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REGULAR SUBGROUPS OF THE AFFINE GROUP AND RADICAL CIRCLE ALGEBRAS

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Abstract

We establish a link between regular subgroups of the affine group and radical circle algebras on the underlying vector space.

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1. Introduction

Caranti *et al.* [1] have obtained a simple description of abelian regular subgroups of the affine group in terms of commutative associative radical algebras. Hegedüs [3] has produced examples of nonabelian regular subgroups of the affine group $AGL(n, p)$ for $p > 2$ and $n > 3$ or $p = 2$ and $n \geq 3$, n odd.

The main purpose of this note is a description of the regular subgroups of the affine group in terms of radical circle algebras. We say that a vector space V over a field F with an operation $\cdot$ is a *circle algebra* if, for all $\lambda \in F$ and $x, y, z \in V$, the following conditions hold:

(1) $(x \cdot y)\lambda = (x\lambda) \cdot y$;
(2) $(x + y) \cdot z = x \cdot z + y \cdot z$;
(3) $x \cdot (y + z + y \cdot z) = x \cdot y + x \cdot z + (x \cdot y) \cdot z$.

Set, for all $x, y \in V$, $x \circ y = x + y + x \cdot y$. The structure $(V, \circ)$ is a semigroup. In particular, if $(V, \circ)$ is a group, then we say that the circle algebra V is *radical*.

It is clear that any associative algebra is a circle algebra and that there are circle algebras which are not associative. Moreover, it is easy to see that a commutative circle algebra is an associative algebra.

Our notation is mostly standard, and for basic results we refer to [2, 5].

2. The correspondence theorem

The main theorem establishes a link between regular subgroups of the affine group and circle algebra structures on the underlying vector space and depends on the techniques developed by Caranti *et al.* in [1].

THEOREM 1. *Let V be a vector space over a field F. Denote by $\mathcal{RC}$ the class of radical circle algebras with underling vector space V and by $\mathcal{T}$ the set of all regular subgroups of the affine group $AGL(V)$.*

(1) *If $V^\bullet$ is a radical circle algebra with underlying vector space V, then $T(V^\bullet) = \{\tau_x \mid x \in V\}$, where $\tau_x : V \longrightarrow V$, $y \mapsto y \circ x$, is a regular subgroup of the affine group $AGL(V)$.*

(2) *The map*

$$f : \mathcal{RC} \longrightarrow \mathcal{T}, \quad V^\bullet \mapsto T(V^\bullet)$$

is a bijection.

In this correspondence, isomorphism classes of circle algebras correspond to conjugacy classes under the action of $GL(V)$ of regular subgroups of $AGL(V)$.

PROOF. (1) First we note that, for all $x \in V$, the map

$$\gamma_x : V \longrightarrow V, \quad y \mapsto y + y \cdot x$$

belongs to $\mathrm{Sym}(V)$. In fact, if $y, z \in V$ are such that $y\gamma_x = z\gamma_x$, then $y \circ x = y\gamma_x + x = z\gamma_x + x = z \circ x$. Since V is radical, we have $y = z$. Moreover, if x^- is the inverse of x with respect to $\circ$, then

$$((y + x) \circ x^-)\gamma_x = (y + x) \circ x^- + ((y + x) \circ x^-) \cdot x = (y + x) \circ x^- \circ x - x = y$$

for every $y \in V$.

Now, since $V^\bullet$ is a circle algebra, the map γ_x belongs to $GL(V)$. Indeed

$$(y + z)\gamma_x = y + z + (y + z) \cdot x = x + z + y \cdot x + z \cdot x = y\gamma_x + z\gamma_x,$$
$$(y\lambda)\gamma_x = y\lambda + (y\lambda) \cdot x = y\lambda + (y \cdot x)\lambda = (y\gamma_x)\lambda,$$

for all $x, y, z \in V$ and $\lambda \in F$. So $\tau_x = \gamma_x t_x \in AGL(V)$, where t_x is the translation by x. Finally, the map

$$\tau : V \longrightarrow AGL(V), \quad x \mapsto \tau_x$$

is a monomorphism of the circle group of $V^\bullet$ into $AGL(V)$ and $V\tau = T(V^\bullet)$. Hence, $T(V^\bullet)$ is a regular subgroup of $AGL(V)$.

(2) Let T be a regular subgroup of $AGL(V)$. For each $x \in V$ there is a unique $\tau_x \in T$ such that $0\tau_x = x$. Thus $T = \{\tau_x \mid x \in V\}$. We remark that, for each $x \in V$, there is a unique $\gamma_x \in GL(V)$ such that $\tau_x = \gamma_x t_x$, where t_x is the translation by x. It follows

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that

$$\begin{aligned}(x\lambda)\tau_y &= (x\tau_y)\lambda - y\lambda + y,\\ (x+y)\tau_z &= x\tau_z + y\tau_z - z,\\ (-x)\tau_z &= -(x\tau_z) + z + z,\\ \tau_y\tau_z &= \tau_{y\tau_z},\end{aligned}$$

for all $\lambda \in F$ and $x, y, z \in V$.

Now, we introduce on V the following operation

$$\forall x, y \in V, \quad x \cdot y = x\tau_y - x - y.$$

The vector space V equipped with this operation is a circle algebra. In fact, for any $x, y, z \in V$ and for any $\lambda \in F$,

$$\begin{aligned}(x\lambda)\cdot y &= (x\lambda)\tau_y - x\lambda - y\\ &= (x\tau_y)\lambda - y\lambda + y - x\lambda - y\\ &= (x\cdot y)\lambda.\end{aligned}$$

Moreover,

$$\begin{aligned}(x+y)\cdot z &= (x+y)\tau_z - (x+y) - z\\ &= x\tau_z + y\tau_z - z - (x+y) - z\\ &= x\cdot z + y\cdot z.\end{aligned}$$

Finally,

$$\begin{aligned}x\cdot y + x\cdot z + (x\cdot y)\cdot z &= x\tau_y - x - y + x\tau_z - x - z + (x\tau_y - x - y)\cdot z\\ &= x\tau_z - x - z + (x\tau_y - x - y)\tau_z - z\\ &= x\tau_z - x - z + x\tau_y\tau_z + (-(x+y))\tau_z - z - z\\ &= x\tau_z - x - z + x\tau_y\tau_z - (x+y)\tau_z\\ &= x\tau_y\tau_z - x - y\tau_z = x\tau_{y\tau_z} - x - y\tau_z = x\cdot y\tau_z\\ &= x\cdot(y + z + y\cdot z).\end{aligned}$$

It follows that $T(V^\bullet) = T$ and that f is onto. On the other hand, it is clear that f is one-to-one, so f is a bijection.

Now, suppose that $V^\bullet$ and V^* are two radical circle algebras with underlying vector space V and suppose that there is an isomorphism φ of $V^\bullet$ onto V^*. In particular, $\varphi \in GL(V)$. Let $T(V^\bullet) = \{\tau_x^{(1)} \mid x \in V\}$ and $T(V^*) = \{\tau_x^{(2)} \mid x \in V\}$ be the corresponding regular subgroups of $AGL(V)$. It is easy see that

$$\forall x \in V, \quad \varphi^{-1}\tau_x^{(1)}\varphi = \tau_{x\varphi}^{(2)}.$$

So that $T(V^*) = \varphi^{-1}T(V^\bullet)\varphi$.

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Conversely, suppose that T_1 and T_2 are two regular subgroups of $AGL(V)$ such that $T_2 = \varphi^{-1} T_1 \varphi$ for some $\varphi \in GL(V)$. Let $V^\bullet$ and V^* be the radical circle algebras such that $T_1 = T(V^\bullet) = \{\tau_x^{(1)} \mid x \in V\}$ and $T_2 = T(V^*) = \{\tau_x^{(2)} \mid x \in V\}$. Let $\psi : V \longrightarrow V$ be the bijection such that

$$\forall x \in V, \quad \varphi^{-1} \tau_x^{(1)} \varphi = \tau_{x\psi}^{(2)}.$$

For all $x \in V$, we have $x\psi = 0(\tau_{x\psi}^{(2)}) = 0(\varphi^{-1}\tau_x^{(1)}\varphi) = x\varphi$, so that $\psi = \varphi$. Now, for all $x, y \in V$,

$$\tau^{(2)}_{x\tau_y^{(2)}} = \tau_x^{(2)}\tau_y^{(2)} = \varphi^{-1}\tau_x^{(1)}\tau_y^{(1)}\varphi = \varphi^{-1}\tau^{(1)}_{x\tau_y^{(1)}}\varphi = \tau^{(2)}_{(x\tau_y^{(1)})\varphi}.$$

Then φ is an isomorphism of $V^\bullet$ onto V^*. □

It is easy to see that the following result holds.

COROLLARY 2. *Let $V^\bullet$ be a radical circle algebra with underlying vector space V over a field F. Let $Tr(V)$ be the group of translations and let $T(V^\bullet) = \{\tau_x \mid x \in V\}$, where $\tau_x : V \longrightarrow V$, $y \mapsto y \circ x$. Then*

$$Tr(V) \cap T(V^\bullet) = \{\tau_x \mid x \in Ann_L(V^\bullet)\}$$

where $Ann_L(V^\bullet)$ is the set of the left annihilators of the circle algebra $V^\bullet$.

In particular, if $V^\bullet$ is a radical associative algebra of finite dimension, then $Tr(V) \cap T(V^\bullet) \neq 1$.

We remark that the regular subgroups given by Hegedüs [3] correspond to the circle algebras produced in the following result.

PROPOSITION 3. *Let p be a prime and let n be an integer. If $p > 2$ and $n > 3$ or $p = 2$ and $n \geq 3$, n odd, there exists a noncommutative radical circle algebra V of dimension n over the field $\mathbb{F}_p$.*

PROOF. Let W be a vector space of dimension $n-1$ over the field $\mathbb{F}_p$ and let Q be a nondegenerate quadratic form of W. Then there exists an element f of the orthogonal group associated with W and Q of order p.

Now, let $V = W \oplus \mathbb{F}_p$ and, if $v \in V$, let $v_r \in W$ and $v_s \in \mathbb{F}_p$ such that $v = v_r + v_s$. We claim that the vector space V is a noncommutative radical circle algebra by setting, for all $x, y \in V$,

$$x \cdot y = (x_r f^{\alpha} - x_r) + (x_r, y_r f^{-\alpha})b,$$

where $\alpha = y_s - y_r Q$ and b is the bilinear form associated with Q. In fact, if $\lambda \in F$ and $x, y, z \in V$, then $(x \cdot y)\lambda = (x\lambda) \cdot y$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. Moreover, if $\alpha = y_s - y_r Q$ and $\beta = z_s - z_r Q$, it is easy to prove that

$$\alpha + \beta = y_s + z_s + (y_r, z_r f^{-\beta})b - (z_r + y_r f^{\beta})Q.$$

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from which it follows that

$$
\begin{aligned}
x \cdot (y + z + y \cdot z) &= (x_r f^{\alpha+\beta} - x_r) + (x_r, (z_r + y_r f^{\beta}) f^{-(\alpha+\beta)})b \\
&= (x_r f^{\alpha+\beta} - x_r) + (x_r f^{\alpha}, z_r f^{-\beta})b + (x_r, y_r f^{-\alpha})b \\
&= x \cdot y + x \cdot z + (x \cdot y) \cdot z.
\end{aligned}
$$

Finally, if $\gamma = x_s - x_r Q$ and $x^- = -x_r f^{-\gamma} - x_s + (x_r, x_r)b$, then

$$x^- \circ x = (x_r, x_r)b + (-x_r f^{-\gamma}, x_r f^{-\gamma})b = 0,$$

and the claim follows. □

Liebeck *et al.* [4] have stated that the regular subgroups given by Hegedüs are examples of transitive subgroups that contain no nontrivial normal subgroup of the socle. A natural question arising from the context is whether other examples could come out from our characterization, and this will be a subject of future work.

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