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ON ε-APPROXIMATE SINGULARITIES OF AUTONOMOUS SYSTEMS OF VORTEX TYPE

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§ 0. Introduction

Let us consider three vortex-filaments $z_j(t)$ with strength Γ_j ($j = 1, 2, 3$) in the complex plane $\mathbf{C}$. Then the system of motion equations is given by

$$\text{(E)} \qquad \frac{dz_j}{dt} = \sqrt{-1} \sum_{\substack{k=1 \\ (k \neq j)}} \frac{\Gamma_k}{\bar{z}_j - \bar{z}_k} \quad (j = 1, 2, 3)\,.$$

This system (E) is defined on $V = \mathbf{C}^3 - \Delta$, where $\Delta = \{(z_1, z_2, z_3) \in \mathbf{C}^3;\ z_j = z_k$ for $j \neq k\}$ is the super-diagonal set of $\mathbf{C}^3$. Let Sol(E) be the space of all smooth solutions of (E) and let $\psi : V \to \mathrm{Sol}(E)$ be a smooth map defined as follows: For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in V$, $\psi(\alpha)$ is the solution with initial values α.

It is well-known (cf. [2], p. 260) that if three points α_j of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ make a regular triangle in $\mathbf{C}$, then $\psi(\alpha)$ becomes a rotational motion about these center of mass, which is called rigid-rotation. This solution $\psi(\alpha)$ has no singular points (cf. Definition 2.1). Now instead of α, let us take $\alpha(\varepsilon) = \alpha + \varepsilon\beta$ as initial values, where ε is a small parameter and $\beta \in \mathbf{C}^3$. Then using computers, we find that $\psi(\alpha(\varepsilon))$ has a singular point at a time $t = T_0(\varepsilon)$, and that $T_0(\varepsilon)$ seems to approach asymptotically to a $\log(1/\varepsilon) + b$ as $\varepsilon \to 0$, for constants a, b (see Figure). We may set the following problems:

(A) Is it true that $T_0(\varepsilon) \sim a \log(1/\varepsilon) + b$ ($\varepsilon \to 0$)?

(B) If (A) is correct, explain how the above constants a and b are determined from the given differential equations (E).

It doesn't seem that such problems have been treated yet.

In this paper we generalize the motion equations (E) on $\mathbf{C}$ to autonomous systems of vortex type on $\mathbf{C}^m$ defined in § 1. We can also consider

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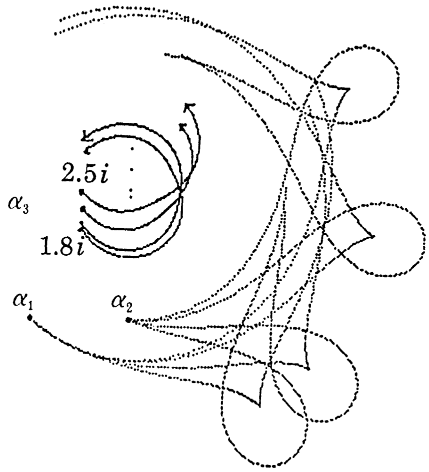

Figure. Integral curves of (E) with initial values $\alpha_1=-1$, $\alpha_2=1$ and (1) $\alpha_3=2.5i$; (2) $\alpha_3=2.2i$; (3) $\alpha_3=1.9i$; (4) $\alpha_3=1.8i$. where $i=\sqrt{-1}$ $\Gamma_1=-2$, $\Gamma_2=1$, $\Gamma_3=4$.

the same problems with respect to ε-approximation of such autonomous systems defined in § 2. Then we prove Theorem 3.6 in § 3 which solves partially our problems.

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§ 1. Vortex-Hamiltonian structures

1.1. Notation. Let $\mathbf{C}^m$ be the space of m complex variables $z_0^1, z_0^2, \cdots, z_0^m$. The elements of $\mathbf{C}^m$ are written as vectors of length m. We put $z_0 = (z_0^1, \cdots, z_0^m)$ and

$$\begin{cases} \bar{z}_0 dz_0 = \sum_{\alpha=1}^{m} \bar{z}_0^\alpha dz_0^\alpha , \\ dz_0 \wedge d\bar{z}_0 = \sum_{\alpha=1}^{m} dz_0^\alpha \wedge d\bar{z}_0^\alpha . \end{cases}$$

For any $\mathbf{C}^\infty$-complex valued function f on $\mathbf{C}^m$, we define the vector-valued function $\partial f/\partial z_0$ by

$$\frac{\partial f}{\partial z_0} = \left(\frac{\partial f}{\partial z_0^1}, \frac{\partial f}{\partial z_0^2}, \cdots, \frac{\partial f}{\partial z_0^m}\right),$$

and for any smooth vector-valued function $X = (X^1, X^2, \cdots, X^m)$ on $\mathbf{C}^m$, the $m \times m$-matrix $\partial X/\partial z_0$ associated with to the function X is defined by

$$\frac{\partial X}{\partial z_0} = \begin{bmatrix} \dfrac{\partial X^1}{\partial z_0^1}, \cdots, \dfrac{\partial X^1}{\partial z_0^m} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\partial X^m}{\partial z_0^1}, \cdots, \dfrac{\partial X^m}{\partial z_0^m} \end{bmatrix}.$$

1.2. Let us set $V_0 = \mathbf{C}^m$. We shall now consider motions of n-points $z_j(t)$ $(j = 1, \cdots, n)$ in V_0. First one notices that there is the canonical Kaehler form Ω_0 on V_0, defined by

(1.1) $$\Omega_0 = \sqrt{-1} dz_0 \wedge d\bar{z}_0$$

and that putting

(1.2) $$\theta_0 = \frac{\sqrt{-1}}{2}(z_0 d\bar{z} - \bar{z}_0 dz_0),$$

it follows that θ_0 is a real 1-form on V_0 such that

$$d\theta_0 = \Omega_0.$$

Set $V_j = \mathbf{C}^m$, $(j = 1, \cdots, n)$ and let $V = V_1 \times \cdots \times V_n$. For each j, let π_j be the j-th projection of V onto V_0, defined by

$$\pi_j(z_1, \cdots, z_n) = z_j \qquad \text{for } (z_1, \cdots, z_n) \in V.$$

DEFINITION 1.1. Let $\Gamma_1, \cdots, \Gamma_n$ be non-zero real constants and put

$$\theta_j = \pi_j(\theta_0), \quad (j = 1, \cdots, n).$$

Then

(1.3) $$\theta = \sum_{j=1}^{n} \Gamma_j \theta_j$$

is called *the fundamental form with strength* $\Gamma_1, \cdots, \Gamma_n$ on V. Further

(1.4) $$\Omega = d\theta$$

is a non-degenerate closed 2-form on V, and so we call (V, Ω) *the symplectic manifold with strength* $\Gamma_1, \cdots, \Gamma_n$.

Let (V, Ω) be a symplectic manifold as in the above definition. We can define the action of the general linear group $GL(m, \mathbf{C})$ and the additive group $\mathbf{C}^m$ on this space V as follows: For all $g \in GL(m, \mathbf{C})$ and $\alpha \in \mathbf{C}^m$,

(i) $g(z_1, \cdots, z_n) = (gz_1, \cdots, gz_n)$,

(ii) $\alpha(z_1, \cdots, z_n) = (\alpha + z_1, \cdots, \alpha + z_n)$

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for any $(z_1, \cdots, z_n) \in V$.

In particular $\mathbf{C}^* = \mathbf{C} - \{0\}$ being regarded as the diagonal subgroup of $GL(m, \mathbf{C})$, V admits $\mathbf{C}^*$-actions. We denote by $U(m)$ the unitary group which acts on V.

Now let Δ be a closed subset of V with the following properties: Δ is invariant under the groups $U(m)$, $\mathbf{C}^*$ and $\mathbf{C}^m$ respectively, and each projection $\pi_j : \tilde{V} = V - \Delta \to V_j$ is onto for $j = 1, \cdots, n$. $\tilde{V}$ is also invariant under these groups. Here instead of (V, Ω) we take this open symplectic submanifold $(\tilde{V}, \Omega)$ of $\tilde{V}$. Finally let $H : \tilde{V} \to R$ be a smooth function (called Hamiltonian function), satisfying the following three conditions:

(a) $U(m)$ and $\mathbf{C}^m$-invariant.

(b) $\mathbf{C}^*$-semiinvariant, that is, for any $a \in \mathbf{C}^*$ and $(z_1, \cdots, z_n) \in \tilde{V}$, $H(az_1, \cdots, az_n, \bar{a}\bar{z}_1, \cdots, \bar{a}\bar{z}_n) = H(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n) + \gamma \log|a|^2$, where γ is a real constant independent of a and $(z_1, \cdots, z_n)$.

(c) $\partial\bar{\partial}H = 0$,

where ∂ and $\bar{\partial}$ mean the derivations of type $(1, 0)$ and $(0, 1)$, respectively.

Thus the triplet $(\tilde{V}, \Omega, H)$ is called *Hamiltonian structure of vortex type.*

DEFINITION 1.2. Let $(\tilde{V}, \Omega, H)$ be as above. A real smooth vector field $\tilde{X}$ is called of *vortex type* if

$$\tilde{X} \lrcorner \Omega = -dH . \tag{1.5}$$

Let $\tilde{X}$ be of vortex type. We express this vector field $\tilde{X}$, using vector-valued coordinates $z_1, \cdots, z_n$ of V. $\tilde{X}$ can be written as

$$\tilde{X} = \sum_{j=1}^{n} \bar{X}_j(z, \bar{z})\partial/\partial z_j + \sum_{j=1}^{n} X_j(z, z)\partial/\partial \bar{z}_j ,$$

where for each j, $z_j = (z_j^1, \cdots, z_j^m)$ and $\bar{X}_j$ is the complex conjugate X_j and $\bar{X}_j\partial/\partial z_j$ stands for $\sum_{\alpha=1}^{m} \bar{X}_j^\alpha \, \partial/\partial z_j^\alpha$.

Then we find from (1.5)

$$\bar{X}_j = -\sqrt{-1}\,\frac{1}{\Gamma_j}\,\frac{\partial H}{\partial \bar{z}_j} \tag{1.6}$$

and

$$X_j = \sqrt{-1}\,\frac{1}{\Gamma_j}\,\frac{\partial H}{\partial z_j} . \tag{1.6'}$$

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Moreover in terms of the condition (c) for H, it follows that the $\overline{X}_j$ are anti-holomorphic vector-valued functions on $\tilde{V}$. Therefore integral curves $z(t)=(z_1(t),\cdots,z_n(t))$ of $\tilde{X}$ satisfy the following system of differential equations, called *an autonomous system of vortex type*

$$\frac{dz_j}{dt}=X_j(z_1,\cdots,z_n)\,,\quad (j=1,\cdots,n)\,. \tag{1.7}$$

§ 2. Singularities and properties of autonomous systems of vortex type

We use the same notations as before.

DEFINITION 2.1. Let $z(t)=(z_1(t),\cdots,z_n(t))$ be a solution of (1.7) and let $\pi_j:\tilde{V}\to\mathbf{C}^m$ be the j-th projection as in 1.2 for $j=1,\cdots,n$. This solution $z(t)$ is *singular*, more precisely *j-singular*, at a time $t=t_0$ if there exists an index j such that the image curve of $z_j(t)=\pi_j(z(t))$ in $\mathbf{C}^m$ has a vanishing derivative at $t=t_0$, that is

$$\left.\frac{dz_j}{dt}\right|_{t=t_0}=0\,.$$

Now we assume that there exists a non-singular solution $z(t)$ of (1.7) with initial values $\alpha=(\alpha_1,\cdots,\alpha_n)\in\tilde{V}$ at $t=0$. Let $z(t;\varepsilon)$ be the solution with initial values $z(0;\varepsilon)=\alpha+\varepsilon\beta$ for a small $|\varepsilon|>0$. Put

$$w(t)=\left.\frac{d}{d\varepsilon}z(t;\varepsilon)\right|_{\varepsilon=0},$$

and

$$\tilde{z}(t;\varepsilon)=z(t)+\varepsilon w(t)$$

which we call *the ε-order approximation* of $z(t;\varepsilon)$.

We now want to obtain a value t_0 of t such that for some k,

$$\frac{d\tilde{z}_k}{dt}(t_0;\varepsilon)=0\,. \tag{2.1}$$

For this purpose we write down a system of differential equations which the above unknown vector-valued function $w(t)$ satisfies. Set

$$\overline{X}=(\overline{X}_1,\cdots,\overline{X}_n)$$

where the $\overline{X}_j$ are defined by (1.6), then $dz(t;\varepsilon)/dt=\overline{X}(z(t;\varepsilon))$. By differentiation in ε,

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$$\frac{dw_j(t)}{dt} = \sum_{j=1}^{n} \frac{\partial \bar{X}_j}{\partial \bar{z}_j} \bar{w}_j(t) \quad (j = 1, \cdots, n), \tag{2.2}$$

or in the matrix form,

$$\frac{d}{dt}\begin{pmatrix} w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix} = \begin{bmatrix} \frac{\partial \bar{X}_1}{\partial \bar{z}_1}, & \cdots, & \frac{\partial \bar{X}_1}{\partial \bar{z}_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial \bar{X}_n}{\partial \bar{z}_1}, & \cdots, & \frac{\partial \bar{X}_n}{\partial \bar{z}_n} \end{bmatrix} \begin{pmatrix} \bar{w}_1 \\ \vdots \\ \bar{w}_n \end{pmatrix} \tag{2.2'}$$

which is the system of differential equations for the w's. Here one notes that the $\partial \bar{X}_j / \partial \bar{z}_k$ are $m \times m$-matrices. For convenience sake, let us put

$$\begin{cases} \bar{A}_{ij}(t) = \dfrac{\partial \bar{X}_j}{\partial \bar{z}_j}(t) \quad (1 \leqq i, j \leqq n), \\ \bar{A}(z) = \begin{pmatrix} \bar{A}_{11}(z), & \cdots, & \bar{A}_{1n}(z) \\ \cdots & \cdots & \cdots \\ \bar{A}_{n1}(z), & \cdots, & \bar{A}_{nn}(z) \end{pmatrix}. \end{cases} \tag{2.3}$$

Then (2.2′) can be written as follows;

$$\frac{dw(t)}{dt} = A(z(t))\bar{w}(t) \tag{2.4}$$

where $w(t) = {}^t(w_1(t), \cdots, w_n(t))$. Putting $z(0; \varepsilon) = \alpha + \varepsilon\beta$. We find that $w(t)$ is a solution of (2.4) with $w(0) = \beta$. From the above discussions our problem is summarized as follows: Let $z(t)$ be a non-singular solution of (1.7) with $z(0) = \alpha$ and $w(t)$ a solution of (2.4) such that $w(0) = \beta$. Then the problem is to find a value t_0 of t satisfying the following equation: For some index k.

$$\frac{d\bar{z}_k}{dt}(t) + \varepsilon \sum_{j=1}^{n} \bar{A}_{kj}(z(t))\bar{w}_j(t) = 0. \tag{2.5}$$

We shall solve this problem in case where the above solution $z(t)$ is $U(m)$- or $\mathbf{C}^*$-solution defined in § 3.

2.2. In this paragraph we examine some properties of the vector field X and the matrix $\bar{A}(z)$ which are defined in 2.1. First of all we obtain the following

LEMMA 2.2. *For $g \in U(m)$ and $a \in \mathbf{C}^*$,*

$$\bar{X}(g\alpha) = g\bar{X}(\alpha) \tag{2.6}$$

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and

$$\overline{X}(a\alpha) = \frac{1}{\overline{a}}\overline{X}(\alpha)\,. \tag{2.7}$$

Proof. Since the Hamiltonian $H(z, \overline{z})$ is $U(m)$-invariant, for any $g = (g_{ab}) \in U(m)$ and $\alpha \in \tilde{V}$, we get

$$\sum_{b=1}^{m} \overline{g}_{ab} \frac{\partial H}{\partial \overline{z}_j^b}(g\alpha) = \frac{\partial H}{\partial \overline{z}_j^a}(\alpha)\,, \quad (j = 1, \cdots, n) \tag{$*$}$$

for $z_j = (z_j^1, \cdots, z_j^m)$.

Using matrix notations, $(*)$ are expressed as

$${}^t\overline{g}\frac{\partial H}{\partial \overline{z}_j}(g\alpha) = \frac{\partial H}{\partial \overline{z}_j}(\alpha)\,, \quad \text{for all } j.$$

Therefore from Definition (1.6) of the $\overline{X}_j$, it follows

$$\overline{X}_j(g\alpha) = {}^t\overline{g}^{-1}X_j(\alpha)\,, \quad (j = 1, \cdots, n)\,. \tag{2.8}$$

As g is unitary, we have (2.6).

Since H is $\mathbf{C}^*$-semiinvariant, (2.8) is also satisfied for $a \in \mathbf{C}^*$, and so (2.7) is proved. Q.E.D.

From this lemma and Definition (2.3) of the matrices $\overline{A}_{ij}$ and $\overline{A}$ we can prove immediately the following

PROPOSITION 2.3. *For $g \in U(m)$ and $a \in \mathbf{C}^*$,*

$$\overline{A}_{ij}(g\alpha) = gA_{ij}(\alpha)\overline{g}^{-1}\,, \tag{2.9}$$

i.e.,

$$\overline{A}(g\alpha) = g\overline{A}(\alpha)\overline{g}^{-1}\,, \tag{2.9$'$}$$

and

$$\overline{A}(a\alpha) = \frac{1}{\overline{a}^2}A(\alpha) \qquad \textit{for any } \alpha \in \tilde{V}\,. \tag{2.10}$$

Finally we obtain the following proposition which states the so-called angular momentum invariance.

PROPOSITION 2.4. *We have*

$$\sum_{j=1}^{n} \Gamma_j \overline{X}_j = 0\,, \tag{2.11}$$

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and

$$\sum_{j=1}^{n} \Gamma_j \bar{z}_j \bar{X}_j = -\sqrt{-1}\gamma , \tag{2.12}$$

where Γ_j *is the strength of the j-th point* z_j $(j = 1, \cdots, n)$ *and* γ *is the constant defined in* (c) *of* 1.2.

Proof. From $\mathbf{C}^m$-invariance of H we get

$$\left.\frac{\partial H(z + a, \bar{z} + \bar{a})}{\partial \bar{a}^\alpha}\right|_{a=0} = \sum_{j=1}^{n} \frac{\partial H}{\partial \bar{z}_j^\alpha} = 0$$

for $a = (a^1, \cdots, a^n)$ and $\alpha = 1, \cdots, m$. Therefore from (1.6) we have

$$\sum_{j=1}^{n} \Gamma_j \bar{X}_j(z) = 0$$

which shows (2.11).

(2.12) can be proved, using

$$\left.\frac{\partial H(az, \bar{a}\bar{z})}{\partial \bar{a}}\right|_{a=1} = \sum_{j=1}^{n} \frac{\partial H}{\partial \bar{z}_j} \bar{z}_j = \gamma \qquad \text{for } a \in \mathbf{C}^* .$$

Q.E.D.

In virtue of (2.11) we have the following

COROLLARY 2.5. *The determinant* $|A|$ *of* A *is zero i.e.,*

$$|A| = 0 .$$

§ 3. The kinds of solutions

3.1. Rigid rotational solutions

3.1.1. We start from the following

DEFINITION 3.1. A solution $z(t)$ of (1.7) is called *a rigid rotational solution* or *U(m)-solution* with initial values $\alpha = (\alpha_1, \cdots, \alpha_n)$ at $t = 0$, if there exists a 1-parameter group $S: R \to U(m)$, that is,

$$S(t) = \exp tC \qquad \text{for all } t \in R$$

such that

$$z(t) = S(t)\alpha , \tag{3.1}$$

where C denotes an anti-hermitian matrix such that $C\alpha_j \neq 0$.

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Let $z(t)$ be a $U(m)$-solution defined by (3.1). Then

$$\dot{S}\alpha = \overline{X}(S\alpha)$$

where $\dot{S} = dS/dt$. It follows from (2.6) and $C = S^{-1}\dot{S}$

$$C\alpha = \overline{X}(\alpha)\,. \tag{3.2}$$

Furthermore differentiating $S(t)^{-1}\overline{X}(S(t)\alpha) = C\alpha$ with respect to t, we find

$$\overline{A}(\alpha)\overline{C}\overline{\alpha} = C^2\alpha\,. \tag{3.3}$$

Now let $\tilde{z}(t;\varepsilon) = z(t) + \varepsilon w(t)$ be an ε-order approximation such that $\tilde{z}(0;\varepsilon) = \alpha + \varepsilon\beta$ as explained in § 2. Then $w(t)$ satisfies

$$\frac{dw(t)}{dt} = S(t)\overline{A}(\alpha)\overline{S}(t)^{-1}\overline{w}(t)\,, \tag{3.4}$$

because of (2.4).

Let us set

$$v(t) = S(t)^{-1}w(t)\,. \tag{3.5}$$

Then the system of linear differential equations for $v(t)$ equivalent to (3.4) is

$$\frac{dv(t)}{dt} = \overline{A}(\alpha)\overline{v}(t) - Cv(t)\,. \tag{3.6}$$

We introduce an R-linear map $B: V \to V$ defined by

$$B(\xi) = -\,C\xi + \overline{A}(\alpha)\overline{\xi}\,, \quad \xi \in V. \tag{3.7}$$

Using this map B, (3.6) is expressed in the form

$$\frac{dv}{dt} = B(v)\,. \tag{3.8}$$

In order to solve (3.8), it is convenient to write down (3.8) in real forms. We identify V with $V_R = R^{mn} \times R^{mn}$ by the map ϕ defined as follows: Let $\xi = x + \sqrt{-1}y \in V$ for x and y real. Then

$$\phi(\xi) = (x, y) \in V_R\,.$$

For simplicity we denote $\phi(\xi) = \hat{\xi}$. Let $\hat{v}(t) = (v_1, v_2) \in V_R$, $C = C_1 + \sqrt{-1}C_2$, and $A(\alpha) = A_1 + \sqrt{-1}A_2$. Then (3.8) is written in the space V_R as follows;

$$\frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \hat{B}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{3.8'}$$

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where

$$\hat{B} = \begin{pmatrix} A_1 - C_1, & -A_2 + C_2 \\ -A_2 - C_2, & -A_1 + C_1 \end{pmatrix}. \tag{3.9}$$

If $B(\xi) = \lambda\xi$ for some vector $\xi \in V$ and a real number λ, then $\hat{\xi} = \phi(\xi)$ is an eigenvector of $\hat{B}$ corresponding to λ. As a consequence of it, we obtain the following

PROPOSITION 3.1. *B has the eigenvalue 0 and the vector $C\alpha$ is the 0-eigenvector.*

Proof. From Definition (3.7) of B and (3.3) we have

$$B(C\alpha) = -C^2\alpha + \overline{A}(\alpha)\overline{C}\overline{\alpha} = 0.$$

But $C\alpha \neq 0$ from the assumption, which implies this proposition. Q.E.D.

Moreover we can show by direct calculations the following

LEMMA 3.2. *Let us assume that*

$$CA(\alpha) = A(\alpha)C. \tag{3.10}$$

Then the characteristic equation of $\hat{B}$ is

$$|(\lambda E + \overline{C})(\lambda E + C) - A\overline{A}| = 0, \tag{3.11}$$

where E is the unit matrix.

In particular in case of $m = 1$ we get following

COROLLARY 3.3. *The matrix $\hat{B}$ has eigenvalues 0, $-c$, and $-\bar{c}$. And 0 is of multiplicity $\geqq 2$, where C reduces to the scalor matrix (c).*

Proof. As $m = 1$, the condition (3.10) is automatically fulfiled. From (3.11) and Corollary 2.5, $-c$ and $-\bar{c}$ are eigenvalues of $\hat{B}$. On the other hand, (3.11) reduces to $|(\lambda^2 + c\bar{c})E - A\overline{A}| = 0$, whence the multiplicity of eigenvalue 0 is not less than 2. Q.E.D.

3.1.2. Now let us return to the discussions of singularities. Let $\lambda_1, \cdots, \lambda_l$ be eigenvalues of $\hat{B}$ and let m_j be the multiplicity of λ_j, $(j = 1, \cdots, l)$. We denote by $\hat{W}(\lambda_j)$ the eigenspace associated with λ_j of multiplicity m_j;

$$\hat{W}(\lambda_j) = \{\hat{\xi} \in V_R; (\lambda_j - \hat{B})^{m_j}\hat{\xi} = 0\}.$$

Remember $v(t)$ is the solution of (3.8) with $v(0) = \beta$ for $\beta = x + \sqrt{-1}y \in V$.

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Since $V_R \otimes C$ is decomposed into the direct sum of $\hat{W}(\lambda_1), \cdots, \hat{W}(\lambda_l)$. $\hat{\beta} = (x, y) \in V_R$ is expressed as a sum of $\hat{W}(\lambda_j)$-components of $\hat{\beta}$. We say that λ_j is associated with β, if the $\hat{W}(\lambda_j)$-component is not zero.

DEFINITION 3.4. Let λ_j be an eigenvalue of $\hat{B}$ associated with β. λ_j is called *dominant* for β, when

(i) $\mathrm{Re}(\lambda_j) > 0$,

(ii) $\mathrm{Re}(\lambda_j)$ is greater than the real part of any other eigenvalue associated with β,

where $\mathrm{Re}(\lambda)$ means the real part of λ.

In order to express the solution $v(t)$ of (3.8), using eigenvalues and eigenvectors of $\hat{B}$, we shall introduce the following notations: Let λ be an eigenvalue of $\hat{B}$ and let $\hat{\beta}_0 \in \hat{W}(\lambda)$. If λ is real, we may assume that $\hat{\beta}_0$ is a real vector. At first in case where λ is real, we can write $\hat{\beta}_0$, β_0 in the forms

$$\hat{\beta}_0 = (x, y) \in V_R \quad \textit{and} \quad \beta_0 = x + \sqrt{-1}y \in V\,.$$

With these notations let $\beta_1, \cdots, \beta_k \in \hat{W}(\lambda)$, and

(I) $P(t) = c_1\beta_1 + tc_2\beta_2 + \cdots + t^{k-1}c_k\beta_k\,.$

On the other hand if $\lambda = a + \sqrt{-1}b$ is imaginary, we may write

$$\hat{\beta}_0 = \hat{\beta}_1 + \sqrt{-1}\hat{\beta}_2 \in V_R \otimes C$$

for $\hat{\beta}_j = (x_j, y_j) \in V_R$, $(j = 1, 2)$. Let

$$\beta_j = x_j + \sqrt{-1}y_j \in V\,, \quad (j = 1, 2)$$

and put for any real number c_j $(j = 1, 2)$,

$$[\hat{\beta}_0 : c_1, c_2] = c_1(\cos bt\cdot\beta_1 - \sin bt\cdot\beta_2) + c_2(\sin bt\cdot\beta_1 + \cos bt\cdot\beta_2)\,,$$

for $a = \mathrm{Re}(\lambda)$ and $b = \mathrm{Im}(\lambda)$. Further for any $\hat{\beta}_1, \cdots, \hat{\beta}_k \in \hat{W}(\lambda)$, we set

(II) $P(t) = [\hat{\beta}_1 : c_{11}, c_{12}] + t[\hat{\beta}_2 : c_{21}, c_{22}] + \cdots + t^{k-1}[\hat{\beta}_k : c_{k1}, c_{k2}]\,.$

We call the above functions $P(t)$ defined by (I), (II) for an eigenvalue λ, $\hat{W}(\lambda)$-polynomial functions of degree $k-1$. With these notations we can express the solution $v(t)$ of (3.8) with initial values β. Let $\{\lambda_1, \cdots, \lambda_s, \bar{\lambda}_1, \cdots, \bar{\lambda}_s, \cdots, \lambda_{s+1}, \cdots, \lambda_r\}$ be all eigenvalues associated with β, where $\bar{\lambda}_j$ is complex-conjugate to λ_j, $(j = 1, \cdots, s)$ and $\lambda_{s+1}, \cdots, \lambda_r$ are real. Then from the well-known theorem of differential equations with constant coefficients (cf. [3]) it follows

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$$v(t) = \sum_{j=1}^{r} e^{a_j t} P_j(t)\,, \tag{3.12}$$

where $\lambda_j = a_j + \sqrt{-1}b_j$ and $P_j(t)$ are $\hat{W}(\lambda_j)$-polynomial functions.

Remark. Let all notations be as above. Let $\hat{\beta} = \sum_{j=1}^{s} \hat{\beta}_j + \sum_{j=1}^{s} \hat{\beta}_j + \sum_{k=s+1}^{r} \hat{\beta}_k$. If $\hat{\beta}_j$ is an eigenvector, that is, $\hat{B}\hat{\beta}_j = \lambda_j \hat{\beta}_j$, then $P_j(t)$ is of degree 0. Therefore for the ε-order approximation $\tilde{z}(t;\varepsilon) = z(t) + \varepsilon S(t)v(t)$, we have from (3.12) and $z(t) = S(t)\alpha$,

$$S(t)^{-1}\frac{d\tilde{z}}{dt} = C\alpha + \varepsilon \sum_{j=1}^{r} e^{a_j t}\overline{A}(\alpha)\overline{P}_j(t)\,. \tag{3.13}$$

Here we need the following.

DEFINITION 3.5. An eigenvalue λ of $\hat{B}$ is *simply dominant* for β if λ is dominant (cf. Definition 3.4) and if the $\hat{W}(\lambda)$-component of β is the eigenvector for λ.

Let us suppose that the above eigenvalue λ_r is simply dominant for β. Then from the preceding remark

$$P(t) = \beta_r\,, \tag{3.14}$$

where $\hat{\beta}_r$ is the $\hat{W}(\lambda_r)$-component of $\hat{\beta}$.

Moreover we introduce a linear map $\overline{A}_k(\alpha) : V \to V_k = C^m$ $(k = 1, \cdots, n)$ defined by

$$\overline{A}_k(\alpha)\beta_0 = \sum_{j=1}^{n} \overline{A}_{kj}(\alpha)\beta_{0j}$$

for any $\beta_0 = (\beta_{01}, \cdots, \beta_{0n}) \in V$. Finally we assume that for some index k, there exists a non-zero real number δ_k such that

$$C\alpha_k = \delta_k \overline{A}(\alpha)\overline{\beta}_r\,, \tag{3.15}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \tilde{V}$.

We say that the vector β satisfying (3.15) is *k-dominant parallel* to α with a ratio-constant δ_k. Under the condition (3.15) for β, we have from (3.13)

$$\frac{d\tilde{z}_k(t;\varepsilon)}{dt} = S(t)\overline{A}_k(\alpha)\left\{\delta_k\beta_r + \varepsilon e^{\lambda_r t}\left[\overline{\beta}_r + \sum_{j=1}^{r-1} e^{(a_j-\lambda_r)t}\overline{P}_j(t)\right]\right\}. \tag{3.16}$$

Let $t = T(\varepsilon)$ be the solution of

$$\delta_k + \varepsilon e^{\lambda_k t} = 0\,, \tag{3.17}$$

that is,

$$T(\varepsilon) = \frac{1}{\lambda_r}\log\left(-\frac{\delta_k}{\varepsilon}\right), \tag{3.17'}$$

where the sign of ε is chosen such that $\delta_k/\varepsilon < 0$.

Now let $\| \ \ \|$ be the usual norm on $\mathbf{C}^m$. Since $S(t)$ is unitary, $P_j(t)$ are $\hat{W}(\lambda_j)$-polynomial functions and $a_j - \lambda_r < 0$ $(j = 1, \cdots, r-1)$, we obtain in terms of (3.16) and (3.17), the following estimates of $\|d\tilde{z}_k/dt\|$ at $t = T(\varepsilon)$ for small $|\varepsilon|$, $0 < |\varepsilon| < \delta$:

$$\left\| \frac{d\tilde{z}_k(t;\varepsilon)}{dt} \right\|_{t=T(\varepsilon)} \leqq K_r|\varepsilon|^{(1-f_r)} \tag{3.18}$$

for an enough small positive number δ, where K_r is a constant independent of ε and f_r denotes $\max\{a_1/\lambda_r, \cdots, a_{r-1}/\lambda_r\}$.

We can now resume the above conclusions in the form of

THEOREM 3.6. *Let $z(t) = S(t)\alpha$ be a $U(m)$-solution and $z(t;\varepsilon)$ a solution with initial values $\alpha + \varepsilon\beta$. Suppose that there exists a simply dominant eigenvalue λ_r for β and that β is k-dominant parallel to α with a real ratio-constant δ_k, $(1 \leqq k \leqq n)$. Then $\tilde{z}(t;\varepsilon)$, the ε-order approximation of $z(t;\varepsilon)$, has the estimate for small $|\varepsilon|$:*

$$\left\| \frac{d\tilde{z}_k}{dt} \right\|_{t=T(\varepsilon)} \leqq K_r|\varepsilon|^{(1-f_r)}, \tag{C}$$

where

$$T(\varepsilon) = \frac{1}{\lambda_r}\log\left(-\frac{\delta_k}{\varepsilon}\right),$$

and K_r, f_r are constant as in (3.18) *such that $f_r < 1$.*

In particular if $s = 0$ and $r = 1$, then

$$\left.\frac{d\tilde{z}_k}{dt}\right|_{t=T(\varepsilon)} = 0. \tag{D}$$

Remark. Suppose $\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 < 0$ in the equation (E). We take $\alpha_1 = -1/2$, $\alpha_2 = 1/2$, $\alpha_3 = \sqrt{-3}$ as initial values. Then $\hat{B}$ has eigenvalues $\lambda = \sqrt{-3(\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1)}$, $-\lambda$, ± 0, and $\pm\sqrt{-1}(\Gamma_1 + \Gamma_2 + \Gamma_3)$. Take $\Gamma_1 = -2$ and $\Gamma_2 = 1$. Then the eigenvector β corresponding to the above simple-dominant root λ is 1-parallel to $\alpha = (\alpha_1, \alpha_2\, \alpha_3)$. It is sufficient

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to take $\Gamma_3 = 2$, a root of the equation $\sqrt{(X+2)}(X^2+4X+4)-(2X^3+9X-2)=0$.

3.2. C*-solutions

3.2.1. In this paragraph we treat an another kind of solutions.

DEFINITION 3.7. Let I be an open interval containing 0. A solution $z(t)$ of (1.7) with $z(0)=\alpha$ is called a **C***-*solution* if there is a smooth function $f: I \to \mathbf{C}^*$ such that

$$z(t) = f(t)\alpha \quad (f(0)=1)\,, \tag{3.19}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in V$ and all vectors α_j are non-zeros.

Let $z(t) = f(t)\alpha$ be a **C***-solution with initial conditions $z(0)=\alpha$. Then we have from (1.7) and (2.7)

$$\bar{f}\dot{f}\alpha = \bar{X}(\alpha)$$

where $\dot{f}$ means df/dt. Therefore $\bar{f}\dot{f}$ being constant, we can set

$$c = \bar{f}\dot{f} \tag{3.20}$$

whence it follows

$$c\alpha = \bar{X}(\alpha)\,. \tag{3.21}$$

Here putting $c = a + \sqrt{-1}b$, we find by (3.20)

$$\frac{d}{dt}|f|^2 = 2a\,.$$

The solution $f(t)$ of this differential equation under the initial condition $f(0)=1$ is

$$\begin{cases} f(t) = \sqrt{2at+1}\exp\left\{\sqrt{-1}\dfrac{b}{2a}\log(2at+1)\right\}, \\ |f|^2 = 2at+1\,. \end{cases} \tag{3.22}$$

If $a = \mathrm{Re}(c)$ is zero, then the solution $z(t)$ reduces to $U(1)$-solution. On the other hand, if $a \neq 0$, then we can state the following

PROPOSITION 3.8. *The Hamiltonian function $H(z, \bar{z})$ is* **C***-*invariant, i.e., the constant γ in* (*b*) *of* § 1.2 *is zero. Moreover it follows*

$$\sum_{j=1}^{n} \Gamma_j \|\alpha_j\|^2 = 0\,. \tag{3.23}$$

Proof. At first it follows from (2.12) and (3.21) that

$$\sqrt{-1}c\sum_{j=1}^{n}\Gamma_j\|\alpha_j\|^2=\gamma\,.$$

Since $\operatorname{Re}(c)=a$ is non-zero and γ is real, we find $\gamma=0$, and so (3.23) is proved. Q.E.D.

Now return to (3.21). Noting $\bar{f}(t)\bar{X}(f(t)\alpha)=c\alpha$, by (2.7) and (2.10)

$$c\alpha+\bar{A}(\alpha)\bar{\alpha}=0\,. \tag{3.24}$$

Here as before let $\tilde{z}(t;\varepsilon)=z(t)+\varepsilon f(t)v(t)$ be an ε-order approximation with initial values $\alpha+\varepsilon\beta$. To obtain differential equations which $v(t)$ satisfies, we take the independent variable τ as

$$\frac{d}{d\tau}=|f|^2\frac{d}{dt}\ ,$$

i.e.,

$$\tau=\frac{1}{2a}\log(2at+1)\,. \tag{3.25}$$

Then the system of differential equations for $v(\tau)$ is

$$\frac{dv}{d\tau}=-cv(\tau)+\bar{A}(\alpha)\bar{v}(\tau)\,. \tag{3.26}$$

Similarly as (3.7) we define an R-linear map $B: V\to V$ by

$$B(x)=-cx+\bar{A}(\alpha)\bar{x} \tag{3.27}$$

for any $x\in V$, and so (3.26) can be written as

$$\frac{dv}{d\tau}=B(v)\,. \tag{3.28}$$

Further we can write (3.28) in the real form

$$\frac{d}{d\tau}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\hat{B}\begin{pmatrix}v_1\\v_2\end{pmatrix} \tag{3.28'}$$

where $v=v_1+\sqrt{-1}v_2$ and $\hat{B}$ is the real matrix of B on V_R. From Lemma 3.2 it follows that the characteristic equation of $\hat{B}$ is

$$|(\lambda+c)(\lambda+\bar{c})E-A\bar{A}|=0\,. \tag{3.29}$$

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Thus we can prove the following

PROPOSITION 3.9. (i) $-(c+\bar{c})$, 0, $-c$ *and* $-\bar{c}$ *are eigenvalues of* $\hat{B}$, *and the vectors* $c\alpha$ *and* $\sqrt{-1}\alpha$ *are eigenvectors corresponding to* $-(c+\bar{c})$ *and* 0, *respectively.*

(ii) *The matrix* $A\bar{A}$ *has eigenvalues* 0 *and* $|c|^2$.

3.2.2. Let us return to the singularities of $\tilde{z}(t;\varepsilon)$. Using $d/d\tau = |f|^2 d/dt$, we find from (3.26)

$$\bar{f}(t)\frac{d\tilde{z}}{dt} = c\alpha + \varepsilon\bar{A}(\alpha)\bar{v}(\tau)\,. \tag{3.30}$$

Assume the following conditions (F) are satisfied: (F) There is a simple-dominant eigenvalue for β, say λ and β is k-dominant parallel to α with a real ratio-constant δ_k. Then put

$$T(\varepsilon) = \frac{1}{2a}\left(\left(-\frac{\delta_k}{\varepsilon}\right)^{2a/\lambda} - 1\right) \tag{3.31}$$

for $a = \mathrm{Re}(c)$. Then we can prove by the same procedures as 3.1.2 the following

THEOREM 3.10. *When the condition* (F) *is satisfied, the* ε-*approximation* $\tilde{z}(t;\varepsilon)$ *has the same estimates as* (C) *in Theorem* 3.6 *at* $t = T(\varepsilon)$.

In particular, if there is only one eigenvalue λ of $\hat{B}$ which is associated with β and s simply dominant, and if β is k-dominant parallel to α with a real ratio-constant, δ_k, then

$$\left.\frac{d\tilde{z}_k}{dt}\right|_{t=T(\varepsilon)} = 0\,.$$

We may conjecture that the constants a, b in the problem (A) for the motion-equation (E) are given by the same relations $a = 1/\lambda_r$, $b = (\log - \delta_k)/\lambda_r$ appearing in $T(\varepsilon)$ in Theorem 3.6.

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