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NON-SYMMETRIC ORNSTEIN-UHLENBECK PROCESSES IN BANACH SPACE VIA DIRICHLET FORMS

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ABSTRACT We use recent advances in the theory of non-symmetric Dirichlet forms to study a class of Banach space valued Ornstein-Uhlenbeck processes As an example, we look at Walsh's stochastic model of neural response and show that it is a continuous process in any Sobolev space $H^{\alpha}(\alpha < 1/2)$, and that it takes values only among functions with unbounded variation

1. **Introduction.** The theory of Dirichlet forms provides analytic tools that help us understand the connection between Markov processes and potential theory. This connection is a two way street; we can use the process to study the potential theory of the form, or we can use the potential theory of the form to study the process. It is the latter direction that we will travel in this paper.

The classical theory of Dirichlet forms, as described for instance in the fundamental works by Fukushima [3] and Silverstein [7], concerns symmetric forms over locally compact state spaces. Unfortunately, the conditions of symmetry and local compactness exclude many interesting examples in Banach spaces and spaces of distributions. Some authors got around these difficulties in particular situations, but it is only recently that a complete theory of non-symmetric Dirichlet forms over more general topological spaces has emerged. Two references on this more general theory are the recent book by Bouleau and Hirsch [2] and the book by Ma and Röckner [6] that will soon appear.

In this paper we show how the well-known and well studied Ornstein-Uhlenbeck process can be fit into the Dirichlet form framework. We will apply the theory of non-symmetric Dirichlet forms to the construction and study of the infinite dimensional Ornstein-Uhlenbeck process X which solves

$$dX = -AX\,dt + dW, \tag{1.1}$$

where A is an operator on a Hilbert space H, and W is the white noise with covariance operator given by the inner product on H. In Sections 3 and 4, we apply the Dirichlet form approach to a specific example. This example is Walsh's stochastic model of neural response [12] and we show that it is a continuous process whose values are functions in L^2 that are not of bounded variation. This extends the results in [9] where these facts are proved in the symmetric case.

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2. **The Dirichlet form associated with an Ornstein-Uhlenbeck process.** In this section we will find the Dirichlet form $\mathcal{E}$ associated with the process X that solves equation (1.1). Here is the outline of our strategy. The two ingredients that are intrinsic to equation (1.1) are the drift operator A and the Hilbert space H. Starting from these two we will first find the invariant measure μ. This measure lives on a Banach space that will eventually serve as the state space for our process. Using μ and the operator A we can then define $\mathcal{E}$, the form associated with X. Finally we will show that $\mathcal{E}$ is closable and that its closure has the Markovian property and the local property.

We begin with a real, separable Hilbert space H whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. Let $\big(A, \mathcal{D}(A)\big)$ be a densely defined operator on H such that $-A^*$ generates a strongly continuous semigroup $\{e^{-tA^*}\}$. We assume that for some constants $c_1, c_2 > 0$ we have

$$\|e^{-tA^*}\|_{\mathcal{L}(H;H)} \leq c_1 e^{-c_2 t} \quad \forall t \geq 0. \tag{2.1}$$

It follows that $T := \int_0^\infty e^{-tA} e^{-tA^*}\, dt$ is a continuous, positive definite, self-adjoint operator and that the corresponding bilinear form

$$(h,k) := \langle h, Tk\rangle = \int_0^\infty \langle e^{-tA^*}h, e^{-tA^*}k\rangle\, dt \tag{2.2}$$

is positive definite and continuous on H. We note that in the case when A is symmetric, then T reduces to $\int_0^\infty e^{-2tA}\, dt = (1/2)A^{-1}$. We now use the bilinear form (2.2) to define the invariant measure μ. Let E be a real separable Banach space that includes H densely and continuously. By duality we have

$$E^* \hookrightarrow H^* \approx H \hookrightarrow E, \tag{2.3}$$

which means $\ell(z) = \langle \ell, z\rangle$ for $\ell \in E^*$ and $z \in H \subseteq E$. We suppose that E is large enough to support the measure μ, that is, there exists a mean zero Gaussian measure μ on $(E, \mathcal{B})$ so that

$$\int_E h(z)k(z)\mu(dz) = (h,k) \tag{2.4}$$

for all $h, k \in E^*$. We can always do this, for example by letting E be the completion of H with respect to a measurable norm [4]; but often, as in the next section, it is more convenient to work on an even smaller space. For any $h \in H$, let h_n be a sequence in E^* that converges to h in H-norm. Then $\{h_n(\cdot)\}$ converges in $L^2(E;\mu)$ to a member of $L^2(\mu)$ that we will call X_h. The collection $\{X_h\}_{h\in H}$ is jointly mean zero Gaussian with

$$\operatorname{cov}(X_h, X_k) = (h,k). \tag{2.5}$$

Before we can define $\mathcal{E}$ we need the following lemma on the bilinear form in (2.2).

LEMMA 2.1. *For $h, k \in \mathcal{D}(A^*)$ we have*

$$(A^*h, k) + (h, A^*k) = \langle h, k\rangle. \tag{2.6}$$

PROOF. From semigroup theory we know that if $h \in \mathcal{D}(A^*)$, then the H-valued function $t \longrightarrow e^{-tA^*}h$ is differentiable and its derivative is

$$\frac{d}{dt}e^{-tA^*}h = -A^*e^{-tA^*}h = -e^{-tA^*}A^*h. \tag{2.7}$$

Let $h, k \in \mathcal{D}(A^*)$ and define the real valued function $f(t) = \langle e^{-tA^*}h, e^{-tA^*}k\rangle$. Then f is differentiable and by (2.1) it goes to zero as $t \longrightarrow \infty$. Thus $f(0) = -\int_0^\infty f'(t)\,dt$, in other words

$$\begin{aligned}\langle h, k\rangle &= \int_0^\infty \langle e^{-tA^*}A^*h, e^{-tA^*}k\rangle + \langle e^{-tA^*}h, e^{-tA^*}A^*k\rangle\,dt\\ &= (A^*h, k) + (h, A^*k).\end{aligned} \tag{2.8}$$

■

Because we want to apply the theory of non-symmetric (sectorial) Dirichlet forms we will have to assume that the bilinear form (A^*h, k) defined on $\mathcal{D}(A^*)$ is sectorial. This means that for some $M > 0$, we have for all $h, k \in \mathcal{D}(A^*)$

$$\begin{aligned}|(A^*h, k)| &\le M(A^*h, h)^{1/2}(A^*k, k)^{1/2}\\ &= (M/2)\|h\|\,\|k\|.\end{aligned} \tag{2.9}$$

This tells us that the form (A^*h, k) has a continuous extension to all of H. We will denote this extension by (A^*h, k) also.

We define a subspace of $L^2(E; \mu)$ by

$$\begin{aligned}\mathcal{F}C_b^\infty = \big\{u: E \longrightarrow \mathbb{R} \mid u(z) = \phi\big(\ell_1(z), \ldots, \ell_n(z)\big) \text{ for some}\\ \phi \in C_b^\infty(\mathbb{R}^n) \text{ and } \ell_i \in E^*\backslash\{0\} \text{ for } i = 1, \ldots, n\big\},\end{aligned} \tag{2.10}$$

where $C_b^\infty(\mathbb{R}^n)$ is the space of all real functions on $\mathbb{R}^n$ that are bounded and have bounded partial derivatives of all orders. Each function is $\mathcal{F}C_b^\infty$ is $\sigma(E, E^*)$-continuous, and so measurable, and therefore can be regarded as an element in $L^2(E; \mu)$. Since μ charges every non-empty $\sigma(E, E^*)$-open set we see that if $u, v \in \mathcal{F}C_b^\infty$ and $u = v$ μ-a.e., then $u = v$ everywhere.

Now for every $u \in \mathcal{F}C_b^\infty$ we define a gradient function $\nabla u: E \longrightarrow E^*$ by

$$(\nabla u)(z) = \sum_{i=1}^n (\partial_i\phi)\big(\ell_1(z), \ldots, \ell_n(z)\big)\ell_i. \tag{2.11}$$

Although the representation of u is not unique, $(\nabla u)(z)$ is well-defined as we have

$$_{E^*}\langle(\nabla u)(z), k\rangle_E = \Big(\frac{\partial}{\partial k}u\Big)(z) \quad z, k \in E. \tag{2.12}$$

where

$$\Big(\frac{\partial}{\partial k}u\Big)(z) = \frac{d}{ds}u(z+sk)|_{s=0} \quad z \in E. \tag{2.13}$$

The partial derivative in (2.13) is representation independent.

Finally we define the form $\mathcal{E}$ on $L^2(\mu)$ by

$$\begin{gathered}\mathcal{D}(\mathcal{E}) = \mathcal{F}C_b^\infty \\ \mathcal{E}(u,v) = \int_E (A^*\nabla u, \nabla v)\,d\mu.\end{gathered} \tag{2.14}$$

This is a densely defined, positive, sectorial form on $L^2(\mu)$, these qualities being inherited from the bilinear form (A^*h,k).

We now need to show that $\mathcal{E}$ is closable, keeping in mind that it suffices to show the closability of its symmetric part $\tilde{\mathcal{E}}(u,v) = 1/2\int\langle\nabla u,\nabla v\rangle\,d\mu$. In order to show that $\tilde{\mathcal{E}}$ is closable we decompose it into closable parts. For $k \in E\setminus\{0\}$, define the form $\mathcal{E}_k$ by

$$\begin{gathered}\mathcal{D}(\mathcal{E}_k) = \mathcal{F}C_b^\infty \\ \mathcal{E}_k(u,v) = 1/2\int\Big(\frac{\partial}{\partial k}u\Big)\Big(\frac{\partial}{\partial k}v\Big)\,d\mu.\end{gathered} \tag{2.15}$$

Now according to Theorem 2.8 of [1], the closability of (2.15) follows if we have integration by parts in the k-direction. That is, if there exists $\hat{\beta}_k \in L^2(\mu)$ so

$$\int\Big(\frac{\partial}{\partial k}u\Big)\,d\mu = -\int u\hat{\beta}_k\,d\mu \tag{2.16}$$

for all $u \in \mathcal{F}C_b^\infty$, then $\mathcal{E}_k$ is closable.

LEMMA 2.2. *If $k \in \mathrm{Range}(T)$ where $T = \int_0^\infty e^{-tA}e^{-tA^*}\,dt$, then (2.16) holds, so the form $\mathcal{E}_k$ in (2.15) is closable.*

PROOF. Let $u \in \mathcal{F}C_b^\infty$ be given by $u(z) = \phi\big(\ell_1(z),\ldots,\ell_n(z)\big)$ where $\ell_1,\ldots,\ell_n$ are linearly independent in E^*, and $\phi \in C_b^\infty(\mathbb{R}^n)$. The random vector $(\ell_1,\ldots,\ell_n)$ has a mean-zero Gaussian distribution (under μ) with a non-singular covariance matrix Σ and hence a density on $\mathbb{R}^n$ given by

$$\varphi(x) = (2\pi)^{-n/2}|\Sigma|^{-1/2}\exp\Big(-\frac{1}{2}x'\Sigma^{-1}x\Big). \tag{2.17}$$

By change of variables, we get

$$\begin{aligned}\int\Big(\frac{\partial}{\partial k}u\Big)\,d\mu &= \int\sum_{\iota=1}^n(\partial_\iota\phi)\big(\ell_1(z),\ldots,\ell_n(z)\big)\ell_\iota(k)\mu(dz) \\ &= \int_{\mathbb{R}^n}\sum_{\iota=1}^n(\partial_\iota\phi)(x)\ell_\iota(k)\varphi(x)\,dx \\ &= \int_{\mathbb{R}^n}\sum_{\iota=1}^n\phi(x)\ell_\iota(k)(-\partial_\iota\varphi)(x)\,dx \\ &= \int_{\mathbb{R}^n}\phi(x)\sum_{\iota=1}^n(\Sigma^{-1}x)_\iota\ell_\iota(k)\varphi(x)\,dx \\ &= \int_E\phi(\ell_1,\ldots,\ell_n)\sum_{\iota=1}^n\big(\Sigma^{-1}(\ell_1,\ldots,\ell_n)\big)_\iota\ell_\iota(k)\,d\mu.\end{aligned} \tag{2.18}$$

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Now since $k \in \mathrm{Range}(T)$ we have $k = Th$ for some $h \in H$. From (2.2) we get $\ell_i(k) = \langle \ell_i, k\rangle = (\ell_i, h)$.

When (x, y) are mean-zero jointly Gaussian vectors we have $E(y|x) = \Sigma_{21}\Sigma_{11}^{-1}x$, where Σ_{11} is the covariance matrix of x, and $(\Sigma_{21})_{ij} = \mathrm{cov}(y_i, x_j)$. Therefore on $(E, \mathcal{B}, \mu)$ we have

$$E(X_h|\ell_1,\ldots,\ell_n) = \sum_{i=1}^{n}\Big(\Sigma^{-1}(\ell_1,\ldots,\ell_n)\Big)_i(\ell_i, h), \tag{2.19}$$

so that (2.18) may be rewritten as

$$\int\Big(\frac{\partial}{\partial k}u\Big)\,d\mu = \int uX_h\,d\mu.$$

This gives (2.16) and so $\mathcal{E}_k$ is closable. ∎

Since $\mathrm{Range}(T)$ is dense in H we may find an orthonormal basis $\{k_1, k_2, \ldots\}$ for H in $\mathrm{Range}(T)$. Then for $u, v \in \mathcal{F}C_b^\infty$

$$\begin{aligned}\int\langle\nabla u,\nabla v\rangle\,d\mu &= \int\sum_{i=1}^{\infty}\langle\nabla u, k_i\rangle\langle k_i,\nabla v\rangle\,d\mu\\ &= \sum_{i=1}^{\infty}\int\Big(\frac{\partial}{\partial k_i}u\Big)\Big(\frac{\partial}{\partial k_i}v\Big)\,d\mu.\end{aligned} \tag{2.20}$$

Since the sum of closable forms is closable, we conclude that $\tilde{\mathcal{E}}$, and hence $\mathcal{E}$ in (2.14), is a closable form. For convenience we let $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ denote the closure also. We note that there are a lot of functions in $\mathcal{D}(\mathcal{E})$ that are not in $\mathcal{F}C_b^\infty$. Suppose $\phi \in C(\mathbb{R}^n)$ and that ϕ and all of its first order partial derivatives are bounded by a polynomial. Let $h_i \in H\backslash\{0\}$ for $i = 1, \ldots, n$ and set $u = \phi(X_{h_1}, \ldots, X_{h_n})$.

Then $u \in \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}(u) = \frac{1}{2}\int\|\nabla u\|^2\,d\mu, \tag{2.21}$$

where $\nabla u := \sum_{i=1}^n(\partial_i\phi)(X_{h_1},\ldots,X_{h_n})h_i$. Furthermore, if the h_i's are orthonormal and belong to $\mathcal{D}(A^*)$, and if we set $\Delta u := \sum_{i=1}^n(\partial_i^2\phi)(X_{h_1},\ldots,X_{h_n})$, then a calculation similar to the one in Lemma 2 gives us

$$\mathcal{E}(u,v) = -\int\Big\{\frac{1}{2}\Delta u - \sum_{i=1}^{n}(\partial_i\phi)(X_{h_1},\ldots,X_{h_n})X_{A^*h_i}\Big\}v\,d\mu. \tag{2.22}$$

If, in addition, we have $A^*h_i \in E^*$ for all i, then this reduces to the more familiar

$$\mathcal{E}(u,v) = -\int\Big\{\frac{1}{2}\Delta u - {}_{E^*}\langle A^*\nabla u,\cdot\rangle_E\Big\}v\,d\mu. \tag{2.23}$$

You can even get away with functions ϕ that are not quite as smooth. In particular if $u, v \in \mathcal{D}(\mathcal{E})$, then $(u \vee v) \in \mathcal{D}(\mathcal{E})$ with

$$\nabla(u \vee v) = 1_{\{u>v\}}\nabla u + 1_{\{u<v\}}\nabla v + \frac{1}{2}1_{\{u=v\}}(\nabla u + \nabla v). \tag{2.24}$$

We will use this result (see also [6; Chapter 4, Lemma 4.1]) in our calculations in Sections 3 and 4. The original form (2.14) on $\mathcal{F}C_b^\infty$ admitted a square of field operator $\Gamma(u,v) := (A^*\nabla u, \nabla v)$, so that $\mathcal{E}(u,v) = \int \Gamma(u,v)\,d\mu$. This operator Γ can be extended to the closure $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ and for convenience we again denote the extension by Γ. The chain rule holds for Γ, so if $u, v \in \mathcal{D}(\mathcal{E})$ and $\phi, \psi \in C_b^1(\mathbb{R})$, then $\phi(u), \psi(v) \in \mathcal{D}(\mathcal{E})$ and

$$\Gamma\big(\phi(u), \psi(v)\big) = \phi'(u)\psi'(v)\Gamma(u,v)\ \mu\text{-a.e.} \tag{2.25}$$

Using this fact it is easy to show, as in [10], that $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ has the following two important properties.

DEFINITION. A form $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ is called *local* if, whenever $u, v \in \mathcal{D}(\mathcal{E})$ satisfy $uv = 0$ μ-a.e., then $\mathcal{E}(u,v) = 0$.

DEFINITION. A form $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ is called *Markovian* if $u \in \mathcal{D}(\mathcal{E})$ implies $u^+ \wedge 1 \in \mathcal{D}(\mathcal{E})$ with $\mathcal{E}(u^+ \wedge 1, u - u^+ \wedge 1) \geq 0$ and $\mathcal{E}(u - u^+ \wedge 1, u^+ \wedge 1) \geq 0$.

$\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ is a closed, sectorial bilinear form on $L^2(\mu)$ with the Markovian property. It is therefore a Dirichlet form, and it is also local. Furthermore, (2.14) combined with Proposition 3.1 of [8] shows that the capacity associated with $\mathcal{E}$ is tight on E.

Since $\mathcal{F}C_b^\infty$ (see (2.10)) is an $\tilde{\mathcal{E}}_1$-dense set of continuous functions that separates points in E, we conclude that $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ on $L^2(E;\mu)$ is quasi-regular in the sense of Ma and Röckner [6; Chapter 4, Definition 3.1]. Consequently we can use their existence result [6; Chapter 4, Theorem 3.5] to obtain an E-valued diffusion

$$\Big(\Omega, \mathcal{F}, \big(X(t)\big)_{t\geq 0}, (P_z)_{z\in E}\Big)$$

associated with $\mathcal{E}$. This is our Ornstein-Uhlenbeck process and it is a weak solution to (1.1) in the sense of [1], to which we refer the interested reader for further details.

3. **Walsh's stochastic model of neural response.**

NOTATION. In Sections 3 and 4 the unlabelled symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will refer to the norm and inner product in $H = L^2\big([0,L]; dx/\eta(x)\big)$. Occasionally we will also use the norm and inner product of $L^2([0,L]; dx)$ and these will be labelled $\|\cdot\|_{L^2(dx)}$ and $\langle\cdot,\cdot\rangle_{L^2(dx)}$. Finally, c will stand for a generic positive constant whose value may change from line to line.

In [12] Walsh proposed a model for a nerve cylinder undergoing random stimulus along its length. The cylinder itself is regarded as the interval $[0,L]$ and $\{X(x,t,\omega) : 0 \leq x \leq L, t \geq 0, \omega \in \Omega\}$ denotes the value of the nerve membrane potential at time t at a location x along the axis. He found that this potential could be approximated by the solution of the stochastic differential equation

$$dX = \Big(\frac{\partial^2 X}{\partial x^2} - X\Big)\,dt + dW \tag{3.1}$$

where the Laplacian $\partial^2/\partial x^2$ is given reflecting boundary conditions at the endpoints 0 and L. Here W is a white noise on $\mathbb{R}_+ \times [0, L]$ based on the measure $\eta(dx)\,dt$, where η models the intensity of the random stimulation acting along the nerve cylinder.

On the space $L^2([0, L]; dx)$, the operator A is self-adjoint and strictly positive definite. This operator has an orthonormal basis $\{e_j\}$ of eigenvectors, namely

$$e_0(x) \equiv L^{-1/2} \text{ and } e_j(x) = 2^{1/2}L^{-1/2}\cos(\pi jxL^{-1}), \quad j \geq 1 \tag{3.2}$$
$$Ae_j = \lambda_j e_j = (1 + \pi^2 j^2 L^{-2})e_j.$$

The operator A generates a strongly continuous semigroup $\{e^{-tA}\}$ and it also generates a closed bilinear form ε. The operator and form can be written explicitly as

$$\mathcal{D}(\varepsilon) = \{f \in L^2 : f' \in L^2\} \quad \varepsilon(f, g) = \int f'g'\,dx + \int fg\,dx \tag{3.3}$$
$$\mathcal{D}(A) = \{f \in L^2 : f'' \in L^2, f'(0) = f'(L) = 0\} \quad Af = f - f''.$$

For $f \in \mathcal{D}(A)$ and $g \in \mathcal{D}(\varepsilon)$ we have $\varepsilon(f, g) = \int Afg\,dx$. It is well-known that every function $f \in \mathcal{D}(\varepsilon)$ is absolutely continuous and satisfies $\|f\|_\infty^2 \leq c\varepsilon(f)$ for some constant $c > 0$. From this it follows that if $f, g \in \mathcal{D}(\varepsilon)$ then $fg \in \mathcal{D}(\varepsilon)$ and

$$\varepsilon(fg) \leq c\varepsilon(f)\varepsilon(g). \tag{3.4}$$

We can now state the assumptions we need in order to apply Dirichlet form techniques to this example. We assume that the measure η is absolutely continuous with $\eta(dx) = \eta(x)\,dx$, where $\eta \in \mathcal{D}(\varepsilon)$ and η is bounded away from zero. To fit Walsh's example into the framework of the previous section we must take

$$A = I - \partial^2/\partial x^2 \text{ and } H = L^2\big([0, L]; dx/\eta(x)\big). \tag{3.5}$$

We see that when we choose a non-constant function η, we really use the same operator A on the same space H, but that H is equipped with a different but equivalent norm. Under this new norm the operators A and e^{-tA} are no longer symmetric, but instead we have

$$e^{-tA^*}f = \eta e^{-tA}(f/\eta) \text{ and} \tag{3.6}$$
$$A^*e^{-tA^*}f = \eta A e^{-tA}(f/\eta).$$

In particular, (2.1) is fulfilled and we have from (2.2), (2.4), (3.2), (3.5) and (3.6) that the invariant measure μ lives on the space H. To see this, we calculate

$$\begin{aligned}
\int \|z\|^2_{L^2(dx)}\mu(dz) &= \int \sum \langle e_j, z\rangle^2_{L^2(dx)}\mu(dz) \\
&= \int \sum \langle \eta e_j, z\rangle^2 \mu(dz) \\
&= \sum (\eta e_j, \eta e_j) \\
&= \sum \int_0^\infty \|e^{-tA^*}(\eta e_j)\|^2\,dt \\
&= \sum \int_0^\infty e^{-2t\lambda_j}\|\eta e_j\|^2\,dt \\
&\leq c \sum (1/\lambda_j) < \infty.
\end{aligned} \tag{3.7}$$

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Thus, in this section we don't need to introduce additional spaces E and E^*, we simply take $E = H$ and do everything directly on this space. The last requirement from Section 2 that we have to fulfill is (2.9), that is, we must to show that the bilinear form (A^*h, k) is continuous on H. To see this, we first note that from (3.4) and (3.6) we have for $f, g \in H$,

$$\begin{aligned}\langle A^* e^{-tA^*} f, e^{-tA^*} g\rangle &= \int A e^{-tA}(f/\eta) e^{-tA}(g/\eta)\eta \, dx \\ &= \varepsilon\big(e^{-tA}(f/\eta), \eta e^{-tA}(g/\eta)\big) \\ &\leq c\varepsilon\big(e^{-tA}(f/\eta)\big)^{1/2}\varepsilon\big(e^{-tA}(g/\eta)\big)^{1/2}\end{aligned} \tag{3.8}$$

Now

$$\begin{aligned}\varepsilon\big(e^{-tA}(f/\eta)\big) &= \int A e^{-tA}(f/\eta) e^{-tA}(f/\eta)\, dx \\ &= \int A e^{-2tA}(f/\eta)(f/\eta)\, dx,\end{aligned} \tag{3.9}$$

so that $\int_0^\infty \varepsilon\big(e^{-tA}(f/\eta)\big)\, dt = \int A(\frac{1}{2}A^{-1})(f/\eta)(f/\eta)\, dx = \frac{1}{2}\|f/\eta\|^2_{L^2(dx)}$.

Finally we have,

$$\begin{aligned}|(A^*f, g)| &= \Big|\int_0^\infty \langle A^* e^{-tA^*} f, e^{-tA^*} g\rangle\, dt\Big| \\ &\leq c\int_0^\infty \varepsilon\big(e^{-tA}(f/\eta)\big)^{1/2}\varepsilon\big(e^{-tA}(g/\eta)\big)^{1/2}\, dt \\ &\leq c\Big[\int_0^\infty \varepsilon\big(e^{-tA}(f/\eta)\big)\, dt\Big]^{1/2}\Big[\int_0^\infty \varepsilon\big(e^{-tA}(g/\eta)\big)\, dt\Big]^{1/2} \\ &= c\|f/\eta\|_{L^2(dx)}\|g/\eta\|_{L^2(dx)} \\ &\leq c\|f\|\,\|g\|.\end{aligned} \tag{3.10}$$

We have satisfied the requirements (2.1) and (2.9) and so the form

$$\begin{aligned}\mathcal{D}(\mathcal{E}) &= \mathcal{F}C_b^\infty \\ \mathcal{E}(u, v) &= \int_H (A^*\nabla u, \nabla v)\, d\mu,\end{aligned} \tag{3.11}$$

is a densely defined, positive definite, sectorial, closable form on $L^2(H;\mu)$. We will denote its closure by $\big(\mathcal{E}, \mathcal{D}(\mathcal{E})\big)$ also. The form $\mathcal{E}$ has the local and Markovian properties, and is the Dirichlet form associated with the weak solution of the equation (3.1). By the results in Section 2 we know that this process

$$\Big(\Omega, \mathcal{F}, \big(X(t)\big)_{t\geq 0}, (P_z)_{z\in H}\Big) \tag{3.12}$$

is an H-valued diffusion, that is,

$$P_z\big(t \longrightarrow X(t) \text{ is } H\text{-continuous}\big) = 1 \quad \mu\text{-a.e.} \quad z \in H. \tag{3.13}$$

4. **Further sample path properties of Walsh's process.** In the previous section we have shown how the theory of Dirichlet forms can be used to construct a function-valued stochastic process $X(t)$ which solves (3.1) and is associated with the form $\mathcal{E}$ in (3.11). So far, we know that $X(t)$ is an H-valued diffusion, that is,

$$P_\mu\Big(t \longrightarrow X(t) \text{ is continuous in } L^2\Big) = 1 \tag{4.1}$$

So we know that the functions $x \longmapsto X(x,t,\omega)$ are square integrable, what more can we say about them? How can we use the form $\mathcal{E}$ to obtain further information on these functions?

We point out that in his original paper Walsh already proved, using Gaussian techniques, that $(t,x) \longmapsto X(x,t,\omega)$ is jointly continuous with probability one, and even gave a modulus of continuity [12; Corollary 4.3]. So we do not consider continuity, but look at other properties of $X(t)$. In fact, we will find that the properties which can be most profitably analyzed using $\mathcal{E}$, are those which can be described in terms of the coefficients in the cosine expansion of $X(t)$. Before we continue then, let us look more closely at the cosine expansion of a randomly selected element $z \in H = L^2\Big([0,L]; dx/\eta(x)\Big)$. Recall that we are working on the measure space (H,μ) where μ is the mean zero Gaussian measure with covariance given by (2.4). Consider the sequence of random variables $\{\langle e_j, z\rangle_{L^2(dx)}; j = 0,1,2,\ldots\}$, where $e_0(x) = L^{-1/2}$ and for $j \geq 1$ we have $e_j(x) = 2^{1/2}L^{-1/2}\cos(\pi j x L^{-1})$ (see (3.2)). This sequence is mean zero Gaussian with covariance

$$\begin{aligned}\int \langle e_i, z\rangle_{L^2(dx)}\langle e_j, z\rangle_{L^2(dx)}\mu(dz) &= \int \langle \eta e_i, z\rangle\langle \eta e_j, z\rangle \mu(dz)\\ &= (\eta e_i, \eta e_j)\\ &= \int_0^\infty \langle e^{-tA^*}(\eta e_i), e^{-tA^*}(\eta e_j)\rangle\, dt\\ &= \int_0^\infty e^{-t(\lambda_i+\lambda_j)}\langle \eta e_i, \eta e_j\rangle\, dt\\ &= \int \eta(x)e_i(x)e_j(x)\,dx/(\lambda_i+\lambda_j).\end{aligned} \tag{4.2}$$

Using the addition law for cosines,

$$\cos(a)\cos(b) = \frac{1}{2}\{\cos(a+b) + \cos(|a-b|)\},$$

we obtain $|\int \eta e_i e_j| \leq c|\eta_{i+j} + \eta_{|i-j|}|$, where we define $\{\eta_j\}$ to the coefficients in the cosine expansion of the function η, *i.e.*,

$$\eta_j := \int \eta(x)e_j(x)\,dx, \quad j \geq 0. \tag{4.3}$$

We have assumed that η is sufficiently smooth, namely that $\eta \in \mathcal{D}(\varepsilon)$ (see (3.3)), so that $\sum_{j=0}^\infty |\eta_j| < \infty$ [5; p. 33]. We can now get a bound on the correlations,

$$\begin{aligned}|\operatorname{corr}(\langle e_i,\cdot\rangle_{L^2(dx)}, \langle e_j,\cdot\rangle_{L^2(dx)})| &\leq c\left|\frac{\eta_{i+j}+\eta_{|i-j|}}{2(\lambda_i+\lambda_j)}\right|\left(\frac{2\lambda_i}{\int \eta e_i^2}\right)^{\frac12}\left(\frac{2\lambda_j}{\int \eta e_j^2}\right)^{\frac12}\\ &\leq c|\eta_{i+j}+\eta_{|i-j|}|.\end{aligned} \tag{4.4}$$

Equation (4.1) told us that for every t, $X(t,x)$ is a square integrable function of x. Let us now look at Sobolev spaces. For $0 \leq \alpha \leq 1$, define the space

$$H^\alpha = \Big\{ z \in L^2 : \sum_{j=0}^{\infty} \lambda_j^\alpha \langle e_j, z\rangle_{L^2(dx)}^2 := \|z\|_\alpha^2 < \infty \Big\}. \tag{4.5}$$

Calculating as in (3.7) we find

$$c_1 \Sigma(1/\lambda_j)\lambda_j^\alpha \leq \int \|z\|_\alpha^2 \mu(dz) \leq c_2 \Sigma(1/\lambda_j)\lambda_j^\alpha, \tag{4.6}$$

for some positive constants c_1, c_2. We know that $\lambda_j \sim cj^2$ (see (3.2)) and so conclude that $\mu(H^{1/2}) = 0$ but $\mu(H^\alpha) = 1$ for $\alpha < \frac{1}{2}$. In addition, it can be shown [11] that the capacity associated with $\mathcal{E}$ is tight on H^α for $\alpha < 1/2$ and so

$$P_\mu\Big(t \longrightarrow X(t) \text{ is continuous in } H^\alpha\Big) = 1. \tag{4.7}$$

What about $H^{1/2}$? We know that

$$P_\mu\Big(X(t) \in H^{1/2}\Big) = \mu(H^{1/2}) = 0, \quad \text{for all } t \geq 0 \tag{4.8}$$

but we'd like to draw the stronger conclusion that the process $X(t)$ fails to enter the space $H^{1/2}$ even at exceptional times, *i.e.*,

$$P_\mu\Big(X(t) \in H^{1/2} \text{ for some } t \geq 0\Big) = 0. \tag{4.9}$$

Unfortunately, at the present time, we are only able to prove it in the symmetric case, *i.e.*, when $\eta(x) \equiv d$ on $[0, L]$. This makes the random variables $\{\langle e_j, z\rangle_{L^2(dx)} : j \geq 0\}$ independent on $(H; \mu)$ by (4.2). However, this independence is only used to establish the bound in (4.14). If a substitute could be found for (4.14), then the rest of the argument would work even in the dependent case.

PROPOSITION 4.1. *If $\eta \equiv d$ for some constant d, then*

$$P_\mu\Big(\|X(t)\|_{1/2} = \infty \text{ for all } t\Big) = 1.$$

PROOF. Recall the definition of $H^{1/2}$,

$$H^{1/2} = \Big\{ z \in L^2 : \|z\|_{1/2}^2 = \sum_{j=0}^{\infty} \lambda_j^{1/2} \langle e_j, z\rangle_{L^2(dx)}^2 < \infty \Big\}. \tag{4.10}$$

For $n \geq 1$ define the continuous function u_n by

$$\begin{aligned} u_n(z) &= \Big(\sum_{j=0}^{n} \lambda_j^{1/2} \langle e_j, z\rangle_{L^2(dx)}^2\Big) \wedge N \\ &= \Big(\sum_{j=0}^{n} \lambda_j^{1/2} \langle \eta e_j, z\rangle^2\Big) \wedge N. \end{aligned} \tag{4.11}$$

Then $u_n \in \mathcal{D}(\mathcal{E})$ and

$$\nabla u_n(z) = \begin{cases} \sum_{j=0}^{n} 2\lambda_j^{1/2}\langle \eta e_j, z\rangle \eta e_j & \text{if } u_n(z) < N \\ 0 & \text{otherwise} \end{cases}$$

Since $|\langle \eta e_i, \eta e_j\rangle| \leq c|\eta_{i+j} + \eta_{|i-j|}|$ is a bounded sequence, we have on the set $\{z : u_n(z) < N\}$

$$\begin{aligned} \|\nabla u_n(z)\|^2 &= 4\sum_{i,j} \lambda_i^{1/2}\lambda_j^{1/2}\langle \eta e_i, z\rangle\langle \eta e_j, z\rangle\langle \eta e_i, \eta e_j\rangle \\ &\leq c(\Sigma 2\lambda_j^{1/2}|\langle \eta e_j, z\rangle|)^2 \\ &\leq c(\Sigma\lambda_j^{1/2})(\Sigma\lambda_j^{1/2}\langle \eta e_j, z\rangle^2) \\ &\leq c \cdot \Sigma_{j=0}^{n}(1+j)\cdot N \\ &= c[1+\cdots+(n+1)] \\ &\leq cn^2 \end{aligned} \tag{4.12}$$

Therefore

$$\begin{aligned} \mathcal{E}(u_n) &= \frac{1}{2}\int \|\nabla u_n(z)\|^2 \mu(dz) \\ &\leq cn^2 \mu\Big(\sum_{j=0}^{n} \lambda_j^{1/2}\langle e_j, z\rangle_{L^2(dx)}^2 \leq N\Big). \end{aligned} \tag{4.13}$$

Using Chebyshev's inequality we get, for any $t > 0$,

$$\begin{aligned} \mu\Big(\sum_{j=0}^{n} \lambda_j^{1/2}\langle e_j, z\rangle_{L^2(dx)}^2 \leq N\Big) &\leq e^{2tN} E\Big(\exp\Big(-2t\sum_{j=0}^{n}\lambda_j^{1/2}\langle e_j, z\rangle_{L^2(dx)}^2\Big)\Big) \\ &= e^{2tN}\prod_{j=0}^{n} E\big(\exp(-2t\lambda_j^{1/2}\langle e_j, z\rangle_{L^2(dx)}^2)\big). \end{aligned} \tag{4.14}$$

If Z is a standard normal random variable and $0 \leq a \leq 1$, then $E\big(\exp(-aZ^2)\big) = (1+2a)^{-1/2} \leq e^{-a/2}$. From (4.2) we know $E(\langle e_j, z\rangle_{L^2(dx)}^2) = \int \eta e_j^2/2\lambda_j$. Choose $T \geq 1$ so large that if $j \geq T$, then $t\lambda_j^{1/2}\int \eta e_j^2/2\lambda_j \leq 1$. Using (4.14) we get, for $n > T$,

$$\begin{aligned} \mathcal{E}(u_n) &\leq ce^{2tN}n^2\prod_{j=T}^{n}\exp\Big(-t\int \eta e_j^2\Big/2\sqrt{\lambda_j}\Big) \\ &= ce^{2tN}n^2\exp\Big(-t\sum_{j=T}^{n}\int \eta e_j^2\Big/2\sqrt{\lambda_j}\Big) \end{aligned} \tag{4.15}$$

Now $\int \eta e_j^2$ is bounded away from zero, and $\sqrt{\lambda_j} \leq cj$ so that for some constant k,

$$\sum_{j=T}^{n}\int \eta e_j^2/2\sqrt{\lambda_j} \geq k\sum_{j=T}^{n}(1/j) \geq k\log(n/T) \tag{4.16}$$

for large n, where k doesn't depend on t. Combining (4.15) and (4.16) yields

$$\mathcal{E}(u_n) \leq ce^{2tN}n^2(n/T)^{-kt}. \tag{4.17}$$

By choosing t so that $kt > 2$, we see that $\mathcal{E}(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, as $n \longrightarrow \infty$, u_n increases pointwise to the function

$$u(z) = \|z\|_{1/2}^2 \wedge N, \tag{4.18}$$

the convergence also taking place in $L^2(H;\mu)$. This means $u_n \longrightarrow u$ in $\mathcal{E}_1$-norm where $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(\mu)}$. Using the fact that u_n is a monotone sequence, along with [6; Chapter 3, Proposition 3.5] and [6; Chapter 4, Lemma 4.5], we obtain

$$P_\mu\Big(u_n\big(X(t)\big) \longrightarrow u\big(X(t)\big) \text{ uniformly on } [0,T] \text{ for all } T\Big) = 1, \tag{4.19}$$

and so

$$P_\mu\Big(t \longrightarrow u\big(X(t)\big) \text{ is continuous}\Big) = 1. \tag{4.20}$$

Since $u = N$ almost everywhere we have

$$P_\mu\Big(u\big(X(t)\big) = N\Big) = \mu(u = N) = 1 \text{ for all } t, \tag{4.21}$$

which combines with (4.20) to give

$$P_\mu\Big(u\big(X(t)\big) = N \text{ for all } t\Big) = 1. \tag{4.22}$$

Letting $N \longrightarrow \infty$ we conclude

$$P_\mu\Big(\|X(t)\|_{1/2}^2 = \infty \text{ for all } t\Big) = 1. \tag{4.23}$$

∎

Two other properties of a function that can be related to its cosine expansion are Hölder continuity and bounded variation. In fact, a slight modification of [5; Chapter 1, Theorem 4.5] and [5; Chapter 1, Section 6.3], where the topology of the unit circle is used, gives the following.

LEMMA 4.2. *For $z \in L^2([0,L];dx)$, let $z_j = \langle e_j, z\rangle_{L^2(dx)}$ for $j \geq 0$.*
(i) If z is of bounded variation, then $|z_j| = O(1/j)$.
(ii) If z is Hölder continuous with coefficient $\alpha > \frac{1}{2}$, then $\Sigma|z_j| < \infty$.

To finish this section we would like to show that

$$\operatorname{Cap}\big(z : |z_j| = O(1/j)\big) = 0 \tag{4.24}$$

and

$$\operatorname{Cap}(z : \Sigma|z_j| < \infty) = 0 \tag{4.25}$$

and so conclude that, with probability one and at all times t, the function $x \longmapsto X(x,t,\omega)$ is of unbounded variation and is not Hölder continuous for $\alpha > \frac{1}{2}$. These complement Walsh's fixed time result [12; Proposition 6.1] which says that, as a function of x, $X(t,x)$ looks like a Brownian motion path plus a C^2-function. In proving (4.25) we must, at the present time, restrict ourselves to the symmetric case (as indeed Walsh did in his fixed time result, see [12; Proposition 6.1, p. 253]). The proof is basically the same as in showing $\operatorname{Cap}(H^{1/2}) = 0$ in Proposition 4.1 so we omit it. Even without the symmetry assumption we can prove (4.24) by using the following lemma which says that if a sequence of Gaussian random variables is weakly correlated, then it behaves much like an independent sequence.

LEMMA 4.3. *Let φ_n be the density on $\mathbb{R}^n$ for a mean-zero Gaussian measure with variances σ_i^2 for $i = 1, \ldots, n$ and correlation satisfying*

$$|\rho_{ij}| \le (1/4)(c_{i+j} + c_{|i-j|}), \tag{4.26}$$

where $\{c_j\}_{j=1}^{\infty}$ is a sequence of positive constants with $\Sigma c_j < 1$. Then

$$\varphi_n(x) \le [1 + (\Sigma c_j)/1 - (\Sigma c_j)]^{n/2} \phi_n(x) \tag{4.27}$$

where ϕ_n is the density for independent normal random variables with mean zero and variances $\hat{\sigma}_i^2 = \sigma_i^2/(1 + \Sigma c_j)$.

PROOF. Without loss of generality we may assume $\sigma_i^2 = 1$ for all i. We have

$$\varphi_n(x) = (2\pi)^{-n/2}|A|^{-1/2}\exp\Big(-\frac{1}{2}x'A^{-1}x\Big), \quad x \in \mathbb{R}^n \tag{4.28}$$

where A is the covariance matrix; $a_{ii} = 1$ for all $i = 1, \ldots, n$ and $a_{ij} = \rho_{ij}$ for $i \ne j$, $i = 1, \ldots, n$ and $j = 1, \ldots, n$. We have the bound

$$\begin{aligned} x'Ax &= \Sigma x_i^2 + \sum_{i \ne j} x_i x_j \rho_{ij} \\ &\le \Sigma x_i^2 + \frac{1}{4}\sum_{i \ne j} |x_i x_j|(c_{i+j}) + \frac{1}{4}\sum_{i \ne j} |x_i x_j| c_{|i-j|}. \end{aligned} \tag{4.29}$$

Now, for example,

$$\sum_{i \ne j} |x_i x_j| c_{|i-j|} = 2c_1 \sum_{j=1}^{n-1} |x_j x_{j+1}| + 2c_2 \sum_{j=1}^{n-2} |x_j x_{j+2}| + \cdots + 2c_{n-1}|x_1 x_n| \le 2(\Sigma c_j)||x||^2 \tag{4.30}$$

and similarly $\sum_{i \ne j} |x_i x_j| c_{i+j} \le 2(\Sigma c_j)||x||^2$. Plugging this information into (4.29) gives

$$x'Ax \le [1 + (\Sigma c_j)]||x||^2. \tag{4.31}$$

In a similar way we obtain the lower bound

$$[1 - (\Sigma c_j)]||x||^2 \le x'Ax \tag{4.32}$$

Now applying (4.31) to $\tilde{x} = A^{-\frac{1}{2}}x$ gives

$$x'A^{-1}x \ge ||x||^2/[1 + \Sigma c_j]. \tag{4.33}$$

Also, we find $|A| = |\lambda_1 \cdots \lambda_n| \ge |\lambda_n|^n$, where

$$\lambda_n = \min_{x \ne 0}(x'Ax/x'x) \tag{4.34}$$

is the smallest eigenvalue of A. Now (4.32) shows that the minimum eigenvalue exceeds $(1 - \Sigma c_j)$ and so

$$|A| \ge (1 - \Sigma c_j)^n. \tag{4.35}$$

Finally, substituting the bounds (4.35) and (4.33) into (4.28) gives the required result. ∎

PROPOSITION 4.4.

$$P_\mu\big(x \longmapsto X(x,t) \textit{ is not of bounded variation for all } t\big) = 1$$

i.e., $\mathrm{Cap}\big(z : |z_j| = O(1/j)\big) = 0$.

PROOF. For $n \geq 1$, define the continuous function

$$u_n(z) = \big(\sup_{j=1}^{n} |jz_j|\big) \wedge N. \tag{4.36}$$

Then $u_n \in \mathcal{D}(\mathcal{E})$ and, almost surely we have

$$\nabla u_n(z) = \sum_{k=1}^{n} k \cdot \eta e_k \cdot \mathrm{sign}(\langle \eta e_k, z\rangle) \cdot 1_{(\sup_{j=1}^{n} |jz_j| = |kz_k| \leq N)}, \tag{4.37}$$

so $\|\nabla u_n(z)\|^2 \leq cn^2 1_{(\sup_{j=1}^{n} |jz_j| \leq N)}$ on H. Therefore

$$\begin{aligned} \mathcal{E}(u_n) &= \frac{1}{2}\int \|\nabla u_n(z)\|^2 \mu(dz) \\ &\leq cn^2 \mu\big(\sup_{j=1}^{n} |jz_j| \leq N\big). \end{aligned} \tag{4.38}$$

We would now like to use (4.4) and apply Lemma 4.3 with $c_j = |4c\eta_j|$, where c is the specific constant in (4.4). Provided $\Sigma|4c\eta_j| < 1$, this gives

$$\mu\big(\sup_{j=1}^{n} |jz_j| \leq N\big) \leq [1 + 4c\Sigma|\eta_j|/1 - 4c\Sigma|\eta_j|]^{n/2} P\big(\sup_{j=1}^{n} |X_j| \leq N\big), \tag{4.39}$$

where X_j are independent, mean zero Gaussian r.v.'s with $\hat{\sigma}_j^2 = \big(j^2/\lambda_j(1+4c\Sigma|\eta_j|)\big)$. The sequence $\hat{\sigma}_j^2$ is bounded away from 0, *i.e.*, $\sigma^2 := \inf \hat{\sigma}_j^2 > 0$. Since $|X_j|/\sigma_j \leq |X_j|/\sigma$ we get

$$\begin{aligned} P\big(\sup_{j=1}^{n} |X_j|/\sigma \leq N/\sigma\big) &\leq P\big(\sup_{j=1}^{n} |X_j|/\sigma_j \leq N/\sigma\big) \\ &= P^n(|Z| \leq N/\sigma). \end{aligned} \tag{4.40}$$

Choose M so large that $\Sigma_{j\geq M} 4c\eta_j < 1$ and that

$$(1 + 4c\Sigma_{j\geq M}\eta_j/1 - 4c\Sigma_{j\geq M}\eta_j)^{1/2} P(|Z| \leq N/\sigma) < 1. \tag{4.41}$$

Applying (4.39) to the tail of the sequence we get, for $n \geq M$

$$\mathcal{E}(u_n) \leq cn^2[(1 + 4c\Sigma_{j\geq M}\eta_j/1 - 4c\Sigma_{j\geq M}\eta_j)^{1/2} P(|Z| \leq N/\sigma)]^{n-M} \tag{4.42}$$

which converges to zero as $n \longrightarrow \infty$, so we have convergence in $\mathcal{E}$-norm. This sequence $\{u_n\}$ of continuous functions also increases pointwise to

$$u(z) = \big(\sup_{j\geq 1}^{\infty} |jz_j|\big) \wedge N. \tag{4.43}$$

The convergence also takes place in L^2, and by (4.39) and (4.41) we see that $u = N$ almost everywhere.

The remainder of the proof is exactly as in Proposition 4.1, so we omit it and conclude with

$$P_\mu\Big(\sup_{j=1}^{\infty} |jX_j(t)| = \infty \text{ for all } t\Big) = 1 \tag{4.44}$$

or

$$P_\mu\Big(x \longrightarrow X(x,t) \text{ is not bounded variation for all } t\Big) = 1. \tag{4.45}$$ ∎

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