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ON THE MAPPING CLASS GROUP OF A HEEGAARD SPLITTING

CHARALAMPOS CHARITOS and IOANNIS PAPADOPERAKIS
Agricultural University of Athens, Iera Odos 75, Athens 118 55, Greece
e-mails: bakis@aua.gr, papadoperakis@aua.gr

GEORGIOS TSAPOGAS
University of the Aegean, Samos 832 00, Greece
e-mail: get@aegean.gr

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Abstract. For the mapping class group of 3-manifold with respect to a Heegaard splitting, a simplicial complex is constructed such that its group of automorphisms is identified with the mapping class group.

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1. Introduction. For a closed 3-manifold M with a fixed Heegaard splitting of genus g, notation $M^3 = H_g \cup_{\Sigma_g} H'_g$ with $\Sigma_g = \partial H_g = \partial H'_g$, consider the group of homeomorphisms of M which preserve the Heegaard splitting. By regarding, as usual, two such homeomorphisms as equivalent if there is an isotopy from one to the other via isotopies that preserve H_g (and thus, H'_g), we obtain a group which is naturally called the mapping class group of the Heegaard splitting of M^3, notation $\mathcal{MCG}\left(M^3, H_g\right)$.

In 1933, Goeritz [**5**] showed that the mapping class group $\mathcal{MCG}\left(\mathbb{S}^3, H_2\right)$ of the standard genus 2 Heegaard splitting of the 3-sphere is finitely generated. Scharlemann in [**12**] gave a modern proof of Georitz's result, and Akbas in [**1**] refined his argument to obtain a finite presentation of the mapping class group $\mathcal{MCG}\left(\mathbb{S}^3, H_2\right)$. Also, Cho in [**3**] recovered Akbas's result using a subcomplex of the disk complex of the handlebody of the splitting.

For genus $g \geq 3$ the question of finite generation of the mapping class group $\mathcal{MCG}\left(M^3, H_g\right)$ is open even in the case $M = \mathbb{S}^3$ (Scharlemann found serious gaps in the proofs of the above statement presented several years ago).

In this work we define a simplicial complex analogous to the curve complex for surfaces and show that the group of automorphisms of this complex is isomorphic to the mapping class group $\mathcal{MCG}\left(M^3, H_g\right)$, provided that $g \geq 3$. The construction of this complex builds on earlier work on the complex of incompressible surfaces for handlebodies defined in [**2**]. For the case $g = 2$, we provide simple examples of automorphisms which are not geometric.

2. Definitions and statements of results. For a compact surface S, the complex of curves $\mathcal{C}(S)$, introduced by Harvey in [**6**], has vertices of isotopy classes of essential, non-boundary-parallel simple closed curves in S. A collection of vertices spans a simplex exactly when any two of them may be represented by disjoint curves, or equivalently when there is a collection of representatives for all of them, any two of which are disjoint. Analogously, for a 3-manifold M, the disk complex $\mathcal{D}(M)$ is defined by using the proper isotopy classes of compressing disks for M as vertices. It was introduced in [**11**], where it was used in the study of mapping class groups of 3-manifolds. In [**10**], it was shown to be a quasi-convex subset of $\mathcal{C}(\partial M)$.

By H_g we denote a 3-dimensional handlebody of genus $g \geq 2$. Recall that a compact connected surface $S \subset H_g$ with boundary is properly embedded if $S \cap \partial H_g = \partial S$ and S is transversed to ∂H_g. A *compressing disk* for S is an embedded disk D such that $\partial D \subset S$ and ∂D is essential in S. A properly embedded surface $S \subset H_g$ is *incompressible* if there are no compressing disks for S. Also recall that a map $F : S \times [0, 1] \to H_g$ is a proper isotopy if for all $t \in [0, 1]$, $F\,|_{S\times\{t\}}$ is a proper embedding. In this case we will say that $F(S \times \{0\})$ and $F(S \times \{1\})$ are properly isotopic in H_g, and we will use the symbol $\simeq$ to indicate isotopy in all cases (curves, surfaces etc) and the symbol $[S]$ to denote the isotopy class of S. We recall the following definition from [**2**].

Definition. Let $\mathcal{I}(H_g)$ be a simplicial complex whose vertices are the proper isotopy classes of compressing disks for ∂H_g and properly embedded boundary-parallel incompressible annuli and pairs of pants in H_g. For a vertex $[S]$, which is not a class of compressing disks, it is also required that S is isotopic to a surface $\overline{S}$ embedded in ∂H_g via an isotopy

$$F : S \times [0, 1] \to H_g$$

with $F(S \times \{0\}) = S$, $F(S \times \{1\}) = \overline{S}$ and F being proper when restricted to $[0, 1)$. A collection of vertices spans a simplex in $\mathcal{I}(H_g)$ when any two of them may be represented by disjoint surfaces in H_g.

Observe that there do exist properly embedded pairs of pants that are not isotopic to a surface entirely contained in ∂H_g. We may regard $\mathcal{D}(H_g)$ as a subcomplex of $\mathcal{I}(H_g)$ or, by taking boundaries of the representative disks, $\mathcal{C}(\partial H_g)$. Also note that the vertices of $\mathcal{I}(H_g)$ represented by annuli exactly correspond to the vertices of $\mathcal{C}(\partial H_g)$ represented by curves that are essential in ∂H_g but are not meridian boundaries. We define the complex of annuli $\mathcal{A}(H_g)$ to be the subcomplex of $\mathcal{I}(H_g)$ spanned by these vertices. Together, the vertices of $\mathcal{D}(H_g) \cup \mathcal{A}(H_g)$ span a copy of $\mathcal{C}(\partial H_g)$ in $\mathcal{I}(H_g)$, and we regard $\mathcal{C}(\partial H_g)$ as a subcomplex of $\mathcal{I}(H_g)$. We will denote by $\mathcal{D}$ (resp. $\mathcal{A}$) the vertex set of $\mathcal{D}(H_g)$ (resp. $\mathcal{A}(H_g)$). A vertex in $\mathcal{D}$ (resp. $\mathcal{A}$) will be called a *meridian* (resp. *annular*) *vertex*. The vertex set of $\mathcal{I}(H_g) \setminus (\mathcal{D}(H_g) \cup \mathcal{A}(H_g))$ will be denoted by $\mathcal{P}$ and a vertex in $\mathcal{P}$ will be called a *pants vertex*. Observe that a vertex v in either $\mathcal{D}$ or $\mathcal{A}$ determines a unique, up to isotopy, simple closed curve in ∂H_g, which will be called the boundary curve of v, denoted by ∂v. Similarly, a vertex in $\mathcal{P}$ determines uniquely, up to isotopy, a pair or a triple of mutually disjoint simple closed curves in ∂H_g.

Remark 1. The complex $\mathcal{I}(H_g)$ can be thought of in the following way: Take the curve complex $\mathcal{C}(\partial H_g)$ and add a vertex for every pair (α_1, α_2) or triple $(\alpha_1, \alpha_2, \alpha_3)$ of non-meridian simple closed curves which bound a pair of pants in ∂H_g. Then add an edge from the new vertex to the vertices α_i as well as to any other vertex in $\mathcal{C}(\partial H_g)$ disjoint from α_i's. In particular, the new vertices are connected to (some) meridian vertices. By construction, such a complex cannot be isomorphic to any kind of subdivision of $\mathcal{C}(\partial H_g)$. For example, subdivisions do not alter dimension, whereas

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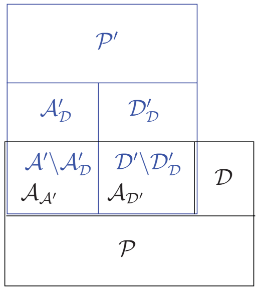

Figure 1. (Colour online) The vertex sets in $\mathcal{I}(M, H_g)$.

$\mathcal{I}\left(H_g\right)$ is not homogeneous with respect to dimension (see properties preceding Lemma 5).

In an identical way the complex $\mathcal{I}\left(H'_g\right)$ is defined and we use the notation $\mathcal{P}'$ (resp. $\mathcal{A}', \mathcal{D}'$) for the vertex set of $\mathcal{I}\left(H'_g\right) \setminus \left(\mathcal{D}\left(H'_g\right) \cup \mathcal{A}\left(H'_g\right)\right)$ (resp. $\mathcal{A}\left(H'_g\right), \mathcal{D}\left(H'_g\right)$).

Observe that an essential simple closed curve in $\Sigma_g = \partial H_g = \partial H'_g$ determines a unique vertex in $\mathcal{I}\left(H_g\right)$ (annular or meridian) and a unique vertex in $\mathcal{I}\left(H'_g\right)$ (possibly of different type). We will also use the following notation:

$$\mathcal{D}'_{\mathcal{D}} := \left\{ v \in \mathcal{D}' |\ \text{the boundary curve of } v \text{ is a meridian in } H_g \right\},$$

$$\mathcal{A}'_{\mathcal{D}} := \left\{ v \in \mathcal{A}' |\ \text{the boundary curve of } v \text{ is a meridian in } H_g \right\},$$

$$\mathcal{A}_{\mathcal{A}'} := \left\{ v \in \mathcal{A} |\ \text{the boundary curve of } v \text{ is nonmeridian in } H'_g \right\},$$

$$\mathcal{A}_{\mathcal{D}'} := \left\{ v \in \mathcal{A} |\ \text{the boundary curve of } v \text{ is a meridian in } H'_g \right\}.$$

We define a simplicial complex $\mathcal{I}\left(M, H_g\right)$ for the manifold M with respect to the Heegaard splitting $M^3 = H_g \cup_{\Sigma_g} H'_g$ by identifying $\mathcal{I}\left(H_g\right)$ with $\mathcal{I}\left(H'_g\right)$ along the vertex set $\mathcal{A}$ of $\mathcal{I}\left(H_g\right)$ as follows.

Definition 2. Let $\mathcal{I}\left(M, H_g\right)$ be the simplicial complex whose

- vertices are all vertices in $\mathcal{I}\left(H_g\right) \cup \mathcal{I}\left(H'_g\right)$ with the exception that a vertex u in $\mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}$ (resp. $\mathcal{A}' \setminus \mathcal{A}'_{\mathcal{D}}$) is identified with the corresponding vertex u' in $\mathcal{A}_{\mathcal{D}'}$ (resp. $\mathcal{A}_{\mathcal{A}'}$), that is, with the unique vertex u' in $\mathcal{A}_{\mathcal{D}'}$ (resp. $\mathcal{A}_{\mathcal{A}'}$) for which $\partial u'$ is isotopic to ∂u in Σ_g;
- edges are all edges in $\mathcal{I}\left(H_g\right) \cup \mathcal{I}\left(H'_g\right)$ with the exception that each edge (u, v) in $\mathcal{I}\left(H_g\right)$ with endpoints $u, v \in \mathcal{A}$ is identified with the (corresponding) edge in $\mathcal{I}\left(H'_g\right)$ with endpoints $u' \equiv u, v' \equiv v \in \left(\mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}\right) \cup \left(\mathcal{A}' \setminus \mathcal{A}'_{\mathcal{D}}\right)$.

Then $\mathcal{I}\left(M, H_g\right)$ is the flag complex with the above vertices and edges, that is, if all the edges of a potential face belong to the complex, then that face is required to belong to the complex.

We will be viewing both $\mathcal{I}\left(H_g\right)$ and $\mathcal{I}\left(H'_g\right)$ as subcomplexes of $\mathcal{I}\left(M, H_g\right)$. In the vertex set of $\mathcal{I}\left(M, H_g\right)$ we clearly have

$$\mathcal{A}_{\mathcal{A}'} \cup \mathcal{A}_{\mathcal{D}'} = \mathcal{A}, \mathcal{D}'_{\mathcal{D}} \cup \mathcal{A}_{\mathcal{D}'} = \mathcal{D}' \text{ and } \mathcal{A}_{\mathcal{A}'} \cup \mathcal{A}'_{\mathcal{D}} = \mathcal{A}'.$$

The above notation is summarized in Figure 1.

REMARK 3. It would be plausible to define $\mathcal{I}\left(M, H_g\right)$ by identifying the copies of $\mathcal{C}\left(\partial H_g\right)$ found inside $\mathcal{I}\left(H_g\right)$ and $\mathcal{I}\left(H_g'\right)$. However, such a complex does not serve our purposes because the pant subcomplexes $\mathcal{P}, \mathcal{P}'$ are not connected and, thus, an automorphism of $\mathcal{I}\left(M, H_g\right)$ may not preserve them in the sense exhibited in Example 4.

Our goal is to show that for any closed 3-manifold M with a fixed Heegaard splitting of genus $g \geq 3$, the automorphisms of the complex $\mathcal{I}\left(M, H_g\right)$ are all geometric, that is, they are induced by homeomorphisms of M that preserve the Heegaard splitting. This can be rephrased by saying that the map

$$A : \mathcal{MCG}\left(M, H_g\right) \to Aut\left(\mathcal{I}\left(M, H_g\right)\right)$$

is an onto map where $Aut\left(\mathcal{I}\left(M, H_g\right)\right)$ is a group of automorphisms of the complex $\mathcal{I}\left(M, H_g\right)$. Moreover, we will show (see Theorem 10) that the map A is 1–1.

For the proof of this result we first show that the dimension of the link of a vertex of $\mathcal{I}\left(M, H_g\right)$ lying in $\mathcal{A}$ is distinct (in fact, bigger) than the dimension of the link of any other vertex of $\mathcal{I}\left(M, H_g\right)$ not contained in $\mathcal{A}$. An important step is to establish that an automorphism ϕ of $\mathcal{I}\left(M, H_g\right)$ must map each vertex v in $\mathcal{P}$ to a vertex $f\left(v\right)$ which also belongs to $\mathcal{P}$ (provided that M is not homeomorphic to the connected sum of copies of $\mathbb{S}^2 \times \mathbb{S}^1$) and similarly for $\mathcal{D}$. In showing this, we use the notion of the pants complex, introduced by Hatcher and Thurston in [**8**] and its connectivity properties (see [**7**]). Finally, we use the corresponding result for handlebodies shown in [**2**], namely, that $\mathcal{MCG}\left(H_g\right)$ is isomorphic to $Aut\left(\mathcal{I}\left(H_g\right)\right)$.

If v is a vertex in $\mathcal{I}\left(M, H_g\right)$, we will denote by $Lk\left(v\right)$ the link of the vertex v in $\mathcal{I}\left(M, H_g\right)$, namely, for each simplex σ containing v consider the faces of σ not containing v and take the union over all such σ. We will use the notation $\ncong$ to declare that two links are not isomorphic as complexes.

We will also use the classical notation $\Sigma_{n,b}$ to denote the surface of genus n with b boundary components.

We conclude this section by demonstrating an example which shows that in the case $g = 2$, non-geometric automorphisms of $\mathcal{I}\left(M, H_g\right)$ may exist.

EXAMPLE 4. Let $M = H_2 \cup_\Sigma H_2'$, where $\Sigma = \partial H_g = \partial H_g'$ is the genus 2 closed surface. One may think of M as the 3-sphere with the standard Heegaard splitting. Choose a non-separating essential simple closed curve α in Σ which is not a generator for $\pi_1\left(H_2\right)$ (for example, choose α to represent the second power of a generator of $\pi_1\left(H_2\right)$). Similarly, choose β in Σ which is not a generator for $\pi_1\left(H_2'\right)$ and, in addition, $\alpha \cap \beta = \varnothing$. Then choose a non-separating essential simple closed curve γ in Σ such that

$$\alpha \cap \gamma = \varnothing = \beta \cap \gamma.$$

Clearly, the curves α, β, γ decompose Σ into two pairs of pants, denoted by P_1, P_2. Observe that P_1, P_2 are not isotopic in H_2. For, if P_1, P_2 were isotopic in H_2, then H_2 would be homeomorphic to $P_1 \times [0, 1]$ making α a generator for $\pi_1\left(H_2\right)$, a contradiction by choice. Similarly, P_1, P_2 are not isotopic in H_2'. Thus, the complex $\mathcal{I}\left(M, H_2\right)$ contains distinct vertices $[P_1], [P_2] \in \mathcal{P}$ and $[P_1]', [P_2]' \in \mathcal{P}'$. Observe that $[P_1]$ is connected by an edge only with the vertices $[\alpha], [\beta], [\gamma], [P_2]$ and similarly for

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$[P_1]'$. Let ϕ be the automorphism of $\mathcal{I}(M, H_2)$ defined by

$$\phi([P_i]) = [P_i]' \text{ and } \phi\left([P_i]'\right) = [P_i]$$

and $\phi(v) = v$ for all $v \neq [P_i], [P_i]'$, $i = 1, 2$.
If ϕ were geometric, then, since ϕ is the identity on $\mathcal{C}(\Sigma)$, ϕ would have to be induced by a homeomorphism $F : M \to M$ with $F|_\Sigma$ being the identity. As any homeomorphism $\Sigma \to \Sigma$ extends uniquely to the handlebody it bounds, F would have to be the identity on M.

3. Properties of the complex $\mathcal{I}\left(M, H_g\right)$. In this section we will calculate the dimension of the link of all types of vetrices in $\mathcal{I}\left(M, H_g\right)$. Although most properties hold for $g = 2$, we will assume throughout this section that $g \geq 3$. We recall certain properties from [**2**]:

(DM) If v is a meridian vertex in $\mathcal{I}\left(H_g\right)$ then its link in $\mathcal{I}\left(H_g\right)$ has dimension $5g - 9$ (Lemma 4).

(DP) If v is a pants vertex in $\mathcal{I}\left(H_g\right)$ then its link in $\mathcal{I}\left(H_g\right)$ has dimension $5g - 7$ (Proposition 2).

(DA) If v is an annular vertex in $\mathcal{I}\left(H_g\right)$ then its link in $\mathcal{I}\left(H_g\right)$ has dimension $5g - 7$ (Lemma 3).

Identical properties hold for the vertices in $\mathcal{I}\left(H_g'\right)$. Analogous properties hold in the complex $\mathcal{I}\left(M, H_g\right)$.

Lemma 5. *If $v \in \mathcal{D} \cup \mathcal{D}'_{\mathcal{D}}$ then its link in $\mathcal{I}\left(M, H_g\right)$ has dimension $5g - 9$. If $v \in \mathcal{A}'_{\mathcal{D}} \cup \mathcal{P} \cup \mathcal{P}'$ then its link in $\mathcal{I}\left(M, H_g\right)$ has dimension $5g - 7$.*

Proof. It is straightforward since, by the definition of $\mathcal{I}\left(M, H_g\right)$, the link of a vertex $v \in \mathcal{D} \cup \mathcal{P}$ in $\mathcal{I}\left(M, H_g\right)$ is identical with the link of v in $\mathcal{I}\left(H_g\right)$. Similarly, the link of a vertex $v \in \mathcal{D}'_{\mathcal{D}} \cup \mathcal{A}'_{\mathcal{D}} \cup \mathcal{P}'$ in $\mathcal{I}\left(M, H_g'\right)$ is identical with the link of v in $\mathcal{I}\left(H_g'\right)$. □

We next examine the dimension of the link of the vertices in $\mathcal{A} = \mathcal{A}_{\mathcal{A}'} \cup \mathcal{A}_{\mathcal{D}'}$.

Lemma 6. *If $v \in \mathcal{A}_{\mathcal{A}'}$, then the dimension of $Lk(v)$ in $\mathcal{I}\left(M, H_g\right)$ is $\geq 7g - 9$. If $v \in \mathcal{A}_{\mathcal{D}'}$, then the dimension of $Lk(v)$ in $\mathcal{I}\left(M, H_g\right)$ is $\geq 5g - 6$.*

Proof. By property (DA) we have that $v \in \mathcal{A}$ is contained in a simplex of dimension $5g - 6$ lying entirely in $\mathcal{I}\left(H_g\right) \subset \mathcal{I}\left(M, H_g\right)$.

Let $v \in \mathcal{A}_{\mathcal{A}'}$. There exist $3g - 2$ simple closed curves $\beta_1, \dots \beta_{3g-2}$ in $\Sigma_g = \partial H_g'$ such that $\left\{\partial v, \beta_1, \dots \beta_{3g-2}\right\}$ is a pants decomposition for Σ_g and each β_i is non-meridian in H_g'. This implies that the pants decomposition $\left\{\partial v, \beta_1, \dots \beta_{3g-2}\right\}$ determines $2g - 2$ pairs of pants which are incompressible in H_g'. Thus, there exist $2g - 2$ vertices in $\mathcal{P}'$ which belong to $Lk(v)$.

Let $v \in \mathcal{A}_{\mathcal{D}'}$. As g is assumed to be ≥ 3, cutting H_g' along the meridian v we always (i.e. v separating or non-separating) obtain a handlebody of genus ≥ 2 with one or two disks marked on its boundary (these being the disks bounded by copies of ∂v). On the boundary of this handlebody we may find non-meridian, simple, mutually disjoint curves $\gamma_1, \gamma_2, \gamma_3$ which form a pair of pants such that each γ_i does not intersect with the marked boundary copies of ∂v. Figure 2. exhibits this in the case $g = 3$ and ∂v is non-separating. It follows that $\gamma_1, \gamma_2, \gamma_3$ determine a pants vertex $w' \in \mathcal{P}'$ which is connected by an edge with v in $\mathcal{I}\left(H_g'\right)$. This completes the proof of the Lemma. □

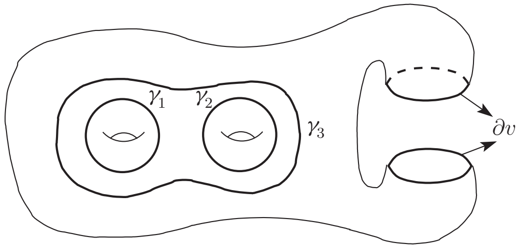

Figure 2.

We will need the following.

LEMMA 7. *If* $\phi \in Aut\left(\mathcal{I}\left(M, H_g\right)\right)$ *and* $v \in \mathcal{P}$, *then* $\phi(v) \notin \mathcal{A}'_{\mathcal{D}}$.

Proof. Let $v \in \mathcal{P}$ and denote by β one of the three boundary components of a pair of pants representing v. The 1-skeleton of $Lk(v)$ is a cone graph, that is, there exists a vertex which is connected by an edge with any other vertex in $Lk(v)$ (the annular vertex v_β with $\partial v_\beta = \beta$ is one such). We will reach a contradiction by showing that for any $u \in \mathcal{A}'_{\mathcal{D}}$ the 1-skeleton of $Lk(u)$ is not a cone graph. For this it suffices to show that

$$\forall w \in Lk(u), \exists r \in Lk(u) : w, r \text{ are not connected by an edge.}$$

For, if β_w is a boundary component of a surface representing $w \in Lk(u)$, then there exists a curve γ such that $\partial u \cap \gamma = \varnothing$ and $\gamma \cap \beta_w \neq \varnothing$. Let r be the vertex in $\mathcal{D}' \cup \mathcal{A}'$ with $\partial r = \gamma$. Then $r \in Lk(u)$ is the required vertex which is not connected by an edge with w. □

PROPOSITION 8. *If* ϕ *is an automorphism of* $\mathcal{I}\left(M, H_g\right)$ *then* $v \in \mathcal{A}$ *if and only if* $\phi(v) \in \mathcal{A}$.

Proof. The conclusion is straightforward by dimension arguments based on Lemmas 5 and 6. □

We conclude this section by showing the following property.

PROPOSITION 9. *The subcomplex of* $\mathcal{I}\left(M, H_g\right)$ *spanned by the vertices* $\mathcal{D} \cup \mathcal{P}$ *is path-connected.*

Proof. By the argument at the end of Lemma 6, if $w \in \mathcal{D}$, there exists a pants vertex $u \in \mathcal{P}$ which is connected by an edge with v. Therefore, it suffices to consider two arbitrary vertices $u, v \in \mathcal{P}$ in order to exhibit path-connectedness of $\mathcal{D} \cup \mathcal{P}$.

We will use the notion of the pants complex for surfaces originally introduced by Hatcher and Thurston in [**8**]. We refer readers to [**9**, Section 2.2] for precise definition and properties. We briefly recall that the 1-skeleton of the pants complex of a (closed for us) surface Σ_g (usually called the pants graph) has one vertex for each pants decomposition of Σ_g (equivalently, for each maximal simplex 1 in $\mathcal{C}\left(\Sigma_g\right)$) and edges joining vertices whose associated pants decomposition differs by elementary moves. More precisely, two vertices $P = \left(\alpha_1, \ldots, \alpha_{3g-3}\right)$ and P' span an edge if P' can be obtained from P by replacing one curve in P, say α_1, by another curve, say α'_1, such that the intersection number of α_1 with α'_1 is 2 if they both belong to a subsurface of Σ_g of type $\Sigma_{0,4}$ and the intersection number is 1 if they both belong to a subsurface of Σ_g of type $\Sigma_{1,1}$.

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Apparently, for each pants vertex $v \in \mathcal{P}$ we may choose a pants decomposition P_v such that the boundary curves of v belong to P_v. It was shown in [7] that the pants complex is connected and simply connected. This means that for arbitrary vertices $u, v \in \mathcal{P}$ there exists pants decompositions $P_0 = P_u, P_1, \ldots, P_{k-1}, P_k = P_v$ such that P_i, P_{i+1} differ by an elementary move for $i = 0, \ldots, k-1$. In particular, P_i, P_{i+1} have $3g-4$ curves in common. It is clear that for each $i = 1, \ldots, k-2$ we may choose a pair of pants p_i in P_i such that p_i, p_{i+1} have disjoint boundary components and similarly for u, p_1 and p_{k-1}, v. If all boundary components of all p_i are non-meridians, the sequence $u, p_1, \ldots, p_{k-1}, v$ gives rise to path of vertices in $\mathcal{P}$ from v to u and we are done. If some p_i is a compressible pair of pants in H_g, we may use a boundary curve of p_i which is meridian. □

4. Proof of the main theorem.

Let

$$A : \mathcal{MCG}\left(M, H_g\right) \rightarrow Aut\left(\mathcal{I}\left(M, H_g\right)\right)$$

be the map sending a mapping class F to the automorphism it induces on $\mathcal{I}\left(M, H_g\right)$, that is, $A(F)$ is given by

$$A(F)[S] := [F(S)],$$

where $[S]$ denotes the isotopy class (vertex) determined by S.

THEOREM 10. *Assume M is not homeomorphic to the connected sum of copies of $\mathbb{S}^2 \times \mathbb{S}^1$. Then the map $A : \mathcal{MCG}\left(M, H_g\right) \rightarrow Aut\left(\mathcal{I}\left(M, H_g\right)\right)$ is an isomorphism for* $g \geq 3$.

Proof. We will use the corresponding result, see [2, Theorem 7], applied to the handlebodies H_g and H'_g.

We first show that every $\phi \in Aut\left(\mathcal{I}\left(M, H_g\right)\right)$ is geometric. We claim that either
Case I: $\phi(\mathcal{D}) = \mathcal{D}$ and $\phi(\mathcal{P}) = \mathcal{P}$
or
Case II: $\phi(\mathcal{P} \cup \mathcal{D}) = \mathcal{P}' \cup \mathcal{D}'_{\mathcal{D}}$, in which case $\mathcal{A}'_{\mathcal{D}} = \varnothing$.

Let $v \in \mathcal{P}$. By dimension considerations (see Lemmas 5 and 6), we have $\phi(v) \in \mathcal{P} \cup \mathcal{P}' \cup \mathcal{A}'_{\mathcal{D}}$, and by Lemma 7, $\phi(v) \in \mathcal{P} \cup \mathcal{P}'$.

Assume first that $\phi(v) \in \mathcal{P}$. By Proposition 9, $\phi(w) \in \mathcal{P}$ for all $w \in \mathcal{P}$. To see the latter, assume that $\phi(w) \in \mathcal{P}'$ for some $w \in \mathcal{P}$. Choose a path σ from v to w whose vertices are in $\mathcal{P} \cup \mathcal{D}$. Then $\phi(\sigma)$ is a path from a vertex in $\mathcal{P}$ to a vertex in $\mathcal{P}'$. It follows that some vertex of σ is mapped to a vertex in $\mathcal{A}$, which is a contradiction by Proposition 8. Thus, we have that if for an arbitrary $v \in \mathcal{P}$, $\phi(v) \in \mathcal{P}$ then $\phi(\mathcal{P}) = \mathcal{P}$ and clearly $\phi(\mathcal{D}) = \mathcal{D}$ as stated in Case I.

Now assume that $\phi(v) \in \mathcal{P}'$. Using Proposition 9 in the same way as above, we have $\phi(\mathcal{P} \cup \mathcal{D}) \subseteq \mathcal{P}' \cup \mathcal{D}'_{\mathcal{D}} \cup \mathcal{A}'_{\mathcal{D}}$. Then by dimension arguments (cf Lemma 5) we have $\phi(\mathcal{D}) = \mathcal{D}'_{\mathcal{D}}$ and $\phi(\mathcal{P}) = \mathcal{P}' \cup \mathcal{A}'_{\mathcal{D}}$. By Lemma 7, we have $\phi(\mathcal{P}) = \mathcal{P}'$ and, again by dimension arguments, we have $\phi\left(\mathcal{A}'_{\mathcal{D}}\right) = \mathcal{A}'_{\mathcal{D}}$. The latter is impossible if $\mathcal{A}'_{\mathcal{D}} \neq \varnothing$: for, if $x \in \mathcal{A}'_{\mathcal{D}}$ and $\phi(x) \in \mathcal{A}'_{\mathcal{D}}$ we may choose a pair of pants $w \in \mathcal{P}'$ in the $Lk(\phi(x))$. Then

$\phi^{-1}(w) \in Lk(x)$ and $\phi^{-1}(w) \in \mathcal{P}$, a contradiction since $x \in \mathcal{A}'_{\mathcal{D}}$ and no vertex in $\mathcal{A}'_{\mathcal{D}}$ is connected by an edge with a vertex in $\mathcal{P}$. Thus, $\mathcal{A}'_{\mathcal{D}} = \varnothing$ as stated in Case II.

We now proceed with the proof of the theorem in Case I. We have $\phi(\mathcal{P}') = \mathcal{P}'$ and $\phi(\mathcal{D}' \cup \mathcal{A}') = \mathcal{D}' \cup \mathcal{A}'$. Thus, ϕ induces an automorphism ϕ' of $\mathcal{I}(H'_g)$, and by [**2**, Theorem 7] ϕ' is geometric, hence we obtain a homeomorphism $F' : H'_g \to H'_g$ realizing ϕ'. Such a homeomorphism F' is unique. Since $\phi'(\mathcal{D}'_{\mathcal{D}} \cup \mathcal{A}'_{\mathcal{D}}) = \mathcal{D}'_{\mathcal{D}} \cup \mathcal{A}'_{\mathcal{D}}$, it follows that F' maps each simple closed curve in $\Sigma = \partial H'_g = \partial H_g$ which bounds a meridian in H_g to another such meridian. Therefore, F' extends to a homeomorphism of H_g. This extension is unique (see, for example, [**4**, Theorem 3.7 p. 94]). In other words, F' defines a homeomorphism

$$F_M : H_g \cup_{\Sigma_g} H'_g \to H_g \cup_{\Sigma_g} H'_g.$$

Clearly, the composition $A(F_M^{-1}) \circ \phi$ is an automorphism of $\mathcal{I}(M, H_g)$, which is the identity on $\mathcal{I}(H'_g)$. Thus, we may assume that the automorphism $\phi \in Aut(\mathcal{I}(M, H_g))$ is the identity on $\mathcal{I}(H'_g)$ and we want to show that it is the identity on the whole complex $\mathcal{I}(M, H_g)$.

We first show that ϕ is the identity on $\mathcal{D}$. Let $w \in \mathcal{D}$, and let D be a meridian in H_g representing w. If $\phi(w) \neq w$, that is, $\phi(w)$ is represented by a meridian D' non-isotopic to D, then we may find a simple, essential curve α in ∂H_g which does not bound a meridian in H_g such that $\partial D \cap \alpha \neq \varnothing$ and $\partial D' \cap \alpha = \varnothing$. Since ϕ fixes the vertex represented by α, we have a contradiction. Thus, ϕ fixes every vertex $w \in \mathcal{D}$.

It follows that ϕ induces an automorphism $\phi|_{\mathcal{I}(H_g)}$ of $\mathcal{I}(H_g)$ which fixes $\mathcal{A} \cup \mathcal{D}$. This automorphism is geometric (see [**2**, Theorem 7]), that is, there exists a homeomorphism $G : H_g \to H_g$ realizing $\phi|_{\mathcal{I}(H_g)}$. As $\phi|_{\mathcal{I}(H_g)}$ fixes every vertex in $\mathcal{A} \cup \mathcal{D}$, G is is the identity on $\Sigma = \partial H_g$. As every homeomorphism of ∂H_g which extends to a homeomorphism of H_g it does so uniquely, it follows that G is the identity. Therefore, $\phi|_{\mathcal{I}(H_g)}$ is the identity on $\mathcal{I}(H_g)$ and, thus, is the identity on the whole complex $\mathcal{I}(M, H_g)$ as required. This completes the proof in Case I.

We proceed with Case II. As $\mathcal{A}'_{\mathcal{D}} = \varnothing$, we have
Case IIa: $\mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}} \neq \varnothing$, and
Case IIb: $\mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}} = \varnothing$, that is, $\mathcal{D}'_{\mathcal{D}} \cap \mathcal{A} = \varnothing$.

We will show that Case IIa does not occur, and in Case IIb M is homeomorphic to the connected sum of copies of $\mathbb{S}^2 \times \mathbb{S}^1$. Let $w \in \mathcal{A}' = \mathcal{A}_{\mathcal{A}'}$. Then $Lk(w)$ contains $2g-2$ pant vertices in $\mathcal{P}$, which form a simplex, and similarly $2g-2$ pant vertices in $\mathcal{P}'$. This implies that $\phi(w) \notin \mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}$ because a meridian vertex in $\mathcal{I}(H'_g)$ cannot have $2g-2$ pant vertices from $\mathcal{P}'$ in its link. It follows that $\phi(\mathcal{A}_{\mathcal{A}'}) = \mathcal{A}_{\mathcal{A}'}$ and $\phi(\mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}) = \mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}$. Let now $v \in \mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}$ and denote by $p_1, \ldots, p_{2g-2}$ a maximal set of pant vertices from $\mathcal{P}$ contained in $Lk(v)$. As $\phi(p_i) = p'_i$ with $p'_i \in \mathcal{P}'$, we have a contradiction because $\phi(v) \in \mathcal{D}' \setminus \mathcal{D}'_{\mathcal{D}}$ and $\partial\phi(v)$ bounds a meridian in H'_g (thus, $\phi(v)$ cannot have $2g-2$ pant vertices from $\mathcal{P}'$ in its link). This shows that Case IIa cannot occur.

We conclude the proof of the theorem by observing that in Case IIb the manifold M is homeomorphic to the connected sum of copies of $\mathbb{S}^2 \times \mathbb{S}^1$. If $H_2 = \mathbb{D}^2 \times \mathbb{S}^1$ is glued with $H'_2 = \mathbb{D}^2 \times \mathbb{S}^1$ along $\mathbb{S}^1 \times \mathbb{S}^1$ so that every curve which is a meridian boundary in H_2 is identified with a meridian boundary in H'_2 then M is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$. Inductively, if a is a separating curve in $\partial H_g = \partial H'_g$ which bounds a meridian D_α in H_g and a meridian D'_α in H'_g, then cutting along the 2-sphere $D_a \cup D'_\alpha$ we obtain 3-manifolds M_1, M_2 each with one boundary component homeomorphic to $\mathbb{S}^2$. By gluing

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a 3-ball along the boundary component of each, we obtain that M is homeomorphic to $M_1 \# M_2$ with M_1, M_2 having Heegaard genus $\leq g - 1$. □

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