define $\bar{c} = \max[C_V, I + C_{V+}]$. The inequality $V(c) \geq V_+(c - I)$ for $c > \bar{c}$ means that $c - C_V + V(C_V) \geq c - C_{V+} - I + V_+(C_{V+})$. Using the definition of $I_V$, the former inequality is equivalent to the inequality $I \geq I_V$, by straightforward calculations.

Next, I prove that $V(c) \geq V_+(c - I_V)$ for any $c \geq I_V$. I exploit the inequalities $C_V < C_{V+} + I_V$ and $\mu_+ - \mu - rI_V > 0$ (these inequalities stem from a slight modification of Lemma C.3 in HMM, so I omit the details). For $c \geq C_V$, the following inequality

$$V_+(c-I_V) \le V_+(C_{V+}) + c - I_V - C_{V+} = c - C_V + V(C_V) = V(c)$$

holds. The first inequality is due to the concavity of $V_+$. The first equality is given by the definition of $I_V$, whereas the second equality is due to the linearity of $V$ above $C_V$. I now need to prove the result for $c \in [I_V, C_V]$. To this end, I define the auxiliary function $u(c) = V(c) - V_+(c-I_V)$. The function $u(c)$ is positive at $C_V$ as argued above, $u'(C_V) < 0$ and $u''(C_V) > 0$. On the interval of interest, it satisfies

$$(\rho+\delta\eta+\lambda)u(c) = (rc+\mu)u'(c) + \frac{\sigma^2}{2}u''(c) + (\mu + rI_V - \mu_+)V'_+(c-I_V)$$
$$+\lambda(V(C_V) - C_V - V_+(C_{V+}) + C_{V+} + I_V),$$

where the last term on the right-hand side is 0 by the definition of $I_V$, whereas the third term is negative. Then, the function cannot have a positive local maximum here, because otherwise $u(c) > 0$, $u''(c) < 0 = u'(c)$, and the ODE above would not hold. Jointly with the fact that $u(C_V)$ is positive, decreasing, and convex means that the function is always decreasing on this interval. Then, $u(c)$ is also always positive, and the claim holds. ■.

### B.5. *Claim (5): Firm Value*

Consider $\eta_1 > \eta_2$ and define $X_1 \equiv C_V(\eta_1)$ and $X_2 \equiv C_V(\eta_2)$. Consider the auxiliary function $h(c) = V(c;X_1,\eta_1) - V(c;X_2,\eta_2)$. Using equation (11), it satisfies the following dynamics:

$$(\text{B-4}) \qquad (\rho+\delta\eta_1)h(c) + \delta(\eta_1-\eta_2)V_2(c;X_1,\eta_1) = (rc+\mu)h'(c) + \frac{\sigma^2}{2}h''(c)$$
$$+\lambda[h(X_2) - h(c)].$$

Because of the boundary condition at 0, $h(0) = 0$ and we have that $h'(X_2) = 0$ and $h'(X_1) = 1 - V'(c,X_2) < 0$ because of the boundary conditions at the thresholds $X_1, X_2$. As the function is non-increasing over $[X_1, X_2]$ and $h(0) = 0$, either the function is negative for any $c \in [0, X_2]$ or it has a positive local maximum over $[0, X_1)$ (at which $h() > 0$, $h'() = 0$, and $h''() < 0$). At such maximum, the left-hand side of equation (B-4) would be positive, whereas the right-hand side would be negative (i.e., if this is a positive maximum, the last term on the right-hand side is negative). Thus, such a maximum cannot exist, meaning that $h$ is decreasing and negative for any $c \in [0, X_2]$. The claim follows. ■.

## Appendix C. Proof of Proposition 2

### C.1. *Claim (1): Target Cash Threshold*

As in Appendix B, I express the function $V(c)$ as a function of $X$, which denotes the threshold satisfying $V'(X,X) - 1 = V''(X,X) = 0$. By a straightforward modification of Lemma B1, it is possible to show that $V(c,X)$ is decreasing in $X$. I exploit the following lemma:

*Lemma C1*. For a given $X$ and two given $v_1 > v_2$, $V(c,X,v_2) > V(c,X,v_1)$ holds for any $c \in [0,X]$.

*Proof.* I define the auxiliary function $h(c) = V(c,X,v_2) - V(c,X,v_1)$. At $c = X$, the function is positive as $h(X) = (rX + \mu - \delta\omega(1-\theta))\left(\frac{1}{\rho+\Psi(v_2)} - \frac{1}{\rho+\Psi(v_1)}\right) > 0$ given the assumption that $v_1 > v_2$. Moreover, $h'(X) = h''(X) = 0$. The function $h(c)$ satisfies $(rc+\mu)h'(c) + \frac{\sigma^2}{2}h''(c) - (\rho+\lambda+\Psi(v_2))h(c) + \lambda h(X) = (1-\theta)(\delta v_2 - \delta v_1)V(c,X;v_1)$, where the right-hand side is negative. Differentiating gives

$$(rc+\mu)h''(c) + \frac{\sigma^2}{2}h'''(c) - (\rho+\lambda+\Psi(v_2) - r)h'(c) = (1-\theta)(\delta v_2 - \delta v_1)V'(c,X;v_1). \tag{C-1}$$

At $X$, using the boundary conditions, I get $\frac{\sigma^2}{2}h'''(X) = (1-\theta)(\delta v_2 - \delta v_1) < 0$. This means that $h''$ is decreasing in a left neighborhood of $X$ which, together with $h''(X) = 0$, implies that $h'' > 0$ in such a neighborhood. In turn, this means that $h'(c)$ is increasing in such a neighborhood which, together with the boundary at $X$, implies that $h'(c) < 0$ in such a neighborhood. By equation (C-1), $h'(c)$ cannot have a negative local minimum (where $h' < 0$, $h'' = 0$, and $h''' > 0$). As $h'(X) = 0$ and $h'$ is negative and increasing in a left neighborhood of $X$, it should be always negative for any $c < X$, so $h$ is always decreasing. As it is positive at $X$, it is then positive for any $c < X$. ■.

Exploiting the result above, I can prove the following:

*Lemma C2.* For any $v_1 > v_2$, then $C_V(v_1) < C_V(v_2)$.

*Proof.* The target cash thresholds are the unique solution to the boundary conditions $V(0,C_V(v_i);v_i) - \ell = 0$. Exploiting Lemma C1, I have $V(0,C_V(v_1);v_2) - \ell > 0 = V(0,C_V(v_1);v_1) - \ell$. As $V$ is decreasing in the payout threshold, this means that $C_V(v_1) < C_V(v_2)$ to get the equality $V(0,C_V(v_2);v_2) - \ell = 0$. The claim follows. ■.

I obtain a similar result when turning to the parameter $\chi$ (which enters the ODE in a way similar to $v$).

*Lemma C3.* For any $\chi_1 > \chi_2$, then $C_V(\chi_1) < C_V(\chi_2)$.

*Proof.* Using the exact same arguments in Lemmas C1 and C2, the result follows. ■.

Next, we turn to $\omega$.

*Lemma C4.* For any $\omega_1 > \omega_2$, $C_V(\omega_1) < C_V(\omega_2)$.

*Proof.* I start by proving that, for a given threshold $X$ so that $V'(X) - 1 = V''(X) = 0$, the inequality $V(c,X,\omega_2) > V(c,X,\omega_1)$ holds for any $c \in [0,X]$. To this end, I follow arguments as in Lemma C1 and define the auxiliary function $h(c) = V(c,X;\omega_2) - V(c,X;\omega_1)$. By calculations, it is possible to show that $h(X) = \delta(1-\theta)(\omega_1 - \omega_2)/(\rho+\Psi) > 0$ as $\omega_1 > \omega_2$ by assumption. Moreover, $h'(X) = h''(X) = 0$ by the boundary conditions and, using standard arguments, $h$ satisfies $(\rho+\lambda+\Psi)h(c) = (rc+\mu)h'(c) + \frac{\sigma^2}{2}h''(c) + \lambda h(X) - \delta(1-\theta)(\omega_2 - \omega_1)$, where the sum of the last two terms is positive; thus, $h(c)$ cannot have a negative local minimum over the interval $[0,X]$ (where $h<0$, $h'=0$, and $h''>0$) as the equation would not hold. Differentiating the above equation, I obtain $(\rho+\lambda+\Psi-r)h'(c)=(rc+\mu)h''(c)+\frac{\sigma^2}{2}h'''(c)$ so that $h'''(X)=0$ (and, continuing differentiating this equation, I can show that all the subsequent derivatives are 0). Thus, $h$ is positive for any $c\in[0,X]$. Using arguments similar to Lemma C2, the claim follows. ■.

We next turn to $\theta$.

*Lemma C5.* For any $\theta_1>\theta_2, C_V(\theta_1)<C_V(\theta_2)$.

*Proof.* As for the previous cases, I start by proving that, for a given threshold $X$ so that $V'(X)-1=V''(X)=0$, the inequality $V(c,X,\theta_2)>V(c,X,\theta_1)$ holds for any $c\in[0,X]$. To this end, I define the auxiliary function $h(c)=V(c,X;\theta_2)-V(c,X;\theta_1)$. By calculations, it is possible to show that

$$\begin{aligned}
h(X)&=\frac{rX+\mu-\delta\omega(1-\theta_2)}{r+\delta[v+\theta_2(\chi-v)]}-\frac{rX+\mu-\delta\omega(1-\theta_1)}{r+\delta[v+\theta_1(\chi-v)]}\\
&=\frac{rX+\mu-\delta\omega(1-\theta_2)}{r+\delta[v+\theta_2(\chi-v)]}\left[\frac{(\theta_1-\theta_2)\delta(\chi-v)}{r+\delta[v+\theta_1(\chi-v)]}\right]-\frac{\delta\omega(\theta_1-\theta_2)}{r+\delta[v+\theta_1(\chi-v)]}\\
&>\frac{rX+\mu-\delta\omega(1-\theta_2)}{r+\delta[v+\theta_2(\chi-v)]}\left[\frac{(\theta_1-\theta_2)\delta(\chi-v)}{r+\delta[v+\theta_1(\chi-v)]}\right]-\frac{\delta\ell(\chi-v)(\theta_1-\theta_2)}{r+\delta[v+\theta_1(\chi-v)]}\\
&=\frac{\delta(\chi-v)(\theta_1-\theta_2)}{r+\delta[v+\theta_1(\chi-v)]}(V(X,X;\theta_2)-\ell)>0,
\end{aligned}$$

where the first inequality is motivated by $\ell(\chi-v)>\omega$ (see Appendix A), whereas the last inequality is motivated by the fact that $V$ is increasing in $c$ (see Lemma A1). Moreover, $h'(X)=h''(X)=0$ by the boundary conditions at $X$. Using standard arguments, $h$ satisfies $(\theta_2-\theta_1)\delta(\chi-v)V(c,X;\theta_1)=(rc+\mu)h'(c)+\frac{\sigma^2}{2}h''(c)-(\rho+\lambda+\delta v+\theta_2\delta(\chi-v))h(c)+\lambda h(X)-\delta\omega(\theta_1-\theta_2)$, where note that the left-hand side is negative. Differentiating the above equation, I obtain

$$\text{(C-2)}\qquad (\theta_2-\theta_1)\delta(\chi-v)V'(c,X;\theta_1)=\begin{aligned}[t]&(rc+\mu)h''(c)+\frac{\sigma^2}{2}h'''(c)\\&-(\rho+\lambda+\delta v+\theta_2\delta(\chi-v)-r)h'(c),\end{aligned}$$

so that $\frac{2}{\sigma^2}(\theta_2-\theta_1)\delta(\chi-v)=h'''(X)<0$ as $(\chi-v)\ell>\omega$ (as per Appendix A). This means that $h''$ is decreasing in a left neighborhood of $X$ which, together with $h''(X)=0$, implies that $h''>0$ in such a neighborhood. In turn, this means that $h'$ is increasing in such a neighborhood which, together with the boundary at $X$, implies that $h'(c)<0$ in such a neighborhood. By equation (C-2), $h'$ cannot have a negative local minimum (where $h'<0$, $h''=0$, and $h'''>0$). As $h'(X)=0$ and $h'$ is negative and increasing in a left neighborhood of $X$, it then should be always negative for any $c<X$, so $h$ is always decreasing. As $h$ is positive at $X$, it is then positive for any $X$. ■.

### *C.2. Claims (2) and (3): Probability of Liquidation and of Payout*

Exploiting that $C_V$ is monotonic in $v$, $\chi$, $\omega$, and $\theta$ as just shown, claims (2) and (3) about the impact of these parameters on the probability of liquidation and of payout follow by using steps similar to those used in Appendices B.3 and B.2.

### C.3. Claim (4): Zero-NPV Cost

To derive the expression of the zero-NPV cost when the bid-ask spread is endogenous, I follow the same steps as in Appendix B.4 and exploit the results in Décamps and Villeneuve (2007) and HMM, namely, the growth option has a non-positive NPV if and only if $V(c) > V_+(c-I)$ for any $c \geq 0$. A straightforward modification of Lemma B10 confirms the claim. Notably, the zero-NPV cost satisfies $I_V \equiv V_+(C_{V+}) - C_{V+} - (V(C_V) - C_V)$, which gives equation (15) by calculations. ■.

### C.4. Claim (5): Firm Value

Consider first the impact of $v$ on firm value. Define $v_1 > v_2$. I define $X_1 \equiv C_V(v_1)$ and $X_2 \equiv C_V(v_2)$ and $X_1 < X_2$ as proved in claim (1). I define the auxiliary function: $h(c) = V(c, X_1, v_1) - V(c, X_2, v_2)$. Using equation (A-1), it satisfies the following dynamics:

$$
\begin{aligned}
&(\rho + \Psi(v_1))h(c) + \delta(1-\theta)(v_1 - v_2)V(c, X_2, v_2) \\
&\quad = (rc + \mu)h'(c) + \frac{\sigma^2}{2}h''(c) + \lambda[h(X_2) - h(c)].
\end{aligned}
$$

Because of the boundary condition at 0, $h(0) = 0$ and we have that $h'(X_2) = 0$ and $h'(X_1) = 1 - V'(c, X_2) < 0$ because of Lemma A1. Thus, either the function is negative for any $c \in [0, X_2]$ or it has at least a positive maximum (at which $h > 0$, $h' = 0$, and $h'' < 0$). At such maximum, the left-hand side of the above equation would be positive, whereas the right-hand side would be negative (i.e., if this is a positive maximum, the last term on the right-hand side is negative). Thus, such a maximum cannot exist, meaning that $h$ is negative for any $c \in [0, X_2]$. Using the same arguments, we can show the claim for $\chi$: That is, firm value decreases in this parameter too.

Consider now $\omega_1 > \omega_2$. As above, define $X_1 \equiv C_V(\omega_1)$ and $X_2 \equiv C_V(\omega_2)$ and $X_1 < X_2$ by claim (1). I define the auxiliary function: $h(c) = V(c, \omega_1) - V(c, \omega_2)$, which satisfies

$$
(\rho + \Psi)h(c) = (rc + \mu)h'(c) + \frac{\sigma^2}{2}h''(c) + \lambda[h(X_2) - h(c)] - \delta(1-\theta)(\omega_1 - \omega_2). \tag{C-3}
$$

As above, because of the boundary condition at 0, $h(0) = 0$ and we have that $h'(X_2) = 0$ and $h'(X_1) = 1 - V'(c, X_2) < 0$. In this case too, a positive maximum cannot exist: The left-hand side equation (C-3) would be positive, whereas the right-hand side would be negative (i.e., if this is a positive maximum, the third term on the right-hand side is negative, and the last term too given the assumption that $\omega_1 > \omega_2$). Thus, such a maximum cannot exist, and the claim follows.

Consider now $\theta_1 > \theta_2$. Define $X_1 \equiv C_V(\theta_1)$ and $X_2 \equiv C_V(\theta_2)$ and $X_1 < X_2$ by claim (1). I define the auxiliary function: $h(c) = V(c, \theta_1) - V(c, \theta_2)$ which, using standard arguments, satisfies

$$
\begin{aligned}
[\rho + \delta v + \delta\theta_1(\chi - v)]h(c) &= (rc + \mu)h'(c) + \frac{\sigma^2}{2}h''(c) + \lambda[h(X_2) - h(c)] \\
&\quad - \delta\omega(\theta_2 - \theta_1) - \delta(\theta_1 - \theta_2)(\chi - v)V(c, \theta_2).
\end{aligned} \tag{C-4}
$$

As above, because of the boundary condition at 0, $h(0) = 0$ and we have that $h'(X_2) = 0$ and $h'(X_1) = 1 - V'(c, X_2) < 0$. Rewriting the last two terms of the above equation gives

$$-\delta\omega(\theta_2-\theta_1)-\delta(\theta_1-\theta_2)(\chi-v)V(c,\theta_2)=\delta(\theta_1-\theta_2)[\omega-(\chi-v)V(c,\theta_2)] < \delta(\theta_1-\theta_2)(\chi-v)[\ell-V(c,\theta_2)]<0.$$

Thus, a positive maximum for $h$ cannot exist: The left-hand side of equation (C-4) would be positive, whereas the right-hand side would be negative (i.e., if this is a positive maximum, the third term on the right-hand side is negative, and the last term too as just shown). Thus, such a maximum cannot exist. As a result, the function $h$ is non-positive for any $c<X_2$ (given its properties at $c=0$ and $c=X_1$), and the claim follows.

## Appendix D. Proof of Proposition 3

I again express $V(c)$ as a function of a given $X$, which denotes the threshold satisfying $V'(X,X)-1=V''(X,X)=0$. Again, $V(c,X)$ is decreasing in $X$ by a straightforward modification of Lemma B1. The following lemma shows that a greater $\lambda$ always leads to a decrease in $C_V$, meaning that the indirect effect of $\lambda$ on $C_V$ via stock illiquidity weakens but does not offset the direct effect of $\lambda$ on $C_V$ via the demand of cash.

*Lemma D1.* A greater $\lambda$ leads to a decrease in $C_V$.

*Proof.* I start by showing that, for a given $X$ satisfying $V'(X,X)-1=V''(X,X)=0$ and two given $\lambda_1>\lambda_2$, then $V(c,X,\lambda_2)>V(c,X,\lambda_1)$ holds for any $c\in[0,X]$. To prove this, I define the auxiliary function $h(c)=V(c,X,\lambda_2)-V(c,X,\lambda_1)$. At $c=X$, the function satisfies $h(X)=0$. Also, by the definition of $X$, $h'(X)=h''(X)=0$. The function $h(c)$ satisfies $(rc+\mu)h'(c)+\frac{\sigma^2}{2}h''(c)-(\rho+\lambda_2+\Psi)h(c)=(\lambda_1-\lambda_2)[V(X,\lambda_1)-V(c,\lambda_1)-X+c]$, where the right-hand side is positive, as $V$ is increasing in $c$ by Lemma A1 and $\lambda_1>\lambda_2$ by assumption. This means that $h(c)$ does not admit a positive local maximum; otherwise, the equation would not hold. Differentiating the above equation gives

(D-1) $$(rc+\mu)h''(c)+\frac{\sigma^2}{2}h'''(c)-(\rho+\lambda_2+\Psi-r)h'(c)=(\lambda_1-\lambda_2)[-V'(c,\lambda_1)+1].$$

At $X$, the boundary conditions imply that $\frac{\sigma^2}{2}h'''(X)=0$. Differentiating equation (D-1) gives

(D-2) $$(rc+\mu)h'''(c)+\frac{\sigma^2}{2}h''''(c)-(\rho+\lambda_2+\Psi-2r)h''(c)=-(\lambda_1-\lambda_2)V''(c,\lambda_1).$$

At $X$, the boundary conditions imply that $\frac{\sigma^2}{2}h''''(c)=-(\lambda_1-\lambda_2)V''(X,\lambda_1)=0$. Differentiating equation (D-2) gives

(D-3) $$(rc+\mu)h''''(c)+\frac{\sigma^2}{2}h'''''(c)-(\rho+\lambda_2+\Psi-3r)h'''(c)=-(\lambda_1-\lambda_2)V'''(c,\lambda_1).$$

At $X$, the boundary conditions imply $\frac{\sigma^2}{2}h'''''(c)=-(\lambda_1-\lambda_2)V'''(c,\lambda_1)$. Differentiating the dynamics of $V(X,\lambda_1)$ (by equation (A-1)) gives $(\rho+\lambda_1+\Psi-r)V'=(rc+\mu)V''+\frac{\sigma^2}{2}V'''+\lambda_1$, so, at $X$, $(\rho+\Psi-r)=\frac{\sigma^2}{2}V'''>0$. Thus,

$h'''''(c)<0$. In turn, given that $h''''(X)=0$, it means that $h''''$ is decreasing and, thus, positive in a left neighborhood of $X$. By applying a similar logic on the third, second, and first derivatives, we get that $h'$ is increasing and, thus, negative in a neighborhood of $X$. Thus, $h$ is decreasing in that neighborhood. Because $h(X)=0$, it means that the function is positive in such a neighborhood. As $h$ cannot have a positive local maximum (as claimed above), then $h$ is positive for any $c<X$.

Next, I prove that for any $\lambda_1>\lambda_2$, then $C_V(\lambda_1)<C_V(\lambda_2)$. $C_V(\lambda_1)$ and $C_V(\lambda_2)$, as opposed to a generic $X$, are the unique solution to the boundary conditions $V(0,C_V(\lambda_i);\lambda_i)-\ell=0$. Above, we just proved that $V(c,X,\lambda_2)>V(c,X,\lambda_1)$. Thus, $V(0,C_V(\lambda_1);\lambda_2)-\ell>0=V(0,C_V(\lambda_1);\lambda_1)-\ell$. As $V$ is decreasing in the payout threshold, then $C_V(\lambda_1)<C_V(\lambda_2)$ to get the equality $V(0,C_V(\lambda_2);\lambda_2)-\ell=0$. The claim follows. ■.

Next, I show that a greater $\sigma$ always leads to an increase in $C_V$, meaning that the indirect effect of $\sigma$ on $C_V$ via stock illiquidity weakens but does not offset the direct effect of $\sigma$ on $C_V$ via the demand of cash.

*Lemma D2.* A greater $\sigma$ leads to an increase in $C_V$.

*Proof.* I start by showing that, for a given $X$ satisfying $V'(X,X)-1=V''(X,X)=0$ and two given $\sigma_1>\sigma_2$, $V(c,X,\sigma_2)<V(c,X,\sigma_1)$ holds for any $c\in[0,X]$. To prove this, I define the auxiliary function $h(c)=V(c,X,\sigma_2)-V(c,X,\sigma_1)$. At $c=X$, the function satisfies $h(X)=0$. Also, by the definition of $X$, $h'(X)=h''(X)=0$. The function $h(c)$ satisfies $(rc+\mu)h'(c)+\frac{\sigma_2^2}{2}h''(c)-(\rho+\lambda+\Psi)h(c)+\lambda h(X)=\frac{\sigma_1^2-\sigma_2^2}{2}V''(c,\sigma_1)$, where the right-hand side is negative, as $V''(c,\sigma_1)<0$ by Lemma A1 and $\sigma_1>\sigma_2$ by assumption. This means that $h(c)$ does not admit a negative local minimum. Differentiating the above equation gives $(rc+\mu)h''(c)+\frac{\sigma_2^2}{2}h'''(c)-(\rho+\lambda+\Psi-r)h'(c)=\frac{\sigma_1^2-\sigma_2^2}{2}V'''(c,\sigma_1)$. At $X$, using the boundary conditions, I get $\frac{\sigma_2^2}{2}h'''(X)=\frac{\sigma_1^2-\sigma_2^2}{2}V'''(X,\sigma_1)$. Using arguments similar to those in Lemma D1, it is possible to show that $V'''>0$, so $h'''(X)>0$ too. Together with $h''(X)=0$, it implies that $h''<0$ in such a neighborhood. In turn, this means that $h'(c)$ is decreasing in such a neighborhood which, together with the boundary at $X$, implies that $h'(c)>0$ in such a neighborhood. Thus, $h$ increases in a neighborhood of $X$. Because $h(X)=0$, then $h$ is negative in such a neighborhood. Because it cannot have a negative local minimum, it means that it is negative for any $c<X$.

Next, I prove that for any $\sigma_1>\sigma_2$, then $C_V(\sigma_1)>C_V(\sigma_2)$. $C_V(\sigma_1)$ and $C_V(\sigma_2)$, as opposed to a generic $X$, are the unique solution to the boundary conditions $V(0,C_V(\sigma_i);\sigma_i)-\ell=0$. Above, we just proved that $V(c,X,\sigma_2)<V(c,X,\sigma_1)$. Thus, $V(0,C_V(\sigma_1);\sigma_2)-\ell<0=V(0,C_V(\sigma_1);\sigma_1)-\ell$. As $V$ is decreasing in the payout threshold, then $C_V(\sigma_1)>C_V(\sigma_2)$ to get the equality $V(0,C_V(\sigma_2);\sigma_2)-\ell=0$. The claim follows. ■.

## Appendix E. Proof of Proposition 4

The proposition exploits the expression for the equilibrium bid price and the bid-ask spread (see equations (6) and (7)) and the concavity of firm value proved in Appendix A. The claim follows. ■.

## Supplementary Material

To view supplementary material for this article, please visit http://doi.org/10.1017/S0022109023001217.

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