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THERE ARE NO n-POINT F_σ SETS IN R^m

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We show that, for any positive integers n and m, if a set $S \subset R^m$ intersects every $m-1$ dimensional affine hyperplane in R^m in exactly n points, then S is not an F_σ set. This gives a natural extension to results of Khalid Bouhjar, Jan J. Dijkstra, and R. Daniel Mauldin, who have proven this result for the case when $m = 2$, and also Jan J. Dijkstra and Jan van Mill, who have shown this result for the case when $n = m$.

1. INTRODUCTION

The subject of n-point sets has been studied since 1914 with the result of Mazurkiewicz [7] that there exists a subset of the plane which intersects every straight line in exactly two points. Kulesza [5] has shown that every 2-point set is zero dimensional, and the authors have shown that every 3-point set is zero dimensional [3], and that any n-point set which is not zero dimensional contains an arc [4] (the latter also independently proven by Kulesza). Much recent emphasis on the topic of n-point sets has been focused on trying to determine whether or not a Borel n-point set can exist. It is not presently known whether there are n-point G_δ sets [6].

An n-point set in R^m is a subset of R^m intersecting every $m-1$ dimensional affine hyperplane in exactly n points. Bouhjar, Dijkstra, and Mauldin [1] have shown that there are no n-point F_σ sets in the plane and Dijkstra and van Mill [2] have also shown that there are no n-point F_σ sets in R^n. We fill in the natural extension by showing also that there can be no n-point F_σ sets in R^m when $n > m$, by forcing the existence of arcs in a similar manner to Dijkstra and van Mill after a separation of points using the Baire category theorem.

2. PROOF OF THEOREM

THEOREM 2.1. *There is no n-point F_σ set in R^m.*

PROOF: First, we let S denote a set which intersects every hyperplane in R^m in exactly n points. Suppose S is an F_σ set. Then S is a countable union of compact sets

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$C_1, C_2, \dots$. We choose an axis of R^m which we shall refer to as the x-axis. We shall refer to a hyperplane with x-coordinate x intersected with S as L_x. For each positive integer k, let E_k denote the set of all points x on the x-axis such that all elements of L_x are a distance at least $1/k$ from each other element of L_x. The union of all E_k is the x-axis, so by the Baire category theorem we can find a closed interval J_0 of points on the x-axis and a dense subset D_0 of J_0 so that for some positive integer k and every point $x \in D_0$, L_x is a set of n points which are at least a distance $d = 1/k$ apart from one another. Furthermore, we can require that D_0 is not the countable union of nowhere dense sets (since if all E_k were the countable union of nowhere dense sets, all of the x-axis would also be), and we can require that there is no open interval contained in J_0 so that the intersection of D_0 with that interval is the countable union of nowhere dense sets. Otherwise, we could remove a countable collection of subintervals whose union is dense, each of which has a nowhere dense intersection with D in J_0, so D_0 would be a countable union of nowhere dense sets which is impossible. For brevity, in future we shall refer to a set D as being DB (densely Baire) in an interval I whenever every open subset of I has intersection with D which is not the countable union of nowhere dense sets.

For a given set W in the real numbers, we shall use the notation B_W to refer to the set of points of R^m whose x-coordinates are contained in W.

Partition B_{J_0} into m-dimensional rectangles having sides parallel to the x-axis of side length d/m except for the sides parallel to the x-axis, which have side length equal to the length of J_0. We shall refer to these rectangles as $K_1, K_2, K_3, \dots$. Then for each $x \in D$, each K_i can only contain at most one point of L_x. There are only countably many $K_i \cap C_k$, and the union of their projections onto the x-axis is all of J_0. By the Baire category theorem there is some $K_{i_1} \cap C_{k_1}$ whose projection onto the x-axis is dense in, and thus contains (since the projection is compact) a non-degenerate closed interval $J_1 \subset J_0$. Let D_1 be the intersection of D with J_1, and note that D_1 is DB in J_1 by construction of D.

Note that each L_x contains points in rectangles K_i, $i \neq i_1$, for all $x \in D_1$. Hence, the union of the projections of all $K_i \cap C_k$, for $i \neq i_1$, onto the x-axis contains J_1. Hence, there is some $K_{i_2} \cap C_{k_2}$ whose projection onto the x-axis is dense in some subinterval $J_2 \subset J_1$ and thus contains J_2. For each $x \in J_2$, there is at least one point in $L_x \cap K_{i_2}$ and at least one point in $L_x \cap K_{i_1}$. We refer to D_2 as the intersection of D with J_2, and note that D_2 is DB in J_2. Similarly, we may find disjoint $C_{i_3} \cap K_{i_3}, \dots, C_{i_n} \cap K_{i_n}$ so that each $C_{i_t} \cap K_{i_t}$ has projection containing an interval J_t on the x-axis so that $J_t \subset J_{t-1}$.

Then define interval $T = J_n$. Hence, for every $x \in T$, $L_x \cap K_{i_t}$ consists of exactly one point for all $1 \leqslant t \leqslant n$. We define $R_t = K_{i_t} \cap B_T$. The projection of each $R_t \cap S$ onto the x-axis is thus a one to one continuous mapping with compact domain, and is thus a homeomorphism. We conclude that $B_T \cap S$ consists of n disjoint arcs, $A_1, A_2, \dots, A_n$. It remains to show that this is impossible.

We let $[a, b]$ be the interval T. We extend the A_t arcs as far as possible along the positive x-coordinate direction. More precisely, let c be the least upper bound of the set of all points r so that $B_{[a,r]} \cap S$ consists of n disjoint arcs (if any). Refer to $E_1, \dots, E_n$ as the union of all such arcs containing $A_1, \dots, A_n$ respectively. The case where there is no largest r so that $B_{[a,r]} \cap S$ will be handled later, and for now we assume that c exists. Each E_t contains at most one limit point on L_c since if an E_t contained two limit points on B_c then a hyperplane separating these two points would necessarily intersect E_t in infinitely many points. We show first notice that each E_t must contain a limit point on B_c. Otherwise E_t is unbounded in some direction, and so by rotating B_c about a point sufficiently high in the direction in which E_t is unbounded, we obtain a hyperplane intersecting E_t in at least two points, and all of the other E_i in points whose x-coordinates are less than c, and thus in at least $n + 1$ points of S.

Choose an upwards direction perpendicular to the x-axis with respect to which the points $\overline{E_t} \cap L_c$ for $1 \leqslant t \leqslant n$ are strictly ordered according to height, and label them from lowest to highest as $p_1, p_2, \dots, p_n$.

Let $E_{i_1}, \dots, E_{i_k}$ be the set of all of the E_t which have the property that there is an arc containing E_t which has x-axis projection $[a, r + \varepsilon]$ for some $\varepsilon > 0$ and has positive distance from each E_i for $i \neq t$. We can pick a number $r' > r$ and disjoint arcs $F_{i_1}, \dots, F_{i_k}$ so that each F_{i_t} has x-axis projection $[a, r']$ and contains the corresponding arc E_{i_t}. Let $p = p_h$ be the highest of $p_1, p_2, \dots, p_n$ having the property that there is no arc F_{i_t} containing p. There must be such a point p or else c is not an upper bound as it was defined.

First, suppose that $p_h = p_j$ for some $j \neq h$. Then we can find a hyperplane P_1 consisting of all points $x - \pi_x(p) + k\big(z - \pi_z(p)\big) = 0$, where z represents the upwards direction and k is a small number so that P_1 intersects E_i, for all i such that p_i is below p, at a point whose x-coordinate is less than c, and intersects F_i, for all i such that p_i is above p, at points whose x-coordinate is less than r'. This hyperplane contains a point $q \in S$ other than the one point on all the previously specified arcs since these points plus p are fewer than n in number. Rotating P_1 about q slightly gives a hyperplane P_2 which intersects all E_i such that p_i has height less than or equal to the height of p, and also intersects all F_i such that p_i is higher than p, and also the point q, and thus contains at least $n + 1$ points of S. This is impossible.

We conclude that $p_1, p_2, \dots, p_n$ are strictly ordered from lowest to highest without duplication. We first find a hyperplane P containing p_h and intersecting all E_i for $i < h$ at points with x-coordinate in (a, r) and intersecting all F_i for $i > h$ at points with x-coordinate in (r, r'). There is then an interval of x-coordinates which we shall call (l, u) so that P intersects the x-axis at l and for all $x \in (l, u)$ the hyperplane P' parallel to P intersecting the x-axis at the point x also has the property that P' intersects all E_i where $i < h$ at a point with x-coordinate in (a, r) and intersects all F_i where $i > h$ at

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a point with x-coordinate in (r, r'). Let P_x refer to the hyperplane parallel to P whose intersection with the x-axis is x. Since there is no F_h extension of E_h we can then find a point $q \in [l, u)$ so that with respect to some new upwards direction z parallel to P, the points $p_1, \ldots, p_n$ are still ordered from lowest to highest and there is an $\varepsilon > 0$ so that for some $q' \in S \cap P_q$ we can find points y as close as we wish to q so that $y \in (l, u)$ and P_y contains no points of S with height within ε of the height of q' (since otherwise we could construct an arc extension F_i containing p_h). By choosing a point y' on a hyperplane P_y sufficiently close to P_q we can find a hyperplane H which contains q' and y' and still intersects all E_i for $i < h$ and all F_i for $i > h$. Hence, H intersects at least $n + 1$ points of S, which is impossible.

We conclude that there is no upper bound c as described above. Similarly, it must follow that there is no lower bound for the set of all real numbers r so that $B_{[r,b]}$ consists of n disjoin arcs. Thus, X consists entirely of n disjoint homeomorphic images of the real numbers, and every hyperplane $x = k$ intersects each of these infinite arcs exactly once. However, since the x-axis direction was chosen arbitrarily in this proof, that means that with respect to every direction each hyperplane intersects each infinite arc in exactly one point. This is impossible. Hence, there are no n-point F_σ sets in R^m. ∎

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