Problem Corner

Solutions are invited to the following problems. They should be addressed to Chris Starr, c/o Bill Richardson, Kintail, Longmorn, Elgin IV30 8RJ (e-mail: czqstarr@gmail.com) and should arrive not later than 10 December 2024.

Proposals for problems are equally welcome. They should also be sent to Chris Starr at the above address and should be accompanied by solutions and any relevant background information.

108.E (Ovidiu Gabriel Dinu)

Find all positive integers such that each of n, $n + 2$, $n + 6$, $n + 8$ and $n + 14$ is a prime number.

108.F (Peter Shiu)

Let $z = x + iy$ and $w = u + iv$ be complex numbers satisfying $z^2 + w^2 = r^2$, where $r > 0$. Show that if (x, y) runs over an ellipse with foci $\pm r$, then (u, v) runs over the same ellipse.

108.G (Sean M. Stewart)

Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$ where n is a positive integer. Evaluate:

$$\lim_{n \to \infty} n \left(\prod_{k=1}^{n} I_k \right)^{\frac{2}{n}}.$$

108.H (Mark Hennings)

Prove the following:

(a) $$\int_0^{\pi} \frac{x(\pi - x)}{\sin x} dx = 7\zeta(3),$$

(b) $$\int_{-\infty}^{\infty} \tan^{-1}(e^x) \tan^{-1}(e^{-x}) \, dx = \frac{7}{4}\zeta(3).$$

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Solutions and comments on **107.I**, **107.J**, **107.K**, **107.L** (November 2023).

107.I (Kieren MacMillan)

For n a positive integer, let Q_n be the sum of the nth powers of the first x even squares. For example, $Q_1(x) = \frac{2}{3}\left(2x^3 + 3x^2 + x\right)$. Find the points which all the graphs $y = Q_n(x)$ have in common.

Answer: The common points are $(-1,\ 0)$, $\left(-\frac{1}{2}, 0\right)$, $(0, 0)$, $\left(-\frac{3}{2}, -1\right)$ and $\left(\frac{1}{2},\ 1\right)$.

The series sums $Q_1(x)$ and $Q_2(x)$ are $\frac{1}{6}x(x + 1)(2x + 1)$ and $\frac{1}{30}x(x + 1)(2x + 1)\left(3x^2 + 3x - 1\right)$ respectively, suggesting the first three coordinates, and the other two points can be found by letting $Q_1(x) = Q_2(x)$. The trick is then to find a method to show that this must be true for all $Q_n(x)$. Most solvers tackled this problem by first setting up a telescoping series that found $Q_n(x)$ in terms of only even powers of x, then using an inductive argument to find all common points. The following argument is from Stan Dolan:

For any positive integer n,

$$\frac{(x+1)^{2n+1} - (x-1)^{2n+1}}{2} = \binom{2n+1}{1}x^{2n} + \binom{2n+1}{3}x^{2n-2} + \ \dots\ + 1.$$

Multiplying by 2^{2n} and summing for $x = 1, \dots, a$, gives

$$2^{2n-1}\left((a + 1)^{2n+1} + a^{2n+1} - (2a + 1)\right)$$
$$= \binom{2n + 1}{1}Q_n(a) + \binom{2n + 1}{3}Q_{n-1}(a) + \ \dots\ .$$

Then, by induction, Q_n is a polynomial of degree $2n + 1$ with no constant term. Also, by induction, Q_n is an odd function of $t = a + \frac{1}{2}$.

We now know that the graph of each Q_n passes through $(-1,\ 0)$, $\left(-\frac{1}{2},\ 0\right)$ and $(0,0)$

Substituting $a = -\frac{1}{2}$ into the equation $Q_n(a+1) - Q_n(a) = (2a+2)^{2n}$ gives $Q_n\left(\frac{1}{2}\right) = 1$. Therefore the graph of each Q_n also passes through $\left(-\frac{3}{2}, -1\right)$ and $\left(\frac{1}{2}, 1\right)$.

Finally note that $Q_2(x) - Q_1(x)$ is a polynomial of degree 5 and therefore we have found all possible common points.

Mark.Hennings proved that $Q_{n+1}(x) - Q_n(x)$ must be divisible by $(2x - 1)2x(2x + 1)(2x + 2)(2x + 3)$ for $n \geqslant 1$, and Nick Lord observed that in the case where the odd powers are summed, the points $(0,\ 0)$, $(-1,\ 0)$, $(1,\ 1)$ and $(-2,\ 1)$ are common.

Correct solutions were received from: M.G. Elliott, C. Starr, S.Dolan, M. Hennings, N. Curwen, J.A. Mundie, and the proposer Kieren MacMillan.

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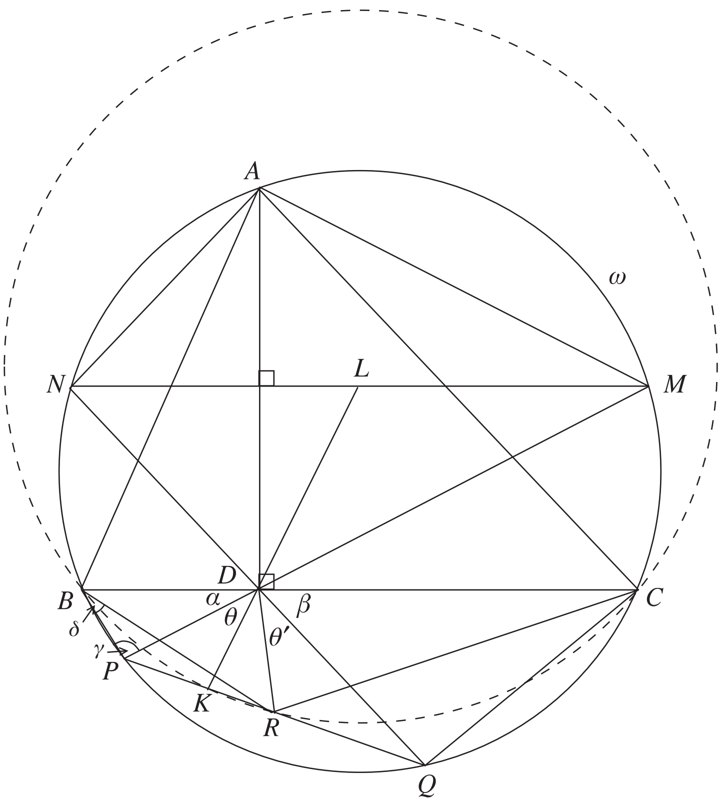
107.J (Tran Quang Hung)

Let ω denote the circumcircle of the acute-angled triangle ABC. Let D be the foot of the altitude from A to BC. The perpendicular bisector of AD meets ω at M and N. Lines MD and ND meet ω again at P and Q respectively. Let R be the midpoint of PQ and let the circumcircle of triangle RBC meet PQ again at K. Prove that the line KD bisects the segment MN.

Solution: A wide range of techniques was used by solvers in their approach to this problem. It was possible to approach this problem through the use of coordinate geometry or complex numbers, but solvers who used a purely geometrical approach tended to arrive at the solution in fewer steps. This solution based on that by Stan Dolan is striking in its brevity.



Using basic angle rules we have $\angle BDP = \angle MDC = \angle NMP = \alpha$, and then using the "same segment rule", $\angle NQP = \alpha$. In a similar manner we also find that $\angle DPQ = \beta$, $\angle DQC = \gamma$ and $\angle RCQ = \delta$. Without loss of generality we let $DP = 1$. Use the sine rule on triangles DPB, DPK and BPK to obtain:

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$$BP = \frac{\sin\alpha}{\sin(180^\circ - \alpha - \gamma)}, \; PK = \frac{\sin\theta}{\sin(180^\circ - \theta - \beta)},$$

$$\frac{BP}{\sin(180^\circ - \beta - \gamma - \delta)} = \frac{PK}{\sin\delta}.$$

Eliminating BP and PK from this system of equations results in the following:

$$\sin(\beta + \gamma + \delta)\sin\theta\sin(a + \gamma) = \sin\alpha\sin\delta\sin(\beta + \theta)$$

$$\sin(\alpha + \gamma)\sin(\beta + \gamma)\cot\delta + \sin(\alpha + \gamma)\cos(\beta + \gamma)$$

$$= \sin\alpha\sin\beta\cot\theta + \sin\alpha\sin\beta.$$

Using the sum-to-product trigonometry identities, this may be rewritten as

$$\sin(\alpha + \gamma)\sin(\beta + \gamma)\cot\delta + \tfrac{1}{2}\sin(\alpha + b + 2\gamma) = \sin\alpha\sin\beta\cot\theta + \tfrac{1}{2}\sin(\alpha + b).$$

Since $\cot\theta$ is given by a formula which is unchanged when α and β are interchanged, this same formula gives θ' and therefore $\theta = \theta'$. Hence, in the case that R is the midpoint of PQ, the line KD will bisect MN.

Correct solutions were received from: M. G. Elliott, S. Dolan, M. Hennings, J. A. Mundie, V. Scindler and the proposer Tran Quang Hung.

107.K (Didier Pinchon and George Stoica)

Let a be a non-zero real number and suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + f(y)) = f(x) + f(y) + ay$ for all $x, y \in \mathbb{R}$. Prove that f is additive, i.e. $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Solution:

Solvers used a variety of techniques to tackle this problem. The following, based on that offered by the proposers caught my eye:

We prove that f is additive by making suitable substitutions into

$$f(x + f(y)) = f(x) + f(y) + ay.$$

(1) Using the substitution $x \to -\frac{x}{a} - f\left(-\frac{x}{a}\right)$, $y \to -\frac{x}{a}$. we have

$$f\left(-\frac{x}{a} - f\left(-\frac{x}{a}\right) + f\left(-\frac{x}{a}\right)\right) = f\left(-\frac{x}{a} - f\left(-\frac{x}{a}\right)\right) + f\left(-\frac{x}{a}\right) + a\left(-\frac{x}{a}\right).$$

After simplifying the left-hand side, and collecting like terms we obtain:

$$f\left(-\frac{x}{a} - f\left(-\frac{x}{a}\right)\right) = x.$$

This proves that f is surjective. Furthermore, if we let $x = 0$, we get $f(-f(0)) = 0$.

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(2) Substituting $x = 0, y = -f(0)$ we get:

$$f\left(0 + f\left(-f(0)\right)\right) = f(0) + f\left(-f(0)\right) + a\left(-f(0)\right).$$

Using the fact that $f\left(-f(0)\right) = 0$ we obtain $f(0) = f(0) - af(0)$. Since $a \neq 0$, this implies that $f(0) = 0$.

(3) Substituting $x = 0$ gives $f\left(f(y)\right) = f(0) + f(y) + ay = f(y) + ay$. But then we can now rewrite the original relationship as $f\left(x + f(y)\right) = f(x) + f\left(f(y)\right)$. Since f is surjective, we therefore have $f(x + z) = f(x) + f(z)$ for all real x, z.

An example of a suitable function over the real numbers is $f(x) = cx$, and substitution into the functional relation gives the further requirement that $a = c^2 - c \neq 0$. It would be interesting to see if there are any further suitable functions.

Correct solutions were received from: P. F. Johnson, M. G. Elliott, S. Dolan, M. Hennings and the proposers Didier Pinchon and George Stoica.

107.L (Toyesh Prakash Sharma)

Prove the inequality:

$$\int_0^{\pi/2} \int_0^{\pi/2} \frac{x\sqrt{y}\,\sin^2 x\,\sin y}{\sqrt{x}\,\sin x + \sqrt{y}\,\sin y}\,dx\,dy < \frac{\pi^3}{64}.$$

Solution:

James Mundie showed by an elaborate argument that the surface described by $f(x, y) = \dfrac{x\sqrt{y}\,\sin^2 x\,\sin y}{\sqrt{x}\,\sin x + \sqrt{y}\,\sin y}$ can be dissected and reconstructed in such a way that it can be contained within a cube of side length $\frac{\pi}{4}$, but most solutions followed a similar line to the following, from the proposer:

Firstly, by symmetry:

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \frac{x\sqrt{y}\sin^2 x \sin y}{\sqrt{x}\sin x + \sqrt{y}\sin y} dx\,dy \equiv \int_0^{\pi/2} \int_0^{\pi/2} \frac{y\sqrt{x}\sin^2 y \cos x}{\sqrt{x}\sin x + \sqrt{y}\sin y} dx\,dy$$

$$\therefore 2I = \int_0^{\pi/2} \int_0^{\pi/2} \frac{x\sqrt{y}\sin^2 x \cos y + y\sqrt{x}\sin^2 y \cos x}{\sqrt{x}\sin x + \sqrt{y}\sin y} dx\,dy$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \frac{\sqrt{x}\sin x\sqrt{y}\sin y\left(\sqrt{x}\sin x + \sqrt{y}\sin y\right)}{\sqrt{x}\sin x + \sqrt{y}\sin y} dx\,dy$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{x}\sin x\sqrt{y}\sin y\,dx\,dy.$$

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Since the limits are constant,

$$\begin{aligned} 2I &= \int_0^{\pi/2} \sqrt{x}\, \sin x\, dx \int_0^{\pi/2} \sqrt{y}\, \sin y\, dy \\ &\equiv \left(\int_0^{\pi/2} \sqrt{x}\, \sin x\, dx\right)^2 \\ &\leqslant \int_0^{\pi/2} x\, dx \int_0^{\pi/2} \sin^2 x\, dx \end{aligned}$$

using the Cauchy-Scharz inequality. Using standard integrals we obtain

$$2I \leqslant \frac{\pi^2}{8} \times \frac{\pi}{4}.$$

Therefore

$$I \leqslant \frac{\pi^3}{64}.$$

Solvers used a variety of estimation techniques to establish the strict inequality. The integral $\int_0^{\pi/2} \sqrt{x}\, \sin x\, dx$ can be transformed by integrating by parts, then using the substitution $x = \frac{1}{2}\pi u^2$ to obtain $I = \frac{1}{4}\pi C(1)^2$, where $C(t) = \int_0^t \cos\left(\frac{1}{2}\pi u^2\right) du$ is the Fresnel cosine integral. This gives $I = 0.478$ to three decimal places, which is close to the bound, 0.484.

Correct solutions were received from: P. F. Johnson, J. D. Mahoney, M. G. Elliott, C. Starr, Z. Retkes, S. Dolan, M. Hennings, N. Curwen, J. A. Mundie and the proposer Toyesh Prakash Sharma.

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