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NEW CRITERIA FOR MEROMORPHIC STARLIKE UNIVALENT FUNCTIONS

B.A. URALEGADDI AND C. SOMANATHA

This paper establishes new criteria for meromorphic starlike univalent functions of the form

$$f(z) = \frac{a_{-1}}{z} + \Sigma_{k=0}^{\infty} a_k z^k, \ (a_{-1} \neq 0).$$

Further property preserving integrals are considered.

1. INTRODUCTION

Let Σ denote the class of functions of the form $f(z) = (a_{-1}/z) + \Sigma_{k=0}^{\infty} a_k z^k$, $(a_{-1} \neq 0)$, regular in the punctured disk $E = \{z : 0 < |z| < 1\}$.

Define

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= \frac{a_{-1}}{z} + 2a_0 + 3a_1 z + 4a_2 z^2 + \ldots, \\ D^2 f(z) &= D\big(D^1 f(z)\big), \end{aligned}$$

and for $n = 1, 2, 3, \ldots$

$$\begin{aligned} D^n f(z) &= D\big(D^{n-1} f(z)\big) \\ &= \frac{a_{-1}}{z} + \Sigma_{m=2}^{\infty} m^n a_{m-2} z^{m-2}. \end{aligned} \tag{1.1}$$

In this paper we shall show that a function $f(z)$ in Σ, which satisfies one of the conditions

$$\operatorname{Re}\{D^{n+1} f(z)/D^n f(z) - 2\} < -\alpha, \ |z| < 1, \ 0 \leqslant \alpha < 1 \text{ and } n \in N_0 = \{0, 1, 2, \ldots\} \text{ is univalent in } 0 < |z| < 1. \tag{1.2}$$

More precisely it is proved that for the classes $B_n(\alpha)$ of functions in Σ satisfying (1.2),

$$B_{n+1}(\alpha) \subset B_n(\alpha) \text{ holds.} \tag{1.3}$$

Since $B_0(\alpha)$ equals $\Sigma^*(\alpha)$ (the class of meromorphic starlike functions of order α) the univalence of members in $B_n(\alpha)$ is a consequence of (1.3). Further property preserving integrals are considered, a known result of Goel and Sohi [**2**, Corollary 1] is obtained as a particular case and a result of Bajpai [**1**, Theorem 1] is extended.

In [4] Ruscheweyh obtained the new criteria for univalent functions.

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2. The class $B_n(\alpha)$

Theorem 2.1. *$B_{n+1}(\alpha) \subset B_n(\alpha)$ for each $n \in N_0$.*

Proof: Let $f(z) \in B_{n+1}(\alpha)$. Then

$$\operatorname{Re}\{D^{n+2}f(z)/D^{n+1}f(z) - 2\} < -\alpha, \quad |z| < 1. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re}\{D^{n+1}f(z)/D^n f(z) - 2\} < -\alpha.$$

Define a regular function $w(z)$ in the unit disk $\Delta = \{z\colon |z| < 1\}$ by

$$D^{n+1}f(z)/D^n f(z) - 2 = -\frac{1 + (2\alpha - 1)w(z)}{1 + w(z)}. \tag{2.2}$$

Clearly $w(0) = 0$.

The equation (2.2) may be written as

$$D^{n+1}f(z)/D^n f(z) = \frac{1 + (3 - 2\alpha)\,w(z)}{1 + w(z)}. \tag{2.3}$$

Differentiating (2.3) logarithmically and using the identity (easy to verify)

$$z(D^n f(z))' = D^{n+1}f(z) - 2D^n f(z) \tag{2.4}$$

we obtain

$$\frac{\left(D^{n+2}f(z)/D^{n+1}f(z)\right) - 2 + \alpha}{1 - \alpha} = \frac{2zw'(z)}{(1 + w(z))(1 + (3 - 2\alpha)w(z))} - \frac{1 - w(z)}{1 + w(z)}. \tag{2.5}$$

We claim that $|w(z)| < 1$ for $z \in \Delta$. Otherwise there exists a point z_0 in $|z| < 1$ such that $\max\limits_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$. From a well-known result due to Jack [3], there is then a real number $k \geqslant 1$ such that

$$z_0 w'(z_0) = kw(z_0). \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$\frac{\left(D^{n+2}f(z_0)/D^{n+1}f(z_0)\right) - 2 + \alpha}{1 - \alpha} = \frac{2kw(z_0)}{(1 + w(z_0))(1 + (3 - 2\alpha)w(z_0))} - \frac{1 - w(z_0)}{1 + w(z_0)}.$$

Thus

$$\operatorname{Re}\frac{\left(D^{n+2}f(z_0)/D^{n+1}f(z_0)\right) - 2 + \alpha}{1 - \alpha} \geqslant \frac{1}{2(2 - \alpha)} > 0$$

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which contradicts (2.1). Hence $|w(z)| < 1$ for $z \in \Delta$ and from (2.2) it follows that $f \in B_n(\alpha)$. ∎

THEOREM 2.2. *Let $f \in \Sigma$ and for a given $n \in N_0$, $c > 0$, let f satisfy the condition*

$$\operatorname{Re}\{D^{n+1}f(z)/D^n f(z) - 2\} < -\alpha + \frac{1-\alpha}{2(1-\alpha+c)} \text{ for } z \in \Delta; \tag{2.7}$$

then $F(z) = \left(c/z^{c+1}\right)\int_0^z t^c f(t)dt \in B_n(\alpha)$.

PROOF: From the definition of F we have

$$z(D^n F(z))' = cD^n f(z) - (c+1)D^n F(z) \tag{2.8}$$

and also

$$z(D^n F(z))' = D^{n+1}F(z) - 2D^n F(z). \tag{2.9}$$

Using (2.8) and (2.9) the condition (2.7) may be written as

$$\operatorname{Re}\left(\frac{D^{n+2}F(z)/D^{n+1}F(z) + (c-1)}{1+(c-1)D^n F(z)/D^{n+1}F(z)} - 2\right) < -\alpha + \frac{1-\alpha}{2(1-\alpha+c)}. \tag{2.10}$$

We have to prove that (2.10) implies the inequality

$$\operatorname{Re}\{D^{n+1}F(z)/D^n F(z) - 2\} < -\alpha.$$

Define a regular function $w(z)$ in the unit disk $\Delta = \{z\colon |z| < 1\}$ by

$$D^{n+1}F(z)/D^n F(z) - 2 = -\frac{1+(2\alpha-1)w(z)}{1+w(z)}; \tag{2.11}$$

clearly $w(0) = 0$.

The equation (2.11) may be written as

$$D^{n+1}F(z)/D^n F(z) = \frac{1+(3-2\alpha)w(z)}{1+w(z)}. \tag{2.12}$$

Differentiating (2.12) logarithmically and simplifying we obtain

$$\begin{aligned}&\frac{D^{n+2}F(z)/D^{n+1}F(z) + (c-1)}{1+(c-1)D^n F(z)/D^{n+1}F(z)} - 2\\ &= -\left[\alpha + (1-\alpha)\frac{1-w(z)}{1+w(z)}\right] + \frac{2(1-\alpha)zw'(z)}{(1+w(z))(c+(2-2\alpha+c)w(z))}.\end{aligned} \tag{2.13}$$

The remaining part of the proof is similar to that of Theorem 2.1. ∎

REMARKS. (i) A result of Goel and Sohi [**2**, Corollary 1] turns out to be a particular case of the above theorem when $a_{-1} = 1$, $n = 0$ and $\alpha = 0$.

(ii) For $a_{-1} = 1$, $n = 0$, $\alpha = 0$ and $c = 1$ the above theorem extends a result of Bajpai [**1**, Theorem 1].

THEOREM 2.3. *$f \in B_n(\alpha)$ if and only if $F(z) = 1/z^2 \int_0^z tf(t)dt \in B_{n+1}(\alpha)$.*

PROOF: From the definition of F we have

$$D^n(zF'(z)) + 2D^nF(z) = D^nf(z).$$

That is,

$$z(D^nF(z))' + 2D^nF(z) = D^nf(z). \tag{2.14}$$

By using the identity (2.4), (2.14) reduces to $D^nf(z) = D^{n+1}F(z)$. Hence $D^{n+1}f(z) = D^{n+2}F(z)$.

Therefore

$$D^{n+1}f(z)/D^nf(z) = D^{n+2}F(z)/D^{n+1}F(z)$$

and the result follows. □

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Department of Mathematics
Karnatak University
Dharwad - 580 003
India