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Chern Classes of Splayed Intersections

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Abstract. We generalize the Chern class relation for the transversal intersection of two nonsingular varieties to a relation for possibly singular varieties, under a splayedness assumption. We show that the relation for the Chern–Schwartz–MacPherson classes holds for two splayed hypersurfaces in a nonsingular variety, and under a strong splayedness assumption for more general subschemes. Moreover, the relation is shown to hold for the Chern–Fulton classes of any two splayed subschemes. The main tool is a formula for Segre classes of splayed subschemes. We also discuss the Chern class relation under the assumption that one of the varieties is a general very ample divisor.

1 Introduction

Let X, Y be nonsingular subvarieties of a nonsingular complex variety V. If X and Y intersect properly and transversally, then the intersection $X \cap Y$ is nonsingular, and an elementary Chern class computation proves that

$$c(X) \cdot c(Y) = c(TV) \cap c(X \cap Y), \tag{1.1}$$

where $c(X)$ denotes the push-forward to V of the total (homology) Chern class of the tangent bundle of X, etc., and $\cdot$ is the intersection product in V. It is natural to ask whether (1.1) holds if X, Y, $X \cap Y$ are allowed to be singular. In [AF13, §3], we proposed the following generalization of (1.1).

Scholium *Let X, Y be (possibly singular) subvarieties of a nonsingular variety V. Assume that X and Y are* splayed. *Then*

$$c_{\mathrm{SM}}(X) \cdot c_{\mathrm{SM}}(Y) = c(TV) \cap c_{\mathrm{SM}}(X \cap Y). \tag{1.2}$$

In (1.2), $c_{\mathrm{SM}}(-)$ denotes the *Chern–Schwartz–MacPherson* class; this is a natural generalization of the total Chern class to singular varieties, and we silently push this class forward to the ambient variety V. The c_{SM} class is defined for more general schemes; X and Y could be reducible, and should not be required to be pure dimensional. The purpose of this note is to investigate (1.2) at this level of generality. For example, we will prove that the Scholium holds for arbitrary splayed hypersurfaces, and more generally for subschemes satisfying a hypothesis of "strong" splayedness. We

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also prove (1.2) for splayed subschemes for a different notion of Chern class defined for arbitrary subschemes of a nonsingular variety. Finally, we will discuss a "Bertini" statement, according to which (1.2) holds if X is a sufficiently general very ample divisor.

The notion of *splayedness* was introduced and studied in the hypersurface case by the second author in [Fab13], and explored further in [AF13]: X and Y are splayed if at each point p of the intersection there exist analytic coordinates $(x_1, \dots, x_r, y_1, \dots, y_s)$ such that X may be defined by an ideal generated by functions in the coordinates x_i and Y by an ideal generated by functions in the coordinates y_j. We also say that two sets $\{X_1, X_2, \dots\}$ and $\{Y_1, Y_2, \dots\}$ are splayed if there are local analytic splittings so that all the X_i are defined in the first set of coordinates, and all the Y_j are defined in the second set of coordinates. These notions generalize to possibly singular varieties and subschemes the notion of proper, transversal intersection of nonsingular varieties.

The reader who is not familiar with characteristic classes may view (1.2) as a very general form of identities involving the topological Euler characteristics of X, Y, $X \cap Y$. For example, let X and Y be splayed surfaces in $\mathbb{P}^3$, of degrees d and e, respectively; assume that the Euler characteristic of a general hyperplane section of X (resp. Y) is a (resp. b). Then it may be checked that the Euler characteristic of the curve $X \cap Y$ is $ea + db - 2de$. Similarly explicit formulas relate the Euler characteristics of general linear sections of X, Y, $X \cap Y$ if these are subsets of projective space and X, Y are splayed (*cf.* [Alu13]). The Scholium reveals the underlying structure of all such identities and generalizes them to splayed subsets in arbitrary nonsingular algebraic varieties.

Note that some transversality hypothesis is certainly needed for (1.2) to hold, as the following example shows.

Example 1.1 Let X be a nonsingular quadric in $V = \mathbb{P}^3$, and let Y be a plane tangent to X. Then $c_{\mathrm{SM}}(X) = 2[\mathbb{P}^2] + 4[\mathbb{P}^1] + 4[\mathbb{P}^0]$ and $c_{\mathrm{SM}}(Y) = [\mathbb{P}^2] + 3[\mathbb{P}^1] + 3[\mathbb{P}^0]$ (since X and Y are nonsingular, these are simply the push-forward to $\mathbb{P}^3$ of the total Chern classes of their tangent bundles). Thus, the left-hand side of (1.2) is

$$\big(2[\mathbb{P}^2] + 4[\mathbb{P}^1] + 4[\mathbb{P}^0]\big) \cdot \big([\mathbb{P}^2] + 3[\mathbb{P}^1] + 3[\mathbb{P}^0]\big) = 2[\mathbb{P}^1] + 10[\mathbb{P}^0].$$

On the other hand, denoting by H the hyperplane class, $c(T\mathbb{P}^3) = 1 + 4H + 6H^2 + 4H^3$; and $X \cap Y$ consists of two lines meeting at a point, a curve of degree 2 and topological Euler characteristic 3, and hence $c_{\mathrm{SM}}(X \cap Y) = 2[\mathbb{P}^1] + 3[\mathbb{P}^0]$. Thus the right-hand side of (1.2) equals

$$c(TV) \cap c_{\mathrm{SM}}(X \cap Y) = 2[\mathbb{P}^1] + 11[\mathbb{P}^0],$$

verifying that (1.2) does not hold in this case.

Several particular cases of the Scholium are proved in [AF13]. In this paper we prove (1.2) under a hypothesis generalizing all those particular cases, but possibly more restrictive than splayedness. We say that X and Y are *strongly splayed* if $X = D_1' \cap \dots \cap D_r'$, $Y = D_1'' \cap \dots \cap D_s''$ where $\{D_1', \dots, D_r'\}$, $\{D_1'', \dots, D_s''\}$ are splayed sets of hypersurfaces. For example, two hypersurfaces are strongly splayed if and only if they are splayed. We do not know if splayed subschemes of higher codimension are necessarily strongly splayed, and this seems an interesting question.

Theorem I *Let X, Y be strongly splayed subschemes of a nonsingular variety V. Then*

$$c_{\mathrm{SM}}(X)\cdot c_{\mathrm{SM}}(Y) = c(TV)\cap c_{\mathrm{SM}}(X\cap Y). \tag{1.3}$$

Example 1.2 Let X be the union of a $\mathbb{P}^4$ and a transversal $\mathbb{P}^3$ in $V = \mathbb{P}^5$; we choose coordinates $(x_0:\cdots:x_5)$ so that X has ideal $(x_0(x_1,x_2))$. Let Y be the quadric cone with ideal $(x_3^2+x_4^2+x_5^2)$. Both X and Y are singular, and X is reducible and not pure-dimensional; X and Y are strongly splayed.

The Macaulay2 code from [Alu03] may be used to compute the c_{SM} classes of X and Y:

$$c_{\mathrm{SM}}(X) = [\mathbb{P}^4]+6[\mathbb{P}^3]+13[\mathbb{P}^2]+13[\mathbb{P}^1]+6[\mathbb{P}^0]$$
$$c_{\mathrm{SM}}(Y) = 2[\mathbb{P}^4]+8[\mathbb{P}^3]+13[\mathbb{P}^2]+11[\mathbb{P}^1]+5[\mathbb{P}^0].$$

According to Theorem I,

$$c(T\mathbb{P}^5)\cap c_{\mathrm{SM}}(X\cap Y) = c_{\mathrm{SM}}(X)\cdot c_{\mathrm{SM}}(Y) = 2[\mathbb{P}^3]+20[\mathbb{P}^2]+87[\mathbb{P}^1]+219[\mathbb{P}^0]$$

from which

$$c_{\mathrm{SM}}(X\cap Y) = 2[\mathbb{P}^3]+8[\mathbb{P}^2]+9[\mathbb{P}^1]+5[\mathbb{P}^0].$$

This can be verified by again using [Alu03].

In fact, the proper level of generality for the result is that of *constructible functions:* a c_{SM} class in A_*V is defined for every constructible function on V; if X is a subvariety of V, $c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(1_X)$, where 1_X is the indicator function of X. Intersection of varieties corresponds naturally to the product of the corresponding constructible functions.

Theorem II *Let ϕ, ψ be constructible functions on a nonsingular variety V, and assume that ϕ and ψ are strongly splayed. Then*

$$c_{\mathrm{SM}}(\phi)\cdot c_{\mathrm{SM}}(\psi) = c(TV)\cap c_{\mathrm{SM}}(\phi\cdot\psi). \tag{1.4}$$

The precise definition of strongly splayed in the context of constructible functions is given in Definition 2.19; it generalizes naturally the notion for subvarieties. Note that (1.4) amounts to the statement that the assignment

$$\phi\mapsto c(TV)^{-1}\cap c_{\mathrm{SM}}(\phi)$$

of a class in A_*V from a constructible function ϕ "preserves multiplication" for strongly splayed constructible functions. Again, this is clearly false without some kind of transversality condition on the constructible functions. It would be interesting to determine weaker conditions than strong splayedness guaranteeing that this multiplicativity property holds.

Our proofs of Theorems I and II rely on intersection-theoretic considerations based on a formula for the Chern–Schwartz–MacPherson class of a hypersurface from [Alu99], and on a general statement about Segre classes proven in this note (Theorem 2.9, which should be of independent interest). Jörg Schürmann has recently shown that a proof of the Scholium for *splayed* (rather than *strongly splayed*) subvarieties may be obtained as a particular case of his Verdier–Riemann–Roch theorem for Chern–Schwartz–MacPherson classes ([Sch]).

It is natural to ask whether a version of the Scholium holds for other characteristic classes for singular varieties. Substantial work has been carried out comparing the Chern–Schwartz–MacPherson class to the Chern–Fulton class, another class agreeing with the Chern class of the tangent bundle for nonsingular varieties. See [Ful84, Example 4.2.6(a)] for the definition (reproduced here in §3). We denote this class by $c_{\rm F}$. The difference $c_{\rm SM}(X) - c_{\rm F}(X)$ is called the *Milnor class* of X, since it generalizes Milnor numbers of isolated hypersurface singularities to arbitrary singularities.

Theorem III *Let X, Y be splayed subschemes of a nonsingular variety V. Then*

$$c_{\rm F}(X)\cdot c_{\rm F}(Y) = c(TV)\cap c_{\rm F}(X\cap Y). \tag{1.5}$$

The proof of this result also follows from Segre class considerations, in fact of a simpler nature than those leading to Theorem 2.9.

It is also natural to ask whether (1.2) and (1.5) hold when X and Y are in general position. The following is a prototype situation where this can be established.

Theorem IV *Let V be a nonsingular variety, and let $X \subseteq V$ be a general very ample divisor on V. Then for all subschemes $Y \subseteq V$,*

$$c(X)\cdot c_{\rm SM}(Y) = c(TV)\cap c_{\rm SM}(X\cap Y),$$
$$c(X)\cdot c_{\rm F}(Y) = c(TV)\cap c_{\rm F}(X\cap Y).$$

Theorem IV hints that a condition analogous to splayedness may satisfy results along the lines of the Bertini or Kleiman–Bertini theorems. It would be interesting to establish a precise result of this type. In general, however, a splayed Bertini theorem cannot hold, as the following example illustrates.

Example 1.3 Let X be the so-called 4-*lines divisor* in $\mathbb{C}^3$, given by the polynomial $xy(x+y)(x+yz)$. It is well known that X is not analytically trivial along the z-axis; *i.e.*, there is no analytic isomorphism between two hyperplane sections X_{t_1} and X_{t_2}, where $X_t := X \cap (\mathbb{C}^2 \times \{t\})$. If X were splayed with a general hyperplane at a general point of the z-axis, then it would be possible to write the equation of the divisor using only two coordinates at that point. This would imply that nearby sections are analytically isomorphic.

2 Proofs of Theorems I and II

2.1 Splayed Blow-ups

Throughout the paper, V will denote a smooth complex algebraic variety; several results extend without change to the context of nonsingular algebraic varieties over an algebraically closed field of characteristic 0. (See *e.g.*, [Ken90] for a treatment of Chern–Schwartz–MacPherson classes in this generality.) We call two subschemes $Z_1, Z_2 \subseteq V$ *splayed* if at every point p in the intersection of Z_1 and Z_2 there is a local analytic isomorphism $\phi\colon V \to V' \times V''$, and subschemes $Z_1' \subseteq V'$, $Z_2'' \subseteq V''$ such that $Z_1 = \phi^{-1}(Z_1' \times V'')$ and $Z_2 = \phi^{-1}(V' \times Z_2'')$. Equivalently, there are analytic coordinates for V at p such that Z_1 and Z_2 are defined in different sets of variables.

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More generally, we will say that two sets of subschemes are splayed in V if at each point there is a local analytic isomorphism ϕ as above such that the schemes in the first set are inverse images from the first factor of the product $V' \times V''$, and the schemes in the second sets are inverse images from the second factor.

Denote by $\pi_i \colon \widetilde{V}_i \to V$ the blowup of V along Z_i. Denote further by $\widetilde{V}_{12}$ the blowup of $\widetilde{V}_1$ along the inverse image $\pi_1^{-1} Z_2$ of Z_2 and by $\widetilde{V}_{21}$ the blowup of $\widetilde{V}_2$ along the inverse image $\pi_2^{-1} Z_1$ of Z_1. We begin by recalling the following fact, for which neither the splayedness assumption on Z_1, Z_2 nor the smoothness of V is needed.

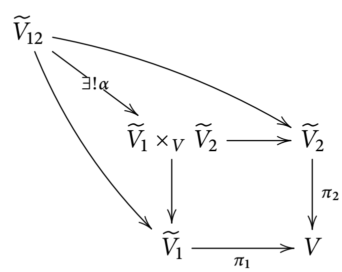
Proposition 2.1 *The blow-ups $\widetilde{V}_{12}$ and $\widetilde{V}_{21}$ are isomorphic, and they are isomorphic to the blow-up of V along $Z_1 \cup Z_2$, where the defining ideal sheaf of $Z_1 \cup Z_2$ in V is the product of the ideal sheaves defining Z_1 and Z_2.*

Proof Both statements follow from the universal property of blow-ups; see, for example, [EH00, Lemma IV-4] and [Li09, Lemma 3.2]. ∎

The blow-up $\widetilde{V}_{12} \cong \widetilde{V}_{21}$ is the joint blow-up of [KT96, §2.7]. Also note that, with this scheme structure, $Z_1 \cup Z_2$ may have embedded components even if Z_1 and Z_2 are reduced.

We want to compare $\widetilde{V}_{12}$ to the fiber product of $\widetilde{V}_1$ and $\widetilde{V}_2$.

By the universal property of fiber products, there is a unique morphism $\alpha \colon \widetilde{V}_{12} \to \widetilde{V}_1 \times_V \widetilde{V}_2$:



Proposition 2.2 *Let V be an irreducible variety. Then α induces an isomorphism from $\widetilde{V}_{12}$ to the unique irreducible component $\widetilde{V}_1 \widehat{\times}_V \widetilde{V}_2$ of $\widetilde{V}_1 \times_V \widetilde{V}_2$ mapping dominantly to V.*

Remark 2.3 M. Kwieciński ([Kwi94]) calls this irreducible component the modified fiber product, and observes that it is a product in the category of proper birational morphisms from varieties to V.

Proof Let $\widehat{\alpha}$ be the induced morphism $\widetilde{V}_{12} \to \widetilde{V}_1 \widehat{\times}_V \widetilde{V}_2$. Since $\pi_1^{-1}(Z_1)$ and $\pi_2^{-1}(Z_2)$ are Cartier divisors, it follows that their inverse images in $\widetilde{V}_1 \widehat{\times}_V \widetilde{V}_2$ are Cartier divisors, and hence so is the inverse image of $Z_1 \cup Z_2$. By the universal property of blow-ups, we obtain a morphism $\widetilde{V}_1 \widehat{\times}_V \widetilde{V}_2 \to \widetilde{V}_{12}$, which is immediately checked to be the inverse of $\widehat{\alpha}$. ∎

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Corollary 2.4 *Assume that V is nonsingular and Z_1 and Z_2 are splayed in V. Then α is an isomorphism $\widetilde{V}_{12} \to \widetilde{V}_1 \times_V \widetilde{V}_2$.*

Proof We claim that $\widetilde{V}_1 \times_V \widetilde{V}_2$ is irreducible. Indeed, it suffices to verify this fact locally analytically over every p in V, so by the splayedness condition we may assume that $Z_1 = Z_1' \times V''$, $Z_2 = V' \times Z_2'$ in $V = V' \times V''$. In this situation, $\widetilde{V}_1 \times_V \widetilde{V}_2 \cong B\ell_{Z_1'} V' \times B\ell''_{Z_2'} V''$.

Since $B\ell_{Z_1'} V'$ and $B\ell_{Z_2'} V''$ are irreducible, this product is irreducible.

With notation as in Proposition 2.2, this shows that $\widetilde{V}_1 \times_V \widetilde{V}_2 = \widetilde{V}_1 \mathbin{\widehat{\times}}_V \widetilde{V}_2$, and the result follows from the proposition. ∎

Remark 2.5 It is worth pointing out that α is not an isomorphism in general. For example, let $V = \mathbb{A}^2$ and let let $Z_1 = Z_2$ be the origin $p = (0,0)$. Then $\widetilde{V}_1 \times_V \widetilde{V}_2$ consists of two components: an isomorphic copy of the blow-up of V at p and a component isomorphic to $E \times E$, where E is the exceptional divisor in $\widetilde{V}_1 = \widetilde{V}_2$. (This is easily verified by a computation with charts.)

Tracing the proof of Proposition 2.2, the problem is that while the inverse image of *e.g.*, Z_1 in $\widetilde{V}_1 \times_V \widetilde{V}_2$ is locally principal, it contains a whole component of the fiber product (*i.e.*, local generators of its ideal are zero-divisors), so this subscheme is not a Cartier divisor of the fiber product. It is, however, a Cartier divisor in the modified fiber product.

On the other hand, α may be an isomorphism even if Z_1 and Z_2 are not splayed. For instance, if Z_1 and Z_2 are any Cartier divisors, then all blow-ups are isomorphisms, and so is the fiber product. For a more substantive example, take two coordinate axes Z_1, Z_2 in $V = \mathbb{A}^3$. It can easily be seen via computation in charts that $\widetilde{V}_1 \times_V \widetilde{V}_2$ is irreducible and isomorphic to the blow-up of V along $Z_1 \cup Z_2$. Thus, in this case, α is an isomorphism, although Z_1 and Z_2 are not splayed according to our definition.

Corollary 2.6 *Let Z_1, Z_2 be splayed in V, and consider the blow-ups along Z_1, Z_2, and $Z_1 \cup Z_2$ as above.*

$$\begin{array}{ccc} \widetilde{V}_{12} & \xrightarrow{\widetilde{\pi}_1} & \widetilde{V}_2 \\ {\scriptstyle \widetilde{\pi}_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \widetilde{V}_1 & \xrightarrow[\pi_1]{} & V \end{array}$$

Then the homomorphisms $\widetilde{\pi}_{2*}\widetilde{\pi}_1^*$ and $\pi_1^*\pi_{2*}$ from $A_*\widetilde{V}_2$ to $A_*\widetilde{V}_1$ coincide.

Proof The maps are all proper local complete intersection morphisms, and the diagram is a fiber square by Corollary 2.4. By [Ful84, Example 17.4.1(a)],

$$\pi_1^*\pi_{2*}(\alpha) = \widetilde{\pi}_{2*}\big(c_e(\mathscr{E}) \cap \widetilde{\pi}_1^*(\alpha)\big)$$

for all $\alpha \in A_*(\widetilde{V}_2)$, where $\mathscr{E}$ is an excess bundle and e is the difference in the codimensions of π_1 and $\widetilde{\pi}_1$. Here both π_1 and $\widetilde{\pi}_1$ are birational, so $e = 0$ and $c_e(\mathscr{E}) = 1$, hence the equality follows. ∎

2.2 A Segre Class Formula

The c_{SM} class of a hypersurface D in a nonsingular variety may be expressed in terms of the *Segre class* of the singularity subscheme JD in V. The precise relationship (from [Alu99]) will be recalled below. The hypersurface case of Theorem I will then follow from a statement on Segre classes of singularity subschemes of splayed hypersurfaces. In this subsection we prove a more general form of this statement (Theorem 2.9).

Reminder. Segre classes are one of the ingredients of Fulton–MacPherson intersection theory, and the reader is referred to [Ful84, Chapter 4] for a thorough treatment of these classes. The following summary should suffice for the purpose of this paper. The Segre class $s(S, X)$ of a proper subscheme S of a scheme X is the class in the Chow group of S determined by the following properties:

- Birational invariance: If $f\colon X' \to X$ is a proper birational morphism, then $s(S, X) = f_* s(f^{-1}(S), X')$ ([Ful84, Proposition 4.2(a)]).
- If S is a Cartier divisor in X, then $s(S, X) = [S] - [S]^2 + [S]^3 - \cdots$ ([Ful84, Corollary 4.2.2]).

We use the shorthand $\frac{[S]}{1+S}$ for the class $[S] - [S]^2 + [S]^3 - \cdots$.

By the first property, blowing-up X along S reduces the computation of $s(S, X)$ to the computation of the Segre class for the exceptional divisor in the blow-up, which may be performed by using the second property. In practice it is often very difficult to carry out this process, but useful formulas for Segre classes may be proved by using this strategy. The second property is a particular case of the following fact:

- If S is regularly embedded in X, with normal bundle $N_S X$, then $s(S, X) = c(N_S X)^{-1} \cap [S]$ ([Ful84, Corollary 4.2.1]).

This will be used below in order to compute the Segre class of the complete intersection of two hypersurfaces.

Let Z_1, Z_2, V, etc. be as in Section 2.1. By the birational invariance of Segre classes recalled above,

$$\pi_{1*} s(\pi_1^{-1}(Z_2), \widetilde{V}_1) = s(Z_2, V).$$

In the splayed situation, a stronger statement holds.

Lemma 2.7 *Let Z_1, Z_2 be splayed in V (as in Section 2.1). Then*

$$s(\pi_1^{-1}(Z_2), \widetilde{V}_1) = \pi_1^* s(Z_2, V).$$

Proof Consider the diagram

$$\begin{array}{ccc} \widetilde{V}_{12} & \xrightarrow{\widetilde{\pi}_1} & \widetilde{V}_2 \\ {\scriptstyle \widetilde{\pi}_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \widetilde{V}_1 & \xrightarrow[\pi_1]{} & V \end{array}$$

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as in §2.1. Let $E_2 = \pi_2^{-1}(Z_2)$ be the exceptional divisor in $\widetilde{V}_2$, and let $E_2' = \widetilde{\pi}_1^{-1}(E_2) = \widetilde{\pi}_2^{-1}(\pi_1^{-1}(Z_2))$. By the birational invariance of Segre classes,

$$s(\pi_1^{-1}(Z_2), \widetilde{V}_1) = \widetilde{\pi}_{2*}\Big(\frac{[E_2']}{1+E_2'}\Big) = \widetilde{\pi}_{2*}\widetilde{\pi}_1^*\Big(\frac{[E_2]}{1+E_2}\Big).$$

Since Z_1 and Z_2 are splayed, by Corollary 2.6, this equals

$$\pi_1^*\pi_{2*}\Big(\frac{[E_2]}{1+E_2}\Big) = \pi_1^* s(Z_2, V)$$

as claimed. ∎

Remark 2.8 The equality stated in Lemma 2.7 does not hold in general: $V = \mathbb{A}^2$, $Z_1 = Z_2 =$ the origin give a simple counterexample. It does hold whenever the fiber product $\widetilde{V}_1 \times_V \widetilde{V}_2$ is irreducible, as the arguments given above show, and this may occur even if Z_1 and Z_2 are not splayed. For example, if Z_1 is a hypersurface of V, then this condition is trivially satisfied regardless of splayedness. For a more interesting example, two lines Z_1, Z_2 meeting at a point in $V = \mathbb{P}^3$ are not splayed according to our definition, yet $\widetilde{V}_1 \times_V \widetilde{V}_2$ is irreducible (*cf.* Remark 2.5).

We will use Lemma 2.7 in the proof of the following more general Segre class formula, which is the key technical result needed for the first proof of Theorem I.

Let D_1, D_2 be hypersurfaces in V, and let $Z_1 \subseteq D_1$, $Z_2 \subseteq D_2$ be subschemes. At the level of ideal sheaves, we have

$$\mathscr{I}_{D_1,V} \subseteq \mathscr{I}_1, \quad \mathscr{I}_{D_2,V} \subseteq \mathscr{I}_2$$

where $\mathscr{I}_1 = \mathscr{I}_{Z_1,V}$, $\mathscr{I}_2 = \mathscr{I}_{Z_2,V}$. We consider the subscheme W of V defined by the ideal sheaf

$$\mathscr{I}_{W,V} := \mathscr{I}_{D_1,V} \cdot \mathscr{I}_2 + \mathscr{I}_{D_2,V} \cdot \mathscr{I}_1.$$

This subscheme is supported on $(D_1 \cup Z_2) \cap (D_2 \cup Z_1) = (D_1 \cap D_2) \cup (Z_1 \cup Z_2)$, with a scheme structure depending subtly on Z_1 and Z_2. Under a splayedness assumption, we will obtain a relation between the Segre classes of Z_1, Z_2, and W. The relation is best expressed in terms of the following notation: for $\iota\colon Z \subset V$ an embedding of schemes, let

$$\widetilde{s}(Z, V) = [V] - \iota_* s(Z, V)^\vee.$$

Here, the dual $(\,\cdot\,)^\vee$ changes the sign of components of odd codimension in V.

Theorem 2.9 *Let D_1, D_2 be hypersurfaces of a smooth variety V, and let $Z_1 \subseteq D_1$, $Z_2 \subseteq D_2$, W be as above. Assume that $\{Z_1, D_1\}$ and $\{Z_2, D_2\}$ are splayed. Then*

$$\frac{\widetilde{s}(W,V) \otimes \mathscr{O}(D_1+D_2)}{1+D_1+D_2} = \Big(\frac{\widetilde{s}(Z_1,V) \otimes \mathscr{O}(D_1)}{1+D_1}\Big) \cdot \Big(\frac{\widetilde{s}(Z_2,V) \otimes \mathscr{O}(D_2)}{1+D_2}\Big) \tag{2.1}$$

in the Chow group of V.

Remark 2.10 In this statement we use the notation introduced in [Alu94, §2]: if $\mathscr{L}$ is a line bundle on V and $A = \sum a^{(i)}$ is a class in the Chow group, where $a^{(i)}$ has *co*dimension i in V, then

$$A \otimes \mathscr{L} := \sum \frac{a^{(i)}}{c(\mathscr{L})^i}.$$

The formula given in Theorem 2.9 is a good example of the usefulness of this notation: the formula (and its proof) would look unintelligibly complicated if it were written out without adopting this shorthand. The notation satisfies simple properties (see [Alu94, Propositions 1 and 2]). These will be used liberally in what follows. It is also useful to observe that if A and B are classes in $A_* V$, then

$$(A \cdot B) \otimes \mathscr{L} = (A \otimes \mathscr{L}) \cdot (B \otimes \mathscr{L})$$

(this is evident from the definition).

Proof of Theorem 2.9 We consider a sequence of three blow-ups over V:

$$\widetilde{V} \xrightarrow{\widetilde{\pi}} \widetilde{V}_{12} \xrightarrow{\widetilde{\pi}_2} \widetilde{V}_1 \xrightarrow{\pi_1} V,$$

where π_1 is the blow-up of V along Z_1 with exceptional divisor E_1; $\widetilde{\pi}_2$ is the blow-up of $\widetilde{V}_1$ along $\pi_1^{-1}(Z_2)$ with exceptional divisor E_2'; and $\widetilde{\pi}$ is the blow-up of $\widetilde{V}_{12}$ along the intersection of the residual subschemes of $\widetilde{\pi}_2^{-1}(E_1)$ (resp. E_2') in the inverse images of D_1 (resp. D_2). Note that under our splayedness hypothesis, this last center is a complete intersection of codimension 2. We let $\widetilde{E}$ be the exceptional divisor of $\widetilde{\pi}$. For notational convenience, we often use the same notation for an object and for its inverse image to a variety in the sequence. For instance, E_1 will also denote its inverse image $\widetilde{\pi}^{-1}\widetilde{\pi}_2^{-1}E_1$ in $\widetilde{V}$. Finally, π will denote the composition $\pi_1 \circ \widetilde{\pi}_2 \circ \widetilde{\pi}\colon \widetilde{V} \to \widetilde{V}_{12} \to \widetilde{V}_1 \to V$.

Claim 2.11 $\pi^{-1}(W) = E_1 \cup E_2' \cup \widetilde{E}$.

The statement of this claim is that the ideal of W pulls back to the product of the ideals of (the inverse images of) E_1, E_2', and $\widetilde{E}$ in $\widetilde{V}$. In $\widetilde{V}_1$,

$$\begin{aligned}\mathscr{I}_{W,V} \cdot \mathscr{O}_{\widetilde{V}_1} &= \mathscr{I}_{\widetilde{D}_1,\widetilde{V}_1}\mathscr{I}_{E_1,\widetilde{V}_1}\mathscr{I}_{\pi_1^{-1}(Z_2),\widetilde{V}_1} + \mathscr{I}_{\pi_1^{-1}(D_2),\widetilde{V}_1} \cdot \mathscr{I}_{E_1,\widetilde{V}_1}\\ &= \mathscr{I}_{E_1,\widetilde{V}_1} \cdot \big(\mathscr{I}_{\widetilde{D}_1,\widetilde{V}_1}\mathscr{I}_{\pi_1^{-1}(Z_2),\widetilde{V}_1} + \mathscr{I}_{\pi_1^{-1}(D_2),\widetilde{V}_1}\big),\end{aligned}$$

where $\widetilde{D}_1$ is the residual of E_1 in $\pi_1^{-1}(D_1)$. In $\widetilde{V}_{12}$,

$$\mathscr{I}_{W,V} \cdot \mathscr{O}_{\widetilde{V}_{12}} = \mathscr{I}_{\widetilde{\pi}_2^{-1}(E_1),\widetilde{V}_{12}} \cdot \mathscr{I}_{E_2',\widetilde{V}_{12}} \cdot \big(\mathscr{I}_{\widetilde{\pi}_2^{-1}(\widetilde{D}_1),\widetilde{V}_{12}} + \mathscr{I}_{\widetilde{D}_2,\widetilde{V}_{12}}\big),$$

where $\widetilde{D}_2$ is the residual of E_2' in the inverse image of D_2. The ideal $\mathscr{I}_{\widetilde{\pi}_2^{-1}(\widetilde{D}_1),\widetilde{V}_{12}} + \mathscr{I}_{\widetilde{D}_2,\widetilde{V}_{12}}$ defines the center of the third blow-up, so this shows that

$$\mathscr{I}_{W,V} \cdot \mathscr{O}_{\widetilde{V}} = \mathscr{I}_{\widetilde{\pi}^{-1}\widetilde{\pi}_2^{-1}(E_1),\widetilde{V}} \cdot \mathscr{I}_{\widetilde{\pi}^{-1}(E_2'),\widetilde{V}} \cdot \mathscr{I}_{\widetilde{E},\widetilde{V}},$$

as claimed.

By the birational invariance of Segre classes,

$$s(W,V) = \pi_* \frac{[E_1] + [E_2'] + [\widetilde{E}]}{1 + E_1 + E_2' + \widetilde{E}},$$

and therefore

$$[V] - s(W,V) = \pi_* \Big(\frac{1}{1 + E_1 + E_2' + \widetilde{E}} \cap [\widetilde{V}] \Big).$$

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Using the $\otimes$ notation recalled after the statement of the proposition,

$$\frac{1}{1+E_1+E_2'+\widetilde{E}}\cap[\widetilde{V}]=\Big(\frac{1-E_1-E_2'}{1+\widetilde{E}}\cap[\widetilde{V}]\Big)\otimes\mathscr{O}(E_1+E_2')$$
$$=\Big((1-E_1-E_2')\Big(1-\frac{\widetilde{E}}{1+\widetilde{E}}\Big)\cap[\widetilde{V}]\Big)\otimes\mathscr{O}(E_1+E_2')$$
$$=\frac{1}{1+E_1+E_2'}\cap\Big([\widetilde{V}]-\frac{[\widetilde{E}]}{1+\widetilde{E}}\otimes\mathscr{O}(E_1+E_2')\Big).$$

The term $[\widetilde{E}]/(1+\widetilde{E})$ pushes forward to the Segre class of the center of the third blow-up, which is the intersection of the (inverse images of the) residual of E_1 in D_1, with class D_1-E_1, and of the residual of E_2' in D_2, with class D_2-E_2'. The intersection is regularly embedded in $\widetilde{V}_{12}$, as noted earlier, and its Segre class equals the inverse Chern class of its normal bundle (by the third property of Segre classes recalled above):

$$\widetilde{\pi}_*\Big(\frac{\widetilde{E}}{1+\widetilde{E}}\Big)=\frac{[D_1-E_1]\cdot[D_2-E_2']}{(1+D_1-E_1)(1+D_2-E_2')},$$

where evident pull-backs are omitted for notational simplicity. Using this fact, properties of the $\otimes$ notation, and the projection formula,

$$\widetilde{\pi}_*\Big(\frac{1}{1+E_1+E_2'+\widetilde{E}}\cap[\widetilde{V}]\Big)$$
$$=\frac{1}{1+E_1+E_2'}\cap\Big([\widetilde{V}_{12}]-\frac{[D_1-E_1]\cdot[D_2-E_2']}{(1+D_1-E_1)(1+D_2-E_2')}\otimes\mathscr{O}(E_1+E_2')\Big)$$
$$=\frac{1}{1+E_1+E_2'}\cap\Big([\widetilde{V}_{12}]-\frac{[D_1-E_1]\cdot[D_2-E_2']}{(1+D_1+E_2')(1+D_2+E_1)}\Big).$$

A remarkable cancellation (and again the projection formula) now gives

$$\widetilde{\pi}_*\Big(\frac{1}{(1+D_1+D_2)(1+E_1+E_2'+\widetilde{E})}\cap[\widetilde{V}]\Big)$$
$$=\frac{1}{(1+D_1+D_2)(1+E_1+E_2')}\cap\Big(1-\frac{(D_1-E_1)\cdot(D_2-E_2')}{(1+D_1+E_2')(1+D_2+E_1)}\Big)\cap[\widetilde{V}_{12}]$$
$$=\frac{[\widetilde{V}_{12}]}{(1+D_1+E_2')(1+D_2+E_1)}.$$

Summarizing, we have shown that

$$\frac{[V]-s(W,V)}{1+D_1+D_2}=\pi_{1*}\widetilde{\pi}_{2*}\Big(\frac{[\widetilde{V}_{12}]}{(1+D_1+E_2')(1+D_2+E_1)}\Big).\tag{2.2}$$

In order to evaluate the right-hand side, note that

$$\frac{[\widetilde{V}_{12}]}{1+D_1+E_2'}=\frac{1}{1+D_1}\Big(\frac{[\widetilde{V}_{12}]}{1+E_2'}\otimes\mathscr{O}(D_1)\Big)=\frac{1}{1+D_1}\Big([\widetilde{V}_{12}]-\frac{[E_2']}{1+E_2'}\otimes\mathscr{O}(D_1)\Big).$$

Pushing this forward by $\widetilde{\pi}_2$ shows that

$$\widetilde{\pi}_{2*}\Big(\frac{[\widetilde{V}_{12}]}{1+D_1+E_2'}\Big)=\frac{[\widetilde{V}_1]-s(\pi_1^{-1}Z_2,\widetilde{V}_1)\otimes\mathscr{O}(D_1)}{1+D_1}.$$

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Since Z_1 and Z_2 are splayed, by Lemma 2.7 this may be rewritten as

$$\widetilde{\pi}_{2*}\Big(\frac{[\widetilde{V}_{12}]}{1+D_1+E_2'}\Big) = \pi_1^*\Big(\frac{[V]-s(Z_2,V)\otimes\mathscr{O}(D_1)}{1+D_1}\Big).$$

By the projection formula and (2.2) we have

$$\frac{[V]-s(W,V)}{1+D_1+D_2} = \pi_{1*}\Big(\frac{[\widetilde{V}_1]}{1+D_2+E_1}\Big)\cdot\frac{[V]-s(Z_2,V)\otimes\mathscr{O}(D_1)}{1+D_1}.$$

The last push-forward is handled similarly to the previous one, giving

$$\pi_{1*}\Big(\frac{[\widetilde{V}_1]}{1+D_2+E_1}\Big) = \frac{[V]-s(Z_1,V)\otimes\mathscr{O}(D_2)}{1+D_2}.$$

Therefore,

$$\frac{[V]-s(W,V)}{1+D_1+D_2} = \frac{([V]-s(Z_1,V))\otimes\mathscr{O}(D_2)}{1+D_2}\cdot\frac{([V]-s(Z_2,V))\otimes\mathscr{O}(D_1)}{1+D_1}.$$

The stated formula follows from this by taking duals and tensoring by $\mathscr{O}(D_1+D_2)$. ■

The argument shows that the formula in Theorem 2.9 holds as soon as $\widetilde{V}_1\times_V\widetilde{V}_2$ is irreducible (*cf.* Remark 2.8), and the residuals of E_1 in $\pi_1^{-1}(D_1)$ and E_2' in $\widetilde{\pi}_2^{-1}\pi_1^{-1}(D_2)$ have no common components. While we focus on splayedness in this paper, the formula in Theorem 2.9 has a substantially more general scope.

Example 2.12 Let Z_1, Z_2 be two lines in $V=\mathbb{P}^3$ intersecting at a point. Then Z_1 and Z_2 are not splayed according to our definition, but $\widetilde{V}_1\times_V\widetilde{V}_2$ is irreducible (Remark 2.5). Choosing coordinates $(x_0:\cdots:x_3)$, we may assume that Z_1 has the ideal (x_0,x_1) and Z_2 has the ideal (x_0,x_2). Then Z_i is contained in $D_i=\{x_i=0\}$; a computation shows that the relevant residuals have no common components. A direct computation of Segre classes, which may, for example, be carried out using [Alu03], confirms that formula (2.1) does hold.

Example 2.13 If $Z_1=Z_2=\varnothing$, then $W=D_1\cap D_2$. Assume that D_1 and D_2 have no common components, so that W is a codimension 2 local complete intersection with normal bundle $\mathscr{O}(D_1)\oplus\mathscr{O}(D_2)$. This is, of course, the case if D_1 and D_2 are splayed, and considerably more generally. We have

$$\begin{aligned}\widehat{s}(W,V) &= [V]-\Big(\frac{D_1\cdot D_2}{(1+D_1)(1+D_2)}\cap[V]\Big)^\vee = \Big(1-\frac{D_1\cdot D_2}{(1-D_1)(1-D_2)}\Big)\cap[V]\\ &= \frac{1-D_1-D_2}{(1-D_1)(1-D_2)}\cap[V],\end{aligned}$$

and hence

$$\widehat{s}(W,V)\otimes\mathscr{O}(D_1+D_2) = \frac{1+D_1+D_2}{(1+D_1)(1+D_2)}\cap[V]$$

(use [Alu94, Proposition 1]). Formula (2.1) follows immediately in this case.

The reader is encouraged to consider the opposite extreme $Z_1=D_1$, $Z_2=D_2$ and verify that (2.1) reduces to $[V]=[V]\cdot[V]$ in this case (regardless of splayedness).

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2.3 Chern Classes of Hypersurface Complements

For a rapid review of *Chern–Schwartz–MacPherson* (c_{SM}) classes, we refer the reader to [AF13, §3.1] and references therein. Briefly, every locally closed subset U of a complete variety V determines a class $c_{\mathrm{SM}}(U)$ in the Chow group of V, such that if $U = Z$ is a nonsingular closed subvariety, then $c_{\mathrm{SM}}(Z)$ equals the push-forward to V of the total Chern class of the tangent bundle to Z. This notion is functorial in a strong sense, and satisfies an inclusion-exclusion property. If U_1, U_2 are locally closed in V, then

$$c_{\mathrm{SM}}(U_1 \cup U_2) = c_{\mathrm{SM}}(U_1) + c_{\mathrm{SM}}(U_2) - c_{\mathrm{SM}}(U_1 \cap U_2).$$

The classes arose in the seminal work of M.-H. Schwartz ([Sch65a, Sch65b]) and R. MacPherson ([Mac74]). See [Ful84, Example 19.1.7] for an efficient statement of MacPherson's definition and result.

We will use the following formula computing the c_{SM} class of a hypersurface D in a nonsingular variety V in terms of the Segre class of the *singularity subscheme* JD, locally defined (as a subscheme of D) by the partial derivatives of a local equation for D.

Lemma 2.14 ([Alu99, Theorem I.4]) *Let D be a hypersurface of a nonsingular variety V, with singularity subscheme JD. Then*

$$c_{\mathrm{SM}}(D) = c(TV) \cap \Big(s(D,V) + c(\mathscr{O}(D))^{-1} \cap (s(JD,V)^\vee \otimes \mathscr{O}(D))\Big). \tag{2.3}$$

This statement again uses the operations $\cdot^\vee$, $\otimes$ employed in §2.2. In terms of the notation introduced before the statement of Theorem 2.9, (2.3) is equivalent to

$$c_{\mathrm{SM}}(V \smallsetminus D) = c(TV) \cap \frac{\widehat{s}(JD,V) \otimes \mathscr{O}(D)}{1+D}. \tag{2.4}$$

Now suppose that D_1 and D_2 are splayed divisors in V and let $D = D_1 \cup D_2$. Note that this implies that $\{D_1, JD_1\}$ and $\{D_2, JD_2\}$ are splayed. It is clear set-theoretically that JD is supported on $(D_1 \cap D_2) \cup (JD_1 \cup JD_2)$. The splayedness condition implies that the scheme structure of JD on this union matches the one studied in §2.2 vis-a-vis W, Z_1, Z_2.

Lemma 2.15 *With notation as above, D_1 and D_2 are splayed if and only if*

$$\mathscr{I}_{JD,V} = \mathscr{I}_{D_1,V} \cdot \mathscr{I}_{JD_2,V} + \mathscr{I}_{D_2,V} \cdot \mathscr{I}_{JD_1,V}.$$

This is a restatement of [AF13, Corollary 2.6]. Only the "only if" part will be needed here.

Corollary 2.16 *Let D_1, D_2 be splayed hypersurfaces in a smooth variety V, and let D be $D_1 \cup D_2$. Then*

$$\frac{\widehat{s}(JD,V) \otimes \mathscr{O}(D)}{1+D} = \Big(\frac{\widehat{s}(JD_1,V) \otimes \mathscr{O}(D_1)}{1+D_1}\Big) \cdot \Big(\frac{\widehat{s}(JD_2,V) \otimes \mathscr{O}(D_2)}{1+D_2}\Big)$$

in A_*V.

Proof This follows from Lemma 2.15 and Theorem 2.9, since if D_1 and D_2 are splayed, then so are $\{D_1, JD_1\}$ and $\{D_2, JD_2\}$. ■

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Corollary 2.17 *Let D_1, D_2 be splayed hypersurfaces in a smooth variety V, and let D be $D_1 \cup D_2$. Then*

$$c(TV) \cap c_{\mathrm{SM}}(V \smallsetminus D) = c_{\mathrm{SM}}(V \smallsetminus D_1) \cdot c_{\mathrm{SM}}(V \smallsetminus D_2). \tag{2.5}$$

Proof This follows from Corollary 2.16 and (2.4). ∎

Formula (2.5) was proposed in [AF13], where it was observed that under a strong freeness assumption on the divisors it follows from an analogous formula for Chern classes of sheaves of logarithmic differentials ([AF13, Proposition 3.2]). Several other particular instances of the formula are studied in [AF13, §3]. Corollary 2.17 proves the formula without extraneous assumptions.

For splayed divisors, Theorem I follows immediately from Corollary 2.17 and the inclusion-exclusion property of c_{SM} classes.

Theorem 2.18 (Theorem I, hypersurface case) *Let D_1, D_2 be splayed divisors in a smooth variety V. Then $c_{\mathrm{SM}}(D_1) \cdot c_{\mathrm{SM}}(D_2) = c(TV) \cap c_{\mathrm{SM}}(D_1 \cap D_2)$ in $A_* V$.*

Proof With $D = D_1 \cup D_2$ as in Corollary 2.17 and noting that $c(TV) \cap \alpha = c(V) \cdot \alpha$ for all $\alpha \in A_* V$,

$$\begin{aligned}
c_{\mathrm{SM}}(D_1) \cdot c_{\mathrm{SM}}(D_2) &= (c(V) - c_{\mathrm{SM}}(V \smallsetminus D_1)) \cdot (c(V) - c_{\mathrm{SM}}(V \smallsetminus D_2)) \\
&= c(TV) \cap \big(c(V) - c_{\mathrm{SM}}(V \smallsetminus D_1) - c_{\mathrm{SM}}(V \smallsetminus D_2)\big) \\
&\quad + c_{\mathrm{SM}}(V \smallsetminus D_1) \cdot c_{\mathrm{SM}}(V \smallsetminus D_2) \\
&= c(TV) \cap \big(c_{\mathrm{SM}}(D_1) + c_{\mathrm{SM}}(D_2) - c(V)\big) \\
&\quad + c(TV) \cap c_{\mathrm{SM}}(V \smallsetminus D) \\
&= c(TV) \cap \big(c_{\mathrm{SM}}(D_1) + c_{\mathrm{SM}}(D_2) - c_{\mathrm{SM}}(D)\big) \\
&= c(TV) \cap c_{\mathrm{SM}}(D_1 \cap D_2),
\end{aligned}$$

where the last equality follows by inclusion-exclusion. ∎

2.4 Strongly splayed varieties and constructible functions

We say that two subvarieties Z_1, Z_2 of V are *strongly* splayed if $Z_1 = D'_1 \cap \cdots \cap D'_r$, $Z_2 = D''_1 \cap \cdots \cap D''_s$, where D'_i, D''_j are hypersurfaces, and $\{D'_1, \ldots, D'_r\}$, $\{D''_1, \ldots, D''_s\}$ are splayed in the sense of §2.1. We do not know if splayed subvarieties are necessarily strongly splayed; the distinction is, of course, immaterial for hypersurfaces.

We can also consider this notion for *constructible functions*. By definition, every constructible function can be written as a linear combination of indicator functions of closed subvarieties. Since every subvariety is an intersection of hypersurfaces, it follows that every constructible function may be written as an integer linear combination of indicator functions of hypersurfaces.

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Definition 2.19 Two constructible functions ϕ, ψ are *strongly splayed* if they admit representations

$$\phi = \sum_i a_i' 1_{D_i'}, \quad \psi = \sum_j a_j'' 1_{D_j''} \tag{2.6}$$

with $\{D_1', \dots, D_r'\}$, $\{D_1'', \dots, D_s''\}$ splayed sets of hypersurfaces and $a_i', a_i'' \in \mathbb{Z}$.

Thus, if Z_1 and Z_2 are strongly splayed, then so are the corresponding indicator functions $1_{Z_1}, 1_{Z_2}$.

Theorem II *Let ϕ, ψ be strongly splayed constructible functions on a nonsingular variety V. Then*

$$c_{SM}(\phi) \cdot c_{SM}(\psi) = c(TV) \cap c_{SM}(\phi \cdot \psi).$$

Proof We will prove this statement by induction on the number of splayed hypersurfaces needed to define ϕ, ψ (as in (2.6)). More precisely, assume that the equality

$$c_{SM}(\phi) \cdot c_{SM}(\psi) = c(TV) \cap c_{SM}(\phi \cdot \psi).$$

is known whenever

$$\phi = \sum_{i=1}^{r} a_i' 1_{D_i'}, \quad \psi = \sum_{j=1}^{s} a_j'' 1_{D_j''}$$

for a given pair (r, s) of positive integers, with $\{D_1', \dots, D_r'\}$ and $\{D_1'', \dots, D_s''\}$ splayed, and for all pairs preceding (r, s) in the lexicographic order. We will show that the equality is then also true for $(r+1, s)$. Since the statement is true for $(r, s) = (1, 1)$ by Theorem 2.18 (and $\cdot$ is symmetric), the general case follows by induction. Thus we are reduced to showing that

$$c_{SM}(a1_D + \phi) \cdot c_{SM}(\psi) = c(TV) \cap c_{SM}((1_D + \phi) \cdot \psi)$$

with ϕ and ψ as above, under the assumption that $\{D, D_1', \dots, D_r'\}$ and $\{D_1'', \dots, D_s''\}$ are splayed. Since c_{SM} is linear,

$$\begin{aligned} c_{SM}(a1_D + \phi) \cdot c_{SM}(\psi) &= a\, c_{SM}(1_D) \cdot c_{SM}(\psi) + c_{SM}(\phi) \cdot c_{SM}(\psi) \\ &= a\, c(TV) \cap c_{SM}(1_D \cdot \psi) + c(TV) \cap c_{SM}(\phi \cdot \psi) \end{aligned}$$

by the induction hypothesis

$$\begin{aligned} &= c(TV) \cap \big(a\, c_{SM}(1_D \cdot \psi) + c_{SM}(\phi \cdot \psi) \big) \\ &= c(TV) \cap c_{SM}\big((a1_D + \phi) \cdot \psi \big) \end{aligned}$$

as needed. ∎

Theorem II implies the full statement of Theorem I from the introduction. Indeed, for $\phi = 1_X$, $\psi = 1_Y$, under the assumption that X and Y (and hence ϕ, ψ) are strongly splayed, Theorem II gives

$$c_{SM}(1_X) \cdot c_{SM}(1_Y) = c(TV) \cap c_{SM}(1_X \cdot 1_Y),$$

which gives (1.3) as $1_X \cdot 1_Y = 1_{X \cap Y}$.

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3 Proof of Theorem III

If X is a subscheme of a nonsingular variety, the *Chern–Fulton* class of X, $c_{\mathrm{F}}(X)$, is defined by

$$c_{\mathrm{F}}(X) := c(TV) \cap s(X, V).$$

W. Fulton introduced this class in [Ful84, Example 4.2.6(a)], and proved that it is in fact independent of the choice of the ambient nonsingular variety V. If X is itself nonsingular, then $s(X, V) = c(N_X V)^{-1} \cap [X]$ (§2.2), so that $c_{\mathrm{F}}(X) = c(X) = c_{\mathrm{SM}}(X)$ in this case. The classes $c_{\mathrm{SM}}(X)$ and $c_{\mathrm{F}}(X)$ differ in general. For example, $c_{\mathrm{F}}(X)$ is sensitive to the scheme structure of X, while $c_{\mathrm{SM}}(X)$ only depends on the support of X.

Theorem III is a straightforward consequence of the following multiplicative formula for Segre classes of splayed subschemes.

Lemma 3.1 *Let Z_1, Z_2 be splayed subschemes of a nonsingular variety V. Then*

$$s(Z_1 \cap Z_2, V) = s(Z_1, V) \cdot s(Z_2, V) \tag{3.1}$$

in $A_*(Z_1 \cap Z_2)$.

Proof Consider again the fiber square of blow-ups

$$\begin{array}{ccc} \widetilde{V}_{12} & \xrightarrow{\widetilde{\pi}_1} & \widetilde{V}_2 \\ {\scriptstyle \widetilde{\pi}_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \widetilde{V}_1 & \xrightarrow[\pi_1]{} & V \end{array}$$

as in §2.1. Let E_1 be the exceptional divisor in $\widetilde{V}_1$, E_2 the divisor in $\widetilde{V}_2$. By splayedness, the inverse images $\widetilde{\pi}_2^{-1}(E_1)$ and $\widetilde{\pi}_1^{-1}(E_2)$ have no components in common. Thus the inverse image of $Z_1 \cap Z_2$ in $\widetilde{V}_{12}$ is the complete intersection of $\widetilde{\pi}_2^{-1}(E_1)$ and $\widetilde{\pi}_1^{-1}(E_2)$. Therefore

$$\begin{aligned} s(Z_1 \cap Z_2, V) &= (\pi_1 \circ \widetilde{\pi}_2)_* \frac{\widetilde{\pi}_2^*(E_1) \cdot \widetilde{\pi}_1^*(E_2)}{(1 + \widetilde{\pi}_2^*(E_1))(1 + \widetilde{\pi}_1^*(E_2))} \\ &= \pi_{1*}\Big(\frac{E_1}{1 + E_1} \cdot \widetilde{\pi}_{2*} \widetilde{\pi}_1^* \frac{E_2}{1 + E_2} \Big), \end{aligned}$$

by the projection formula

$$, = \pi_{1*}\Big(\frac{E_1}{1 + E_1} \cdot \pi_1^* \pi_{2*} \frac{E_2}{1 + E_2} \Big)$$

since the diagram is a fiber square

$$\begin{aligned} &= \pi_{1*}\Big(\frac{E_1}{1 + E_1} \cdot \pi_1^* s(Z_2, V) \Big) \\ &= s(Z_1, V) \cdot s(Z_2, V) \end{aligned}$$

again by the projection formula. ∎

Remark 3.2 Formula (3.1) also follows formally by setting $D_1 = D_2 = 0$ in (2.1); note that if $\mathscr{I}_{D_1,V}$ and $\mathscr{I}_{D_1,V}$ are trivial, then the the scheme W appearing in Theorem 2.9 equals $Z_1 \cap Z_2$. However, then the proof given for Theorem 2.9 does then not work: one has to assume that D_1 and D_2 are hypersurfaces containing Z_1, Z_2 respectively, and this is in general incompatible with assuming that their classes vanish. Also, (2.1) only holds in A_*V, while Lemma 3.1 proves (3.1) in $A_*(Z_1 \cap Z_2)$.

Theorem III, stated in the introduction, follows immediately from Lemma 3.1. Assuming X and Y are splayed,

$$\begin{aligned} c_{\mathrm{F}}(X) \cdot c_{\mathrm{F}}(Y) &= (c(TV) \cap s(X,V)) \cdot (c(TV) \cap s(Y,V)) \\ &= c(TV) \cap (c(TV) \cap (s(X,V) \cdot s(Y,V))) \\ &= c(TV) \cap (c(TV) \cap s(X \cap Y, V)) \\ &= c(TV) \cap c_{\mathrm{F}}(X \cap Y). \end{aligned}$$

Example 3.3 With X and Y as in Example 1.2, we have

$$\begin{aligned} c_{\mathrm{F}}(X) &= [\mathbb{P}^4] + 6[\mathbb{P}^3] + 11[\mathbb{P}^2] + 12[\mathbb{P}^1] + 3[\mathbb{P}^0], \\ c_{\mathrm{F}}(Y) &= 2[\mathbb{P}^4] + 8[\mathbb{P}^3] + 14[\mathbb{P}^2] + 12[\mathbb{P}^1] + 6[\mathbb{P}^0] \end{aligned}$$

(obtained using the code from [Alu03]). According to Theorem III,

$$c(T\mathbb{P}^5) \cap c_{\mathrm{F}}(X \cap Y) = c_{\mathrm{F}}(X) \cdot c_{\mathrm{F}}(Y) = 2[\mathbb{P}^3] + 20[\mathbb{P}^2] + 84[\mathbb{P}^1] + 208[\mathbb{P}^0]$$

from which

$$c_{\mathrm{SM}}(X \cap Y) = 2[\mathbb{P}^3] + 8[\mathbb{P}^2] + 6[\mathbb{P}^1] + 12[\mathbb{P}^0].$$

Again, this can be verified in this example by using the code in [Alu03].

4 Proof of Theorem IV

We now assume that X is a general section of a very ample line bundle on V; in particular, X is itself nonsingular. If a Bertini theorem for splayedness held, then one would expect that for any $Y \subseteq V$, the formulas established in Theorems I and III would hold. We prove these formulas independently of such Bertini statements (and without invoking splayedness); as we pointed out in the introduction, a simple-minded 'splayed Bertini' statement in fact does not hold (Example 1.3).

Our main tool is again a formula for Segre classes, which we reproduce here for the convenience of the reader.

Lemma 4.1 *Let $Z \subseteq W$ be schemes and let D be a Cartier divisor in W, meeting properly the support of every component of the normal cone of Z in W. Then*

$$s(D \cap Z, D) = D \cdot s(Z, W). \tag{4.1}$$

Proof Under the hypothesis of this statement, the blow-up of D along $D \cap Z$ is the inverse image of D in the blow-up of W along Z. The statement follows then from the projection formula. ∎

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The formula for the Chern–Fulton class in Theorem IV follows easily. Indeed, if X is general and very ample, then it can be chosen to intersect properly the components of the normal cone of Y; further, X is nonsingular, so applying (4.1) and the definition of Chern–Fulton class,

$$c_{\mathrm{F}}(X\cap Y)=c(TX)\cap s(X\cap Y,X)=c(TX)\cap(X\cdot s(Y,V))=c(X)\cdot s(Y,V).$$

Thus,

$$c(TV)\cap c_{\mathrm{F}}(X\cap Y)=c(X)\cdot c(TV)\cap s(Y,V)=c(X)\cdot c_{\mathrm{F}}(Y),$$

as stated in Theorem IV.

For the proof of the corresponding statement about c_{SM} classes, after applying a Veronese embedding we may assume that $V\subseteq\mathbb{P}^n$ and X is a general hyperplane section. In this situation,

$$c_{\mathrm{SM}}(X\cap Y)=\frac{X}{1+X}\cdot c_{\mathrm{SM}}(Y). \tag{4.2}$$

This follows from [Alu13, Proposition 2.6]. (The proof of this proposition may be summarized as follows: by inclusion-exclusion it can be reduced to the case in which Y is a hypersurface. Using Lemma 2.14, the formula amounts then to a relation for Segre classes that ultimately depends again on Lemma 4.1.) From (4.2),

$$c(TV)\cap c_{\mathrm{SM}}(X\cap Y)=\Big(c(TV)\cap\frac{X}{1+X}\Big)\cdot c_{\mathrm{SM}}(Y)=c(X)\cdot c_{\mathrm{SM}}(Y),$$

completing the proof of Theorem IV. ■

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