ON UNIFORM APPROXIMATIONS OF ABSTRACT FUNCTIONS

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As is well known, every real function is the pointwise (uniform) limit of a sequence of functions with a finite (countable) range of values. Monna [5] and Kvačko [4] suggested some extensions of this theorem to functions with values in a separable metric space.[1] In the present note we give some further generalizations, with an emphasis on <u>uniform</u> approximations which have many applications in the generalized theory of measure and integration. In particular, we consider <u>measurable</u> abstract functions (mappings).

In order to be able to deal with uniform approximations in an arbitrary space, it is convenient to use "indexed neighborhood systems" which, as was shown by Davis [2], can be introduced in <u>any</u> space, in such a manner as to preserve its topology.[2] In this connection, we shall formulate a few definitions:

TERMINOLOGY AND NOTATION. A topological space T is said to be <u>semi-uniform</u> if, for some fixed index set I, each point $x \in T$ has a local base of open indexed neighborhoods $N^i(x)$ (briefly N_x^i), $i \in I$, such that:

(N1) <u>For any</u> $i, j \in I$, <u>there is</u> $k \in I$, <u>with</u> $N_x^k = N_x^i \cap N_x^j$ <u>for all</u> $x \in T$. <u>Equivalently,</u> $N^k = N^i \cap N^j$ <u>where</u>

[1] Unfortunately, both papers contain errors which we rectify below.

[2] In a sense, this note is also a contribution to the theory of such neighborhood systems.

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$N^i = \{(x, y) \mid x \in T,\ y \in N_x^i\}$.

(N2) $y \in N_x^i$ <u>always implies</u> $x \in N_y^i$ (symmetry axiom).

(N3) <u>For each</u> N_x^i, <u>there is</u> $j = j(i, x) \in I$ <u>such that</u> $N^j(N_x^j) \subseteq N_x^i$, <u>where</u> $N^j(A) = \bigcup_{z \in A} N_z^j$ $(A \subseteq T)$.

If in (N3), j depends on i only (not on x), then T becomes a <u>uniform</u> space in the sense of A. Weil [6, pages 7-8]. It is worth noting that the uniform limit of a net of continuous functions is continuous in semi-uniform spaces, as it is in uniform spaces. Thus the former are a natural generalization of the latter.

If only (N1) is assumed, T is called a <u>graded</u> space; I is its <u>grader</u>, and $N = \{N_x^i \mid i \in I,\ x \in T\}$ is its <u>graded base (structure)</u>, also denoted by (N, I). Similarly, (T, N, I) is a space T with graded base (N, I). If T satisfies (N1) and (N2), we call it a <u>symmetric space</u>, and (N, I) a <u>symmetric base</u>. We say that T <u>admits</u> a structure (N, I) if the latter preserves the topology of T . (T, N, I) is said to be <u>totally</u> (σ-<u>totally</u>) <u>bounded</u> if, for each $i \in I$, $T = N^i(A_i)$ for some finite (at most countable) set $A_i \subseteq T$. T is called an R_o-<u>space</u> (Davis) if each open set $G \subseteq T$ contains the closure $\bar{x}$ of every one-point set $\{x\} \subseteq G$. Other topological concepts (e.g. <u>nets</u>) are defined as in [3]. Uniform convergence of nets of functions $\{f_k\}$ is defined for graded spaces exactly as in uniform spaces. (notation: $f_k \to f$ (unif.)); similar notation holds for pointwise <u>(ptw.)</u> and <u>(a.e.)</u> convergence. The grader I of T is <u>directed</u> by setting $i \geq j$ $(i, j \in I)$ iff $N^i \subseteq N^j$; thus we may consider nets of the form $\{f_i \mid i \in I\}$. As will be seen, all nets obtained in our theorems can be chosen to be of that type. The importance of graded structures is evident from the following Lemma (due to Davis):

LEMMA 1 . (a) <u>Every topological space</u> T <u>admits some graded base</u> (N, I).

(b) <u>This</u> (N, I) <u>can be chosen symmetric if</u> T <u>is an</u> R_o-<u>space</u> (<u>e.g., a</u> T_1-<u>space</u>).

(c) If T is regular, (T, N, I) can be made semi-uniform.[1]

Indeed, all this is proved in [2, Theorems 1, 2 and 4].

Below, m will denote a non-negative countably additive measure defined on a σ-algebra $\mathcal{M}$ of subsets ("measurable sets") of a set S. A mapping (function) $f: S \to T$ is called measurable (or $\mathcal{M}$-measurable) if $f^{-1}(G) \in \mathcal{M}$ for every open set $G \subseteq T$; f is said to be simple (elementary) if $f(S)$ is a finite (at most countable) set. Clearly, such an f is measurable if and only if it is constant on certain measurable sets $A_1, A_2, \ldots$.

We now proceed to prove our theorems.

THEOREM 1. Every regular topological space T admits a totally bounded semi-uniform structure (N, I) under which every mapping $f: S \to T$ is the uniform limit of a net $\{f_i \mid i \in I\}$ of simple maps. If further f is $\mathcal{M}$-measurable, the simple maps f_i can be made measurable as well.

Proof. By part (c) of Lemma 1, T admits a semi-uniform base (N, I). Thus, to prove our first assertion, it suffices to show that (N, I) can be transformed into a semi-uniform structure (N', I') which is also totally bounded and is likewise admitted by T (though possibly not equivalent to (N, I)). This can be done by using a method outlined by Behrend [1], with only slight modifications. Thus one need not assume, as Behrend does, that T is a uniform Hausdorff space; our axioms (N1-N3) suffice for his proof of the fact that (N', I') preserves the topology of T and is semi-uniform. Also, instead of the filter of all entourages, it suffices to use the filter base $\{N^i \mid i \in I\}$ consisting of open symmetric entourages and closed under finite intersections (the fact that the sets N^i defined in (N1) are indeed such "entourages", i.e. neighborhoods of the diagonal of $T \times T$, is true in every symmetric structure, by Davis' Theorem 2). The rest of Behrend's proof carries over to our case almost verbally; so we omit its further details and assume that (N, I) is itself totally bounded.

Thus, for each $i \in I$, there is a finite point set $\{p_{ik}\}$,

[1] A strengthening of (c) is contained in Theorem 1 below.

$k = 1, 2, \ldots, n_i$, such that $T = \bigcup_{k=1}^{n_i} N^i(p_{ik})$. Let

$A_{ik} = f^{-1}[N^i(p_{ik}) - \bigcup_{j=1}^{k-1} N^i(p_{ij})]$, $k = 2, \ldots, n_i$, and

$A_{i1} = f^{-1}[N^i(p_{i1})]$. Then, for each (fixed) $i \in I$, the sets A_{ik} are disjoint and $S = \bigcup_{k=1}^{n_i} A_{ik}$. We now define a net of simple functions f_i, $i \in I$, by setting $f_i(x) = p_{ik}$ for $x \in A_{ik}$, $k = 1, 2, \ldots, n_i$. Then, by the definition of the sets A_{ik}, $f(x) \in N^i(f_i(x))$ or, by (N2), $f_i(x) \in N^i(f(x))$, for all $x \in S$ and $i \in I$. This, however, easily implies that $f_i \rightarrow f$ (unif.), as required. Moreover, the A_{ik} are f^{-1}-images of Borel sets (for the sets $N^i(p_{ik})$ are open). Thus, if f is $\mathcal{M}$-measurable, then $A_{ik} \in \mathcal{M}$, i.e. the maps f_i are measurable as well. Q.E.D.

COROLLARY 1. Every map $f: S \rightarrow T$ (where T is an arbitrary topological space) is the pointwise limit of a net of simple maps.

This follows by applying Theorem 1 to T with discrete topology. (suggested by the referee).

COROLLARY 2. If (T, N, I) is symmetric and totally σ-totally) bounded, then every map $f: S \rightarrow T$ is the uniform limit of a net $\{f_i \mid i \in I\}$ of simple (elementary) maps $f_i: S \rightarrow T$ (all measurable if f is).

Indeed, if the total boundedness of (T, N, I) is assumed a priori, the proof of Theorem 1 only requires the use of (N2), not (N3). In the σ-totally bounded case, one only has to replace finite sets $\{p_{ik}\}$ by countable ones.

If $I = \{1, 2, \ldots\}$, the net $\{f_i\}$ becomes a sequence. We use this to obtain a slightly stronger (and rectified) version of a theorem by Monna [5]:

COROLLARY 3. If T is a regular space with countable

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base (e.g., a separable pseudometric space), then every map $f: S \to T$ is the uniform limit of a sequence of simple maps (measurable if f is), under a suitable totally bounded pseudometric d preserving the topology of T (hence it also is a pointwise limit under that topology).

Proof. As T is regular, it is an R_o-space. Thus, for any $p, q \in T$, the closures $\bar{p}$, $\bar{q}$ are either identical or disjoint [2, Theorem 2(e)]. Hence T has a separated quotient space $\bar{T}$ whose elements are such closures $\bar{p}$, with topology defined as follows: a set of elements $\bar{p}$ is open in $\bar{T}$ if and only if its union is open in T. Then our assumptions imply that $\bar{T}$ is a regular T_1-space with countable base. Thus by Urysohn's theorem (cf. [3, p. 125]), $\bar{T}$ is topologically embedded in the Hilbert cube Q^{ω}. Under the metric of Q^{ω}, $\bar{T}$ is totally bounded, while T itself becomes a totally bounded pseudometric space. The result then follows by Cor. 2, with $I = \{1, 2, \ldots\}$ and N_p^i = the sphere of radius 2^{-i} about p, under the pseudometric d inherited from Q^{ω}.

NOTE 1. The idea of using Urysohn's theorem here is due to Monna who, however, considers only metric spaces T and erroneously claims uniform approximation under the original metric of T (cf. his "Conséquence" on p. 405).

THEOREM 2. Every separable space T admits the uniform approximation of any map $f: S \to T$ by a net $\{f_i \mid i \in I\}$ of elementary functions, under any graded base (N, I) for the topology of T.[1] If further T is an R_o-space and f is measurable, then all f_i, for a suitable choice of (N, I), can be made so.

Proof. Let $\{p_k\}$ be a dense sequence in T, and (N, I) a graded base for T. For each $i \in I$ and $k = 1, 2, \ldots$, let $B_{ik} = \{y \in T \mid p_k \in N_y^i\}$. Then, by the density of the p_k,

[1] Such a base exists by part (a) of Lemma 1. Note that here (unlike Theorem 1), (N, I) may be chosen at will.

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$T = \bigcup_{k=1}^{\infty} B_{ik}$ for each $i \in I$. The rest of the proof now proceeds as in Theorem 1, with the sets $N^i(p_{ik})$ replaced by B_{ik}. This yields here a net of elementary maps, with $f_i \to f$ (unif.) on S. If further T is an R_o-space, then T admits a symmetric base (N, I), by part (b) of Lemma 1. Thus, by (N2), $B_{ik} = \{y \in T \mid p_k \in N^i_y\} = \{y \in T \mid y \in N^i(p_k)\} = N^i(p_k)$, so that the B_{ik} are open sets, and the measurability of the f_i results as in Theorem 1. Q.E.D.

NOTE 2. If, in Theorem 2, T also satisfies the first axiom of countability then, clearly, T admits a graded base (N, I), with $I = \{1, 2, \ldots\}$; thus the net $\{f_i\}$ becomes a sequence. However, in general, such a base (N, I) is not symmetric, and the measurability of the f_i fails.

Theorems 1 and 2 may be summarized thus: the regularity (separability) of T ensures a uniform approximation (under a suitable graded base) of any map $f: S \to T$ by simple (elementary) functions f_i. In general, elementary functions cannot be replaced by simple ones. The following error (occurring in Kvačko's Lemma 3 [4, p.89]) should be avoided: given a sequence of elementary functions $f_i \to f$ (unif.), with $f_i \equiv p_k$ on $A_{ik} \subseteq S$ $(i, k = 1, 2, \ldots)$, one can certainly define simple functions g_i $(i = 1, 2, \ldots)$ by setting $g_i = f_i$ on $\bigcup_{j=1}^{k+1} A_{ik}$ and $g_i \equiv p_{k+2}$ on $S - \bigcup_{j=1}^{k+1} A_{ij}$. However, Kvačko's seemingly plausible inference that $g_i \to f$ (ptw.) unfortunately fails to materialize, even if T is a metric space and the sets A_{ik} are defined as in the proof of Theorem 1. As an alternative, avoiding this error, we give below a proposition (Theorem 3) which suffices for most measure-theoretical applications.[1] First we prove:

[1] In particular, it suffices for all of Kvačko's paper, including generalized theorems of Lusin and Egoroff and some applications in integration. Despite its simplicity, the theorem seems to be new in the proposed generality.

LEMMA 2. *If* $(S, \mathcal{M}, m)$ *is a measure space with* $m(S) < \infty$, *and if* $f_i : S \to T$ $(i = 1, 2, \ldots)$ *are measurable elementary mappings, then for every* $\epsilon > 0$ *there is a set* $D \in \mathcal{M}$ *such that* $m(S-D) < \epsilon$ *and such that all* f_i *are measurable and simple on* D *(i.e., they become so when restricted to* D).

Proof. By assumption, each f_i is constant on some disjoint sets $A_{ik} \in \mathcal{M}$ $(k = 1, 2, \ldots)$, with $S = \bigcup_{k=1}^{\infty} A_{ik}$, so that $m(\bigcup_{k=1}^{\infty} A_{ik}) = m(S) < \infty$ and $\lim_{n \to \infty} m(S - \bigcup_{k=1}^{n} A_{ik}) = 0$. Hence, given $\epsilon > 0$, we can find for each i an integer $n_i > 0$ with $m(S - \bigcup_{k=1}^{n_i} A_{ik}) < \epsilon / 2^i$. Let $D_i = \bigcup_{k=1}^{n_i} A_{ik}$ and $D = \bigcap_{i=1}^{\infty} D_i \in \mathcal{M}$. Then $m(S-D) < \epsilon$, and each f_i is measurable and simple on D_i. Hence *all* f_i become so when restricted to D. Q.E.D.

We shall say that $g_i \to f$ *almost uniformly* (a.unif.) on S if for every $\epsilon > 0$ there is a set $D \in \mathcal{M}$ such that $m(S-D) < \epsilon$ and $g_i \to f$(unif.) on D. Then:

THEOREM 3. *If the measure space* $(S, \mathcal{M}, m)$ *is* σ*-finite, every sequence of measurable elementary maps* $f_i : S \to T$ *(where* T *is any graded space) can be replaced by simple measurable maps* g_i *such that* $g_i = f_i$ *on some* $D_i \in \mathcal{M}$ *with* $D_i \subseteq D_{i+1}$, $i = 1, 2, \ldots,$ *and with* $m(S - \bigcup_{i=1}^{\infty} D_i) = 0$. *Moreover, the* g_i *satisfy:*

(a) $g_i \to f$ (a.e.) *on* S *whenever* $f_i \to f$ (a.e.) *on* S, *and*

(b) $g_i \to f$ (a.unif.) *on* S *whenever* $f_i \to f$ (a.unif.) *on* S *and* $m(S) < \infty$.

Proof. (a) By σ-finiteness, $S = \bigcup_{n=1}^{\infty} E_n$ for some sets $E_n \in \mathcal{M}$, $m(E_n) < \infty$. Thus Lemma 2 (with S replaced by E_n) yields for every $n, k = 1, 2, \ldots,$ some $D_{nk} \subseteq E_n$ $(D_{nk} \in \mathcal{M})$

such that $m(E_n - D_{nk}) < 1/k$ and all f_i are simple and measurable on each D_{nk}. We may assume that $D_{nk} \subseteq D_{n,k+1}$ (otherwise replace $D_{n,k+1}$ by $\bigcup_{j=1}^{k+1} D_{nj}$). Clearly, $m(E_n - \bigcup_{k=1}^{\infty} D_{nk}) = 0$. Thus, setting $D_k = \bigcup_{n=1}^{k} D_{nk}$, $k = 1, 2, \ldots$, and neglecting a set of measure zero, we may write $E_n \approx \bigcup_{k=1}^{\infty} D_{nk} \subseteq \bigcup_{k=1}^{\infty} D_k$, $n = 1, 2, \ldots$,[1] whence $S = \bigcup_{n=1}^{\infty} E_n \approx \bigcup_{k=1}^{\infty} D_k$, i.e., $m(S-D) = 0$ where $D = \bigcup_{k=1}^{\infty} D_k \in \mathcal{M}$. Also, since $D_{nk} \subseteq D_{n,k+1}$, we have $D_k \subseteq D_{k+1}$, $k = 1, 2, \ldots$. The formula $S \approx \bigcup_{k=1}^{\infty} D_k$ then yields $\lim_{k\to\infty} m(D_k) = m(S)$. We also note that all f_i are simple and measurable on each $D_k = \bigcup_{n=1}^{k} D_{nk}$ (being so on each D_{nk}).

Thus, setting $g_k = f_k$ on D_k and $g_k \equiv$ const. on $S - D_k$, $k = 1, 2, \ldots$, we obtain a sequence of simple measurable functions g_k on S, satisfying the first clause of Theorem 3. Moreover, we clearly have

(i) $\qquad g_k(x) = f_k(x)$ whenever $x \in D_{nk}$ and $n \leq k$.

Now, if $f_k \to f$ (a.e.) on S, we lose no generality by assuming that $f_k \to f$ (ptw.) on S. We then complete the proof of (a) by showing that $g_k \to f$ (ptw.) on $D = \bigcup_{k=1}^{\infty} D_k$. Fix any $x \in D$: Then there are integers n and $k_0 \geq n$ such that $x \in D_{nk}$ for $k \geq k_0$ (since $D_{nk} \subseteq D_{n,k+1}$). Hence, by (i), $g_k(x) = f_k(x)$

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[1] We write "$\approx$" for an equality valid to within a set of measure 0.

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for $k \geq k_o$, and thus $\lim_{k \to \infty} g_k(x) = \lim_{k \to \infty} f_k(x) = f(x)$, proving (a).

(b) Now suppose that $f_k \to f$ (a.unif.) on S and $m(S) < \infty$. Fix any $\epsilon > 0$. Then, with g_k and D_k as above, the formula $\lim_{k \to \infty} m(D_k) = m(S)$ yields a $k = \underline{k}$ with $m(S - D_{\underline{k}}) < \epsilon/2$. Also, as $f_k \to f$ (a.unif.), there is a set $E \in \mathcal{M}$ such that $m(S-E) < \epsilon/2$ and $f_k \to f$ (unif.) on E and, a fortiori, on $E \cap D_{\underline{k}}$. By the definition of the maps g_k, we have $g_k = f_k$ on $D_k \supseteq D_{\underline{k}}$ for $k \geq \underline{k}$. Hence $g_k \to f$ (unif.) on $E \cap D_{\underline{k}}$. Since $m(S - E \cap D_{\underline{k}}) < \epsilon$, assertion (b) is proved. Q.E.D.

Thus, in σ-finite measure spaces, measurable elementary functions may be replaced by simple ones in practically all cases of interest.

It might be of interest to investigate necessary and sufficient conditions for the replacement of nets by sequences, as well as to analyze point-wise approximations more closely. We leave these questions for a separate paper.

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