ON SOME NON-LINEAR PROBLEMS

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Non-linear problems have been studied by Krasnoselski, Browder, and others; in fact Browder and independently Kirk (cf., **1**; **5**) have proved the following remarkable theorem: let X be a uniformly convex Banach space, U a non-expansive mapping of a bounded closed convex subset C of X into C, i.e., $||Ux - Uy|| \leqslant ||x - y||$ for $x, y \in C$; then U has a fixed point in C. The aim of this paper is to give some existence theorems for non-linear functional equations in uniformly convex Banach spaces. Similar results may be found in (**3**; **6**).

1. Let X, Y be Banach spaces and f: $X \to Y$ be a non-linear mapping; f is said to be *linearly upper bounded* if there exist numbers α, $\gamma > 0$ such that $||f(x)|| \leqslant \gamma||x||$ for $||x|| \geqslant \alpha$. If f: $X \to Y$ is a mapping such that

$$|f| = \inf_{0<\alpha<\infty} \left\{ \sup_{||x||\geqslant\alpha} ||f(x)||/||x|| \right\}$$

is finite, then f is linearly upper-bounded; the number $|f|$ is called the quasi-norm of f (cf., **3**, p. 63).

Let f be a mapping of an open subset V of X into Y and let $x_0 \in V$; if there exists a bounded linear operator S: $X \to Y$ such that

$$\lim_{t\to 0} \frac{f(x_0 + tx) - f(x_0)}{t} = S(x)$$

for every $x \in X$, we say that f has the Gateaux derivative **S** at x_0. If the convergence is uniform with respect to x with $||x|| = 1$, then f is said to be Fréchet differentiable at x_0.

THEOREM 1. *Let f: $X \to X$ be a mapping of a uniformly convex Banach space X into itself such that it has the Gateaux derivative $f'(x)$ for every $x \in X$. Let T: $X \to X$ be a linear mapping of X onto X having an inverse and let $F = I - Tf$, where I is the identity mapping of X. Assume further that* $\sup_{x\in X} ||F'(x)|| \leqslant 1$. *If $|F| < 1$, then the equation $f(x) = y$ has at least one solution for each y in X.*

Proof. By definition $F(x) = x - Tf(x)$ and $F(x)$ has the Gateaux derivative $F'(x)$ given by $F'(x) = I - Tf'(x)$; since $||F(x) - F(y)|| \leqslant ||F'(z)||\ ||x - y||$ for some z on the segment $[x, y]$ and $\sup_{x\in X} ||F'(x)|| \leqslant 1$, we see that F is a non-expansive mapping. Let y^* be an arbitrary point of X and let $T(y^*) = z^*$;

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the equation $f(x) = y^*$ is equivalent to $x - F(x) = z^*$. We prove that $f(x) = y^*$ has a solution x^* in X; to prove this, we define a mapping $\bar{F}: X \to X$ by $\bar{F}(x) = F(x) + z^*, x \in X$. Since $|F| < 1$, we have $||F(x)||/||x|| < \epsilon < 1$ for all x with $||x|| \geqslant \alpha_1$. Let $\delta > 0$ be such that $\epsilon + \delta < 1$ and let $\alpha_2 = ||z^*||/\delta$; put $\alpha = \alpha_1 + \alpha_2, B = \{x \mid ||x|| \leqslant \alpha\}$. Clearly B is bounded, closed, and convex and $||\bar{F}(x)|| \leqslant ||F(x)|| + ||z^*|| \leqslant (\epsilon + \delta)||x|| < ||x||$ for $x \in B$. Moreover,

$$||\bar{F}(x_1) - \bar{F}(x_2)|| \leqslant ||x_1 - x_2||$$

for every $x_1, x_2 \in B$; hence $\bar{F}$ has at least one fixed point $x^* \in B$ by the theorem of Browder–Kirk (cf., **1**; **5**). Thus $\bar{F}(x^*) = x^*$ or equivalently $f(x^*) = T^{-1}(z^*) = y^*$.

2. Following Krasnoselskii (**6**, p. 207), we say that a mapping $f: X \to Y$ is *asymptotically close* to a bounded linear operator $T: X \to Y$ if

$$\lim_{||x|| \to \infty} ||f(x) - T(x)||/||x|| = 0;$$

in particular if

$$\lim_{||x|| \to \infty} ||f(x)||/||x|| = 0,$$

we say that f is asymptotically close to zero.

THEOREM 2. *Let X be a uniformly convex Banach space; let T be a linear mapping with domain $D \subset X$ and onto X and suppose that T has a bounded inverse. Let f: $X \to X$ be a mapping asymptotically close to zero having the Gateaux derivative $f'(x)$ for every $x \in X$. If $\sup_{x \in X} ||T^{-1}f'(x)|| \leqslant 1$, then $(T + f)D = X$, i.e., the equation $T(x) + f(x) = y$, $x \in D$, has at least one solution for every $y \in X$.*

Proof. The equation $T(x) + f(x) = y$, $x \in D$, $y \in X$, is equivalent to $x + T^{-1}f(x) = T^{-1}(y)$. Since f is asymptotically close to zero, for any $\epsilon > 0$ there exists an $m > 0$ such that $||f(x)||/||x|| < \epsilon$ for $||x|| > m$. Thus.

$$0 \leqslant ||T^{-1}f(x)||/||x|| \leqslant ||T^{-1}||\ ||f(x)||/||x|| < \epsilon||T^{-1}||$$

for every $x \in X$ with $||x|| > m$ and hence $T^{-1}f$ is asymptotically close to zero. We proceed now as in Theorem 1.

Next, we prove the following.

THEOREM 3. *Let X be a uniformly convex Banach space; let $f: X \to X$ be a mapping such that f has the Gateaux derivative for every $x \in X$, f is Frechet differentiable at* 0, *and* $f(0) = 0$. *Let $T: X \to X$ be a bounded linear mapping of X onto X having an inverse and let $F = I - Tf$, where I is the identity; assume further that $||F'(0)|| < 1$ and $\sup_{x \in X} ||F'(x)|| \leqslant 1$. Let $\epsilon > 0$ be such that $\epsilon < 1 - ||F'(0)||$; then there exists a $\delta > 0$ such that for any y satisfying $||y|| \leqslant \delta(1 - d)||T^{-1}||$, where $d = ||F'(0)|| + \epsilon$, the equation $f(x) = y$ has at least one solution in $B = \{x| \ ||x|| \leqslant \delta\}$.*

Proof. We have $F(x) = x - Tf(x)$, $x \in X$; clearly F is Gateaux differentiable on X, $F(0) = 0$, and $F'(x) = I - Tf'(x)$ for $x \in X$. Moreover, F is Fréchet differentiable at 0. Since $\sup_{x \in X} ||F'(x)|| \leqslant 1$, F is non-expansive on X. Let $\epsilon > 0$ be such that $\epsilon < 1 - ||F'(0)||$; then there exists an $\eta > 0$ such that

$$F(x) - F(0) = F'(0)x + \omega(0, x),$$

where $||\omega(0, x)|| < \epsilon||x||$ whenever $||x|| < \eta$. Thus

$$||F(x)|| \leqslant ||F'(0)x|| + ||\omega(0, x)|| < ||F'(0)||\,||x|| + \epsilon||x|| \leqslant d\delta$$

for $||x|| \leqslant \delta$, where $d = ||F'(0)|| + \epsilon$. Define $\bar{F}: X \to X$ by $\bar{F}(x) = F(x) + T(y^*)$ where y^* is a fixed element with $||y^*|| \leqslant \delta(1 - d)||T||^{-1}$. We have

$$||\bar{F}(x)|| \leqslant ||F(x)|| + ||T(y^*)|| \leqslant d\delta + \delta(1 - d) = \delta$$

and

$$||\bar{F}(x_1) - \bar{F}(x_2)|| \leqslant ||x_1 - x_2||$$

for x_1, x_2 in $B = \{x|\ ||x|| \leqslant \delta\}$. Thus, by the theorem of Browder–Kirk **(1; 5)**, $\bar{F}$ has a fixed point in B; hence $f(x^*) = y^*$, as required.

Finally we prove the following result.

THEOREM 4. *Let f be a weakly continuous mapping of a reflexive Banach space X into itself such that f has the Frechet derivative at 0 and $f(0) = 0$; let T be a bounded linear mapping of X onto itself having an inverse. Assume that*

$$||F'(0)|| < 1$$

where $F = I - Tf$; then for any positive $\epsilon < 1 - ||F'(0)||$, there exists a $\delta > 0$ such that, for any $y \in X$ satisfying

$$||y|| \leqslant \delta(1 - (||F'(0)|| + \epsilon))||T||^{-1},$$

the equation $f(x) = y$ has at least one solution in $B = \{x|\ ||x|| \leqslant \delta\}$.

Proof. $F(x) = x - Tf(x)$, $F(0) = 0$, and F is Frechet differentiable at 0. Let $\epsilon > 0$ be such that $\epsilon < 1 - ||F'(0)||$; then there exists an $\eta > 0$ such that

$$F(x) = F'(0)x + \omega(0, x),$$

where $||\omega(0, x)|| < \epsilon||x||$ whenever $||x|| < \eta$. Thus if $0 < \epsilon < \eta$, then

$$||F(x)|| < (||F'(0)|| + \epsilon)\delta$$

for $x \in B$. Let $\bar{F}: X \to X$ be defined by $\bar{F}(x) = F(x) + T(y)$ where y is a fixed element with

$$||y|| \leqslant \delta(1 - (||F'(0)|| + \epsilon))||T||^{-1}.$$

For every $x \in B$, we have

$$||\bar{F}(x)|| \leqslant ||F(x)|| + ||T(y)|| < \delta;$$

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consequently $\bar{F}$ is a weakly continuous mapping of B into itself. Since X is reflexive, every bounded set is weakly relatively compact; moreover, every closed convex set is weakly closed **(4)**. Hence by Schauder's fixed point theorem there exists at least one $x^* \in B$ such that $\bar{F}(x^*) = x^*$; consequently, we have $f(x^*) = y$.

Remarks. (i) It is not clear to the author if the condition "$\sup_{x\in X}$ ( ) $\leqslant 1$" in Theorems 1, 2, and 3 can be replaced by some other weaker condition. (ii) It will be very *interesting to know* whether the fixed point theorem of Browder–Kirk can be proved for strictly convex reflexive Banach spaces. It may be pointed out that some very interesting results connected with this conjecture have been obtained by M. Edelstein **(2)**.

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