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STRUCTURE OF CORADICAL FILTRATION AND ITS APPLICATION TO HOPF ALGEBRAS OF DIMENSION pq

DAIJIRO FUKUDA
e-mail: d-fukuda@world.odn.ne.jp

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Abstract. This paper contributes to the classification problem of pq dimensional Hopf algebras H over an algebraically closed field $\mathbf{k}$ of characteristic 0, where p, q are odd primes. It is shown that such Hopf algebras H are semisimple for the pairs of odd primes $(p, q) = (3, 11)$,$(3, 13)$,$(3, 19)$,$(5, 17)$,$(5, 19)$,$(5, 23)$,$(5, 29)$,$(7, 17)$,$(7, 19)$,$(7, 23)$, $(7, 29)$,$(11, 29)$,$(13, 29)$.

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1. Introduction. From the beginning of the 1990s, the classification of finite dimensional Hopf algebras over an algebraically closed field was actively studied, and remarkable developments were accomplished. Especially, for a Hopf algebra H of dimension pq over an algebraically closed field of characteristic 0, P. Etingof and S. Gelaki showed that if H is semisimple then it is isomorphic to a group algebra or the dual of a group algebra [**4**]. And D. Ştefan showed that if H is pointed then it is semisimple [**10**]. However the classification of general non-semisimple, non-pointed Hopf algebras of dimension pq is still open.

The following results are also obtained about the classification problem in the dimension pq. S.-H. Ng showed that a Hopf algebra H of dimension p^2 is either semisimple or isomorphic to a Taft algebra [**7**]. N. Andruskiewitsch and S. Natale solved problems for 15, 21, 25, 35 and 49 dimensions using a decomposition of the first term H_1 of the coradical filtration [**1**]. M. Beattie and S. Dăscălescu solved problems for 14, 55, 77, 65, 91 and 143 dimensions by examining the first term of the coradical filtration more closely [**2**]. Recently S.-H. Ng solved the case where p, q are twin primes in [**8**], and the case for dimension $2p$ in [**9**]. Furthermore P. Etingof and S. Gelaki solved the case where $q \leq 2p + 1$ [**5**]. For distinct prime numbers p, q, all the pq dimensional examples known so far are semisimple. So it is natural to ask whether that all pq dimensional Hopf algebras are semisimple.

In this paper we apply the technique of the decomposition of the first term H_1 of the coradical filtration used in [**1**] and [**2**] to general n-th term H_n of the coradical filtration, and contribute to the question above by proving the following main Theorem.

THEOREM 1.1. *Let H be a pq dimensional Hopf algebra over an algebraically closed field of characteristic* 0 *where p, q are odd primes with $p < q < 4p + 10$, $q \leq 29$ such that q can not be expressed as $q = 4p + 2 + pz_1 + 4z_2$ with positive integers z_1, z_2. Then H is semisimple.*

As a corollary to Theorem 1.1, our new classification results are the following.

COROLLARY 1.2. *A Hopf algebra of dimension pq where* $(p, q) = (3, 11)$, $(3, 13)$, $(3, 19)$, $(5, 17)$, $(5, 19)$, $(5, 23)$, $(5, 29)$, $(7, 17)$, $(7, 19)$, $(7, 23)$, $(7, 29)$, $(11, 29)$, $(13, 29)$ *is semisimple and isomorphic to a group algebra or the dual of a group algebra.*

2. Preliminaries. Throughout this paper, H is a finite dimensional Hopf algebra over an algebraically closed field $\mathbf{k}$ of characteristic 0, and Δ, ϵ, S denote the comultiplication, the counit and the antipode respectively. The n-th term of the coradical filtration of H is $H_n = \wedge^{n+1} H_0$, where H_0 is the coradical of H. As $\mathbf{k}$ is algebraically closed, there exists a coalgebra projection $\pi : H \to H_0$ and $H = H_0 \oplus I$, where $\ker \pi = I$(see [**6**, 5.4.2]). Refer to [**3**] for general results of Hopf algebras.

Set $\rho_l = (\pi \otimes id)\Delta$ and $\rho_r = (id \otimes \pi)\Delta$. H is a H_0-bicomodule with the structure maps ρ_l and ρ_r. H_0, H_n, I are H_0-subbicomodules of H. Any H_0-bicomodule is a direct sum of simple H_0-subbicomodules and a simple H_0-bicomodule has coefficient coalgebras $(C_i\ ,\ C_j)$ and its dimension is $n_i n_j$, where n_i, n_j are the dimensions of associated comodules of C_i, C_j respectively.

Let P_n , $n = 0, 1, 2, \ldots$ be defined inductively by:

$$P_0 = 0,$$
$$P_1 = \{x \in H\,;\, \Delta(x) - \rho_l(x) - \rho_r(x) = 0\},$$
$$P_n = \{x \in H\,;\, \Delta(x) - \rho_l(x) - \rho_r(x) \in \textstyle\sum_{1 \le i \le n-1} P_i \otimes P_{n-i}\},\ n \ge 2.$$

Then P_n is a H_0-subbicomodule of I and $P_n = H_n \cap I$, due to Nichols (see [**1**, Lemma 1.1]).

We denote by $P_n^{C_i,\, C_j}$ the isotypic component of the simple subbicomodule of P_n with coalgebra of coefficients $(C_i\ ,\ C_j)$. We say the subspace $P_n^{C_i,\, C_j}$ is non-degenerate if $P_n^{C_i,\, C_j} \not\subset P_{n-1}$.

We list several key results which we use in this paper for the readers convenience.

PROPOSITION 2.1 ([Lemma 1.2, **1**]). *The first term of the coradical filtration can be expressed as* $C_1 = \sum_{i,j} C_i \wedge C_j$ *and* $C_i \wedge C_j = C_i \oplus C_j \oplus P_1^{C_i, C_j}$.

PROPOSITION 2.2 ([Crollary 1.3, **1**]). *For a grouplike element g and the antipode S,*

$$\dim P_1^{C,\, D} = \dim P_1^{gC,\, gD} = \dim P_1^{Cg,\, Dg} = \dim P_1^{S(D),\, S(C)}.$$

The following results are from [**2**].

PROPOSITION 2.3 ([Crollary 4.2, **2**]). *If there is no non-trivial skew primitives then there exists a simple subcoalgebra C* $(\dim C \ge 4)$ *of H such that* $P_1^{1,\, C} \ne 0$.

PROPOSITION 2.4 ([Lemma 5.1, **2**]). *No non-semisimple Hopf algebra H of square-free dimension can be generated as a Hopf algebra by a simple subcoalgebra of dimension* 4 *that is stable under the antipode.*

In general the orders of the grouplike elements and the antipode are important factors for discussion of finite dimensional Hopf algebras. The following result gives this for non-semisimple Hopf algebras of dimension pq where p, q $(p < q)$ are odd primes.

PROPOSITION 2.5 ([Proposition 6.2, **7**]). *If H is a non-semisimple Hopf algebra of dimension pq then the order of any grouplike element is 1 or p and the order of the antipode is* $4p$.

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3. Some properties of coradical filtration. In this section we will obtain some properties of the space P_n. Throughout this section, C, D, Z, W and $X_1, .., X_i, \ldots$ are simple subcoalgebras of H. We denote $\Delta - \rho_l - \rho_r$ by $\widehat{\Delta}$. Then the kernel of $\widehat{\Delta}$ is P_1. $\widehat{\Delta}$ has a coassociativity inherited from the coassociativity of Δ.

Suppose $\widehat{\Delta}^{(n)}(x) = 0$ for $x \in H$. Setting a base of P_k by extending the base of P_{k-1} inductively, $\widehat{\Delta}^{(m)}(x)$ is contained in $\sum_{\substack{1\le i_j \le n-1 \\ i_1+\cdots+i_{m+1}=n}} P_{i_1} \otimes \cdots \otimes P_{i_{m+1}}$ by induction on m. Thus x is contained in P_n, and so the space P_n can be expressed as $\{x \in H | \widehat{\Delta}^{(n)}(x) = 0\}$.

LEMMA 3.1. *If $x \in P_n$, $x \notin P_{n-1}$ then $\widehat{\Delta}(x) \notin \sum_{\substack{1\le i \le n-1 \\ i\neq m}} P_i \otimes P_{n-i}$ for all $m \in \{1, .., n-1\}$.*

Proof. We suppose that $\widehat{\Delta}(x)$ is contained in $\sum_{\substack{1\le i \le n-1 \\ i\neq m}} P_i \otimes P_{n-i}$ for some $m \in \{1, .., n-1\}$. Thus $\widehat{\Delta}^{(n-1)}(x) = (\widehat{\Delta}^{(m-1)} \otimes \widehat{\Delta}^{(n-m-1)})\widehat{\Delta}(x) = 0$. This means $x \in P_{n-1}$. □

LEMMA 3.2. *If the subspace $P_n^{C,D}$ is non-degenerate then there exists a set of simple coalgebras $\{X_1, .., X_{n-1}\}$ such that P_i^{C,X_i}, $P_{n-i}^{X_i,D}$ are non-degenerate for all $i \in \{1, .., n-1\}$.*

Proof. Since $\Delta(P_n^{C,D}) \subset C \otimes P_n^{C,D} + P_n^{C,D} \otimes D + \sum_{\substack{1\le i \le n-1 \\ j,k,l,m}} (P_i^{X_j,X_k} \otimes P_{n-i}^{X_l,X_m})$, and using the coassociativity, $\Delta^{(2)}(P_n^{C,D})$ is contained in the following space.

$$\begin{aligned} &C \otimes C \otimes P_n + C \otimes P_n \otimes D + P_n \otimes D \otimes D + I \otimes I \otimes I \\ &\quad + \sum_{i,j,k,l,m} (C \cap X_j) \otimes P_i^{X_j,X_k} \otimes I + P_i^{X_j,X_k} \otimes (X_k \cap X_l) \otimes P_{n-i}^{X_l,X_m} \\ &\quad + I \otimes P_{n-i}^{X_l,X_m} \otimes (D \cap X_m). \end{aligned}$$

Thus the following result can be obtained by applying the counit ϵ.

$\Delta(P_n^{C,D}) \subset C \otimes P_n + P_n \otimes D + \sum_{i,k} P_i^{C,X_i^{(k)}} \otimes P_{n-i}^{X_i^{(k)},D}$.

If it lacks even one of the simple coalgebras $X_1, .., X_{n-1}$ such that P_i^{C,X_i}, $P_{n-i}^{X_i,D}$ are non-degenerate, then $P_n^{C,D}$ is contained in P_{n-1} by Lemma 3.1. This contradicts non-degeneracy of $P_n^{C,D}$. □

LEMMA 3.3. *The space $C \wedge (\wedge^{n-1} H_0) \wedge D$ which is defined by the wedge product of simple coalgebras can be decomposed as follows.*

$$C \wedge (\wedge^{n-1} H_0) \wedge D = \begin{cases} C \oplus D \oplus P_1^{C,D} & \text{for } n = 1 \\ H_0 \oplus \left(P_{n-2} + \sum_{i,j}\left(P_{n-1}^{C,X_i} + P_{n-1}^{X_j,D}\right) + P_n^{C,D}\right) & \text{for } n \ge 2. \end{cases}$$

Proof. N. Andruskiewitsch, S. Natale proved the case $n = 1$ (see Proposition 2.1). We suppose $n \ge 2$.

We first show that the left-hand side is contained in the right-hand side by induction on n. We assume $P_n^{Z,W}$ is non-degenerate and $x \in P_n^{Z,W} \cap (C \wedge (\wedge^{n-1} H_0) \wedge D)$. By Lemma 3.2, there exist a set of simple coalgebras $\{X_1^{(k)}, .., X_{n-1}^{(k)}\}$ such that $\Delta(x)$ is contained in the intersection of the following two spaces

$$\begin{gathered} C \otimes H + H \otimes (\wedge^{n-1} H_0 \wedge D), \\ Z \otimes P_n^{Z,W} + P_n^{Z,W} \otimes W + \sum_{i,k} P_i^{Z,X_i^{(k)}} \otimes P_{n-i}^{X_i^{(k)},W}. \end{gathered}$$

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Thus $\Delta(x) \in (Z \cap C) \otimes P_n^{Z,W} + Z \otimes (P_{n-2} + P_{n-1}^{Z,(W \cap D)}) + P_n^{Z,W} \otimes W + I \otimes I$. If $Z \neq C$ then $\Delta(x)$ is contained in $Z \otimes P_{n-1} + I \otimes W + I \otimes I$. Therefore $x = (\epsilon \otimes id)\Delta(x) \in P_{n-1}$ and so $C \wedge (\wedge^{n-1} H_0) \wedge D$ is contained in $H_0 \oplus (P_{n-1} + \sum_i P_n^{C, X_i})$. We can show that $C \wedge (\wedge^{n-1} H_0) \wedge D \subset H_0 \oplus (P_{n-1} + \sum_i P_n^{X_i, D})$ in the same way. Moreover $P_{n-1} \cap (C \wedge (\wedge^{n-1} H_0) \wedge D)$ is contained in $P_{n-2} \oplus \sum_{ij} P_{n-1}^{C, X_i} + P_{n-1}^{X_j, D}$ similarly. Therefore the left-hand side is contained in the right-hand side.

We next show that the right-hand side is contained in the left-hand side. By the definition of wedge and the inductive argument, the spaces $P_{n-1}^{C,X}$, $P_{n-1}^{X,D}$, $P_n^{C,D}$ are contained in $C \wedge (\wedge^{n-1} H_0) \wedge D$ for all simple coalgebras X. Hence the right-hand side is contained in the left-hand side. □

COROLLARY 3.4. *The space $C \wedge (\wedge^{n-1} H_0) \wedge D$ can be decomposed as following,*

$$H_0 \oplus \left(\bigoplus_{X \neq C, Y \neq D} P_{n-2}^{X,Y} \right) \oplus \left(\bigoplus_{X \neq D} P_{n-1}^{C,X} \right) \oplus \left(\bigoplus_{X \neq C} P_{n-1}^{X,D} \right) \oplus P_n^{C,D} \qquad \textit{for } n \geq 2.$$

LEMMA 3.5. *For a grouplike element g and the antipode S,*

$$\dim P_n^{C,D} = \dim P_n^{gC, gD} = \dim P_n^{Cg, Dg} = \dim P_n^{S(D), S(C)}.$$

Proof. Since $g(C \wedge H_0 \wedge \cdots \wedge H_0 \wedge D) = gC \wedge H_0 \wedge \cdots \wedge H_0 \wedge gD$, the first equality is obtained by the counting dimensions of $gC \wedge H_0 \wedge \cdots \wedge H_0 \wedge gD$ and $C \wedge H_0 \wedge \cdots \wedge H_0 \wedge D$ with Corollary 3.4. The rest can be shown similarly. □

This Lemma is a generalization of Proposition 2.2 which deals with the case $n = 1$. And the proof is based on the proof of Proposition 2.2(see [**1**, Corollary 1.4]).

Let the socle of H be $Soc(H) = \oplus_{i,j} M_{C_i}^j$ where $M_{C_i}^1, \ldots, M_{C_i}^k$ are all simple subcomodules of C_i as a right H-comodule. Let $E(M_{C_i}^j)$ be an injective envelope of $M_{C_i}^j$. Then H can be expressed as $H = \oplus_{i,j} E(M_{C_i}^j)$ as a right H-comodule [**3**, Theorem 2.4.16]. To simplify the description, we denote the sum of above injective envelopes $\oplus_j E(M_{C_i}^j)$ by $E(C_i)$ for fixed C_i.

LEMMA 3.6. *The space $E(C)$ can be expressed as $C \oplus \sum_{n,D} P_n^{C,D}$.*

Proof. We first show that the left-hand side is contained in the right-hand side. Since $H = H_0 \oplus I = H_0 \oplus \sum_{n,X,Y} P_n^{X,Y}$, it is clear that $E(C) \subset C \oplus \sum_{n,X,Y} P_n^{X,Y}$. Thus it is sufficient to show $X = C$. We consider $E(C) \cap P_n^{X,Y}$ under the assumption $X \neq C$.

$$\begin{aligned} E(C) \cap P_n^{X,Y} &= (\epsilon \otimes id)\Delta(E(C) \cap P_n^{X,Y}) \\ &\subset (\epsilon \otimes id)((E(C) \otimes H) \cap (X \otimes P_n + P_n \otimes Y + I \otimes I)) \\ &\subset (\epsilon \otimes id)(I \otimes H) = 0. \end{aligned}$$

Next we show that the right-hand side is contained in the left-hand side. Let M_D be a simple subcomodule of the simple coalgebra D which is not equal to C. If $E(M_D) \cap (C \oplus \sum_{n,X} P_n^{C,X}) \neq 0$ then such intersection is a subcomodule of $E(M_D)$ and it contains M_D, contradicting $D \neq C$. Therefore $E(M_D) \cap (C \oplus \sum_{n,X} P_n^{C,X}) = 0$. □

LEMMA 3.7. *Let C, D be simple subcoalgebras of H such that $P_m^{C,D}$ is non-degenerate. If $\widehat{\Delta}(P_n^{C,X}) \subset \sum_{\substack{1 \leq i \leq n-1 \\ Y \neq D}} (P_i^{C,Y} \otimes P_{n-i}^{Y,X}) + P_{m-1}^{C,D} \otimes P_{n-m+1}^{D,X}$ for all simple coalgebras X*

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and all $n \in \mathbb{N}$. Then $Head(E(C)) = E(C)/\mathbf{J}E(C)$ contains D-simple comodules as a direct summand where $\mathbf{J} = H_0^{\perp}$ is the Jacobson radical of H^*.

Proof. Set $\Phi = C \oplus (\sum_{n,\, Y \neq D} P_n^{C,\, Y}) \oplus P_{m-1}^{C,\, D}$. The space Φ is a subcomodule of $E(C)$ by the condition above.

Since $\widehat{\Delta}(E(C)) \subset \sum_{\substack{n,\, X,\, Y \\ Y \neq D}} (P_n^{C,\, Y} \otimes P_n^{Y,\, X}) + P_{m-1}^{C,\, D} \otimes P_n^{D,\, X}$, the following holds.

$$
\begin{aligned}
JE(C) &= \sum \langle E(C)_{(2)}\,,\ J\rangle E(C)_{(1)} \\
&\subset \sum_{\substack{n,\, X,\, Y \\ Y \neq D}} \left(\langle H_0\,,\ H_0^{\perp}\rangle I + \langle I\,,\ H_0^{\perp}\rangle C + \langle P_n^{Y,\, X}\,,\ H_0^{\perp}\rangle P_n^{C,\, Y} + \langle P_n^{D,\, X}\,,\ H_0^{\perp}\rangle P_{m-1}^{C,\, D} \right) \\
&\subset C \oplus (\sum_{n,\, Y \neq D} P_n^{C,\, Y}) \oplus P_{m-1}^{C,\, D} = \Phi.
\end{aligned}
$$

Thus there exists a natural projection $E(C)/\mathbf{J}E(C)$ to $E(C)/\Phi$.

Since $E(C)/\mathbf{J}E(C)$ and $E(C)/\Phi$ are semisimple and $E(C)/\Phi$ contains D-simple comodules as a direct summand, $E(C)/\mathbf{J}E(C)$ also contains D-simple comodules as a direct summand. □

LEMMA 3.8. *Let C, D be simple subcoalgebras such that $P_m^{C,\, D}$ is non-degenerate. If $\dim C \neq \dim D$ or $\dim P_m^{C,\, D} - \dim P_{m-1}^{C,\, D} \neq \dim C$ then there exists a simple subcoalgebra E such that $P_l^{C,\, E}$ is non-degenerate for some $l \geq m+1$.*

Proof. We assume that there is no simple subcoalgebra E such that $P_l^{C,\, E}$, $(l \geq m + 1)$ is non-degenerate. Since the simple coalgebra D satisfies the condition in Lemma 3.7, $Head(E(C))$ contains D-simple comodules as a direct summand. On the other hand $Head(E(C)) = Soc(E(\pi(C))) = \pi(C)$ where π is a permutation on the set of all simple coalgebras of H, since H is coFrobenius. Then $\pi(C) = D$ and thus $\dim C = \dim D$.

Moreover, if $\dim C = \dim D$ and $\dim P_m^{C,\, D} - \dim P_{m-1}^{C,\, D} \neq \dim C$ then the following holds where Φ is the subspace in the proof of Lemma 3.7.

$$
\begin{aligned}
\dim C &< \dim P_m^{C,\, D} - \dim P_{m-1}^{C,\, D} = \dim P_m^{C,\, D}/P_{m-1}^{C,\, D} = \dim E(C)/\Phi \\
&\leq \dim Head(E(C)) = \dim C.
\end{aligned}
$$
□

4. Proof of Theorem 1.1. In this section we will show Theorem 1.1 using Lemmas 3.2, 3.5 and 3.8. Let H be a non-semisimple Hopf algebra of dimension pq where p, q are odd primes such that $p < q < 4p + 10$, $q \leq 29$. We can assume the order of the antipode is $4p$ and $G(H)$ is isomorphic to the cyclic group $\mathbf{C}_p$ by Proposition 2.5, and there exists a simple coalgebra C such that $P_1^{1,\, C} \neq 0$, $\dim C \geq 4$ by Proposition 2.3. Let $\dim C = m^2$, and set $C_1 = C$, $C_2 = gC, \ldots, C_p = g^{p-1}C$ where $g \neq 1_H \in G(H)$. If $(m, p) = 1$ then $C_i \neq C_j$ for $i \neq j$.

LEMMA 4.1. *If $P_1^{1,\, C_1} \neq 0$, $\dim C_1 = 4$ then there exists $k \in \{1, \ldots, p\}$ such that $S(C_k) \notin \{C_1, \ldots, C_p\}$.*

Proof. The proof is by contradiction. Suppose that the antipode S induces a permutation σ on the set $\{C_1, \ldots, C_p\}$. Each C_i is not stable under the antipode S from Proposition 2.4, hence the permutation σ has no fixed points. The order of σ is a divisor of $4p$, so any cycle has length 2 or 4 or p. However, if there exists a cycle with length

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2 or 4 then σ has fixed points since p is an odd prime. Thus σ has a cycle of length p. Then $P_1^{1,\,C_1} \neq 0$ implies that there exist $2p$ disjoint $P_1^{S(C_1),\,1}$, $P_1^{1,\,S^2(C_1)}, \ldots, P_1^{S^{p-1}(C_1),\,1}$ by using Lemma 3.5, and so P_1 contains $2p^2$ disjoint subspaces $P_1^{g^i,\,C_j}$, $P_1^{C_i,\,g^j}$, using Lemma 3.5 again. The dimension of each $P_1^{g^i,\,C_j}$ is greater than or equal to 2, hence $\dim P_1 \geq 4p^2$. Furthermore, by Lemma 3.8, there exist 4 dimensional simple subcoalgebras $\{\bar{C}_i\}$ and grouplike elements $\{g^i\}$ such that $P_n^{C_i,\,\bar{C}_j}$, $P_{n'}^{g^i,\,g^j}$ $(n, n' \geq 2)$ are non-degenerate. Thus $\dim I - \dim P_1 \geq p + 4p$ and so $\dim H \geq (p + 4p) + 4p^2 + (p + 4p) = 4p^2 + 10p$. This contradicts the condition $q < 4p + 10$. $\square$

We denote ${}^X I^Y$ the total sum of the subspaces of I whose existence is induced from the fact $P_1^{X,\,Y} \neq 0$ by repeated application of Lemmas 3.2, 3.5 and 3.8, as in the second half of the proof of Lemma 4.1.

Let C_1 be a simple subcoalgebra such that $P_1^{1,\,C_1} \neq 0$, and m^2 is the dimension of C_1. The fact that $P_1^{1,\,C_1} \neq 0$ implies the existence of no less than $2p$ subspaces $P_1^{g^i,\,C_{i+1}}$, $P_1^{S(C_{i+1}),\,g^{p-i}}$, each of which has the same dimension as $P_1^{1,\,C_1}$ by Lemma 3.5. Moreover, each $P_1^{g^i,\,C_{i+1}} \neq 0$ implies the existence of a grouplike element h_i such that $P_n^{g^i,\,h_i}$ $(n \geq 2)$ is non-degenerate, by applying Lemma 3.8. And each $P_1^{S(C_{i+1}),\,g^{p-i}} \neq 0$ implies the existence of an m^2 dimensional simple subcoalgebra $\bar{C}_i$ such that $P_{n'}^{S(C_i),\,\bar{C}_i}$ $(n' \geq 2)$ is non-degenerate by Lemma 3.8 again.

On the other hand, non-degeneracy of $P_n^{g^i,\,h_i}$ $(n \geq 2)$ implies $P_1^{g^i,\,X}$ and $P_{n-1}^{X,\,h_i}$ are non-degenerate by Lemma 3.2. Since $\dim X \neq 1$, we can assume $X = C_{i+1}$ without loss of generality.

We consider the case $\dim C_1 = 4$. By the result of Lemma 4.1, there exists a positive integer k such that $S(C_k) \notin \{C_1, \ldots, C_p\}$, $P_1^{S(C_k),\,g^{p-k+1}} \neq 0$. Set $g^{k-1}S(C_k) = D_1$, $D_j = g^{j-1}D_1$ for fixed k above. Then $\{C_1, \ldots, C_p\} \cap \{D_1, \ldots, D_p\} = 0$ and $P_1^{D_i,\,g^{i-1}}$, $P_{n-1}^{h_i',\,D_i}$, $P_{n'}^{D_i,\,\bar{D}_i}$ are non-degenerate where $h_i' = g^{i+k-2}h_{k-1}^{-1}$, $\bar{D}_i = g^{i+k-2}\bar{C}_k$. Moreover, $P_{n''}^{C_i,\,\bar{\bar{C}}_i}$ $(n'' \geq 2)$ is non-degenerate for each $i \in \{1, \ldots, p\}$ where $\bar{\bar{C}}_i$ is a simple coalgebra of dimension 4, since $P_{n-1}^{C_{i+1},\,h_i}$ is non-degenerate and using Lemma 3.8. As a result of the discussion above, disjoint non-degenerate subspaces

$$P_1^{g^i,\,C_{i+1}},\ P_1^{D_i,\,g^{i-1}},\ P_{n-1}^{C_{i+1},\,h_i},\ P_{n-1}^{h_i',\,D_i},\ P_n^{g^i,\,h_i},\ P_{n''}^{C_i,\,\bar{\bar{C}}_i},\ P_{n'}^{D_i,\,\bar{D}_i}\ (n, n', n'' \geq 2)$$

exist where g^i, h_i, h_i' are grouplikes, $C_i, D_i, \bar{D}_i, \bar{\bar{C}}_i$ are 4 dimensional simple coalgebras.

Therefore we obtain the following. If $(m, p) = 1$ then $\dim {}^1I^{C_1} \geq 2pm + p + m^2p$, and if $m = kp$ then $\dim {}^1I^{C_1} \geq 2p^2k + p + k^2p^2$. Moreover, $\dim {}^1I^{C_1} \geq (4p \cdot 2) + (p + p \cdot 4 + p \cdot 4) = 17p$ for the case $\dim C_1 = 4$.

We set ${}^X H^Y = \sum_{i,j}(g^iS^j(X) + g^iS^j(Y)) \oplus {}^X I^Y$.

If $(m, p) = 1$ then ${}^1H^{C_1}$ contains disjoint $C_1, \ldots, C_p$. Moreover, if $m = 2$ then ${}^1H^{C_1}$ contains disjoint $C_1, \ldots, C_p, D_1, \ldots, D_p$. Thus we obtain the following.

LEMMA 4.2. *Let C_1 be a simple subcoalgebra such that $P_1^{1,\,C_1} \neq 0$, and m^2 be the dimension of C_1. If $m = 2$ then $\dim {}^1H^{C_1} \geq 26p$, and if $(m, p) = 1$ then $\dim {}^1H^{C_1} \geq 2(m^2 + m + 1)p$, and if $m = kp$ then $\dim {}^1H^{C_1} \geq 2(k^2 + k)p^2 + 2p$.*

COROLLARY 4.3. *The dimension of any simple subcoalgebra of H is 1 or 4 or 9 or p^2. Moreover the following holds for the coradical H_0 of H.*

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If $\dim C_1 = 4$ *then* $H_0 = \mathbf{k}\mathbf{C}_p \oplus C_1 \oplus \cdots \oplus C_p \oplus D_1 \oplus \cdots \oplus D_p$.
If $\dim C_1 = 9$, $(p \neq 3)$ *then* $H_0 = \mathbf{k}\mathbf{C}_p \oplus C_1 \oplus \cdots \oplus C_p$.
If $\dim C_1 = p^2$ *then* H_0 *is isomorphic to one of the following,*
(*i*) $\mathbf{k}\mathbf{C}_p \oplus C_1$ (*ii*)$\mathbf{k}\mathbf{C}_p \oplus C_1 \oplus \bar{C}$ (*iii*) $\mathbf{k}\mathbf{C}_p \oplus C_1 \oplus \bar{C} \oplus \tilde{C}$
(*iv*) $\mathbf{k}\mathbf{C}_p \oplus C_1 \oplus B_1 \oplus \cdots \oplus B_p$ (*v*) $\mathbf{k}\mathbf{C}_p \oplus C_1 \oplus \bar{C} \oplus B_1 \oplus \cdots \oplus B_p$,
where $\bar{C}$, $\tilde{C}$ *are simple coalgebras of dimension* p^2, $B_1, \ldots, B_p$ *are simple coalgebras of dimension* 4.

Proof. It is obvious by counting dimensions using Lemma 4.2. □

LEMMA 4.4. *The space* $P_n^{g,h}$ *degenerates to* $P_2^{g,h}$ *for* $n \geq 3$ *where* $g, h \in G(H)$.

Proof. The proof of Lemma 4.4 is by contradiction.

Suppose $P_n^{g,h}$ is non-degenerate for $n \geq 3$. From Lemma 3.2, there exist simple coalgebras X_1, X_2 such that P_1^{g,X_1}, $P_{n-1}^{X_1,h}$, P_{n-1}^{g,X_2}, $P_1^{X_2,h}$ are non-degenerate. Since the space $P_{n-1}^{X_1,h}$ is non-degenerate, applying Lemma 3.2 again, there exists a simple coalgebra X_3 such that $P_{n-2}^{X_1,X_3}$, $P_1^{X_3,h}$ are non-degenerate. Since H has no non-trivial skew primitive elements, dimensions of X_1, X_2 and X_3 are greater than or equal to 4. Then $\dim X_1 = \dim X_2 = \dim X_3$ by the counting dimensions with Corollary 4.3.

(i) The case $\dim X_1 = 4$ or 9. Since $P_{n-2}^{X_1,X_3}$ is non-degenerate, ${}^gH^{X_1}$ contains disjoint p subspaces $\{P_{n-2}^{\bar{g}^iX_1,\bar{g}^iX_3}\}$ which have not appeared in the counting argument of the proof of Lemma 4.2 where $\bar{g}$ is a non-trivial grouplike element of H. Therefore $\dim H \geq 30p$, contradicting $q \leq 29$.

(ii) The case $\dim X_1 = p^2$. Since P_1^{g,X_1}, $P_{n-1}^{X_1,h}$, P_{n-1}^{g,X_2}, $P_1^{X_2,h}$, $P_{n-2}^{X_1,X_3}$ are non-degenerate, ${}^gH^{X_1}$ contains disjoint $4p+1$ subspaces

$$\{P_1^{\bar{g}^ig,\bar{g}^iX_1}\},\ \{P_{n-1}^{\bar{g}^iX_1,\bar{g}^ih}\},\ \{P_{n-1}^{\bar{g}^ig,\bar{g}^iX_2}\},\ \{P_1^{\bar{g}^iX_2,\bar{g}^ih}\},\ P_{n-2}^{X_1,X_3}$$

where $\bar{g}$ is a non-trivial grouplike element of H. Therefore $\dim H \geq 7p^2 + 2p$, contradicting $q < 4p + 10$. □

Let C_1 be a simple coalgebra such that $P_1^{1,C_1} \neq 0$. We suppose that $\dim C_1 = 4$ or $\dim C_1 = 9$, $p \neq 3$. Then Corollary 4.3 and Lemma 4.4 imply that there is no subspace which have not appeared in the proof of Lemma 4.2. Thus $H = {}^1H^{C_1}$ and so the dimension of ${}^1H^{C_1}$ can be expressed as $26p + 4z_1 + 6z_2 + 9z_3$ with positive integers z_i, by Lemma 4.4. This contradicts the condition $q \leq 29$. Therefore $\dim C_1 = p^2$. And thus the dimension of H can be expressed as $4p^2 + 2p + p^2z_1 + 4pz_2$ with positive integers z_1, z_2.

This completes the proof of Theorem 1.1.

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