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QUASI-HEREDITARY ENDOMORPHISM ALGEBRAS

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ABSTRACT. Quasi-hereditary algebras were introduced by Cline-Parshall-Scott (see [CPS] or [PS]) to deal with highest weight categories which occur in the study of semi-simple complex Lie algebras and algebraic groups. In fact, the quasi-hereditary algebras which appear in these applications enjoy a number of additional properties. The objective of this brief note is to describe a class of lean quasi-hereditary algebras [ADL] which possess such typical characteristics. A study of these questions originated in collaboration with C. M. Ringel (see [DR]).

1. **Introduction.** Throughout the paper, R denotes a (finite dimensional) commutative local self-injective K-algebra with a splitting field K, and A the endomorphism algebra of a (finite) direct sum $X = \bigoplus_{\lambda\in\Lambda} X(\lambda)$ of pair-wise non-isomorphic (finite dimensional) local-colocal R-modules $X(\lambda)$, *i.e.* such that both $X(\lambda)/\operatorname{rad} X(\lambda)$ and $\operatorname{soc} X(\lambda)$ are simple. Write, for each λ, $e_\lambda = m_\lambda p_\lambda$, where $p_\lambda\colon X \longrightarrow X(\lambda)$ and $m_\lambda\colon X(\lambda) \longrightarrow X$ are the canonical projection and embedding, respectively. Thus, for all $\lambda \in \Lambda$, $S(\lambda) = Ae_\lambda/\operatorname{rad} Ae_\lambda$ are the (pair-wise non-isomorphic) left simple A-modules, $P(\lambda) = Ae_\lambda$ their projective covers and $I(\lambda) = \operatorname{Hom}_K(e_\lambda A, K)$ their injective hulls.

Observe that, for each $X(\lambda)$, there is a (unique) embedding into R and that every R-homomorphism $f\colon X(\lambda) \longrightarrow X(\kappa)$ is induced by multiplication by an element $r \in R$: Given f, there is an extension $\bar f\colon R \longrightarrow R$ and every endomorphism of R_R is given by multiplication,

$$\begin{array}{ccccc} 0 & \longrightarrow & X(\lambda) & \longrightarrow & R_R \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f = r}\, . \\ 0 & \longrightarrow & X(\kappa) & \longrightarrow & R_R \end{array}$$

Thus, in particular the image $\operatorname{Im} f$ is isomorphic to a submodule of $X(\lambda)$. As a result, the following three statements which will be used repeatedly, are equivalent:

(a) $R \supseteq X(\kappa) \supseteq X(\lambda)$;

(b) there is a monomorphism from $X(\lambda)$ to $X(\kappa)$;

(c) there is an epimorphism from $X(\kappa)$ to $X(\lambda)$.

Furthermore, each $X(\lambda)$ is a factor module of R and as such has a natural structure of a local commutative self-injective K-algebra; thus $\operatorname{Hom}_K\big(X(\lambda), K\big) \simeq X(\lambda)$. As a consequence, $A = \operatorname{End}_R X$ is an algebra with involution and thus there is a duality functor

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$D: A\text{-mod} \longrightarrow A\text{-mod}$ satisfying $D(S) \simeq S$ for all simple A-modules S. Indeed, the map $*: A \longrightarrow A$ defined for

$$f: X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{f_{\kappa\lambda}} X(\kappa) \xrightarrow{m_\kappa} X$$

by

$$f^*: X \xrightarrow{p_\kappa} X(\kappa) \simeq \mathrm{Hom}_K\big(X(\kappa), K\big) \xrightarrow{\mathrm{Hom}(f_{\kappa\lambda}, K)} \mathrm{Hom}_K\big(X(\lambda), K\big) \simeq X(\lambda) \xrightarrow{m_\lambda} X$$

is an involution. In addition to the relations $(ab)^* = b^*a^*$ and $(a^*)^* = a$, we have also $e_\lambda^* = e_\lambda$ for all $\lambda \in \Lambda$. Hence, we get a duality functor D if, for every right A-module Y_A we define the left module ${}_AY^*$ by putting $Y^* = Y$ and $ay = ya^*$, and set $D(Y_A) = \mathrm{Hom}_K({}_AY^*, K)$. Thus $D\big(P(\lambda)\big) \simeq I(\lambda)$ and $D\big(S(\lambda)\big) \simeq S(\lambda)$.

The main result of this paper is the following theorem.

THEOREM. *Let R be a commutative local self-injective K-algebra over a splitting field K; $\dim_K R = n$. Let $\mathcal{X} = \{X(\lambda) \mid \lambda \in \Lambda\}$ be a set of local ideals of R indexed by a finite partially ordered set Λ reflecting inclusions: $X(\lambda') \subset X(\lambda'')$ if and only if $\lambda' > \lambda''$. Let $R = X(\lambda_1)$ belong to $\mathcal{X}$. Then $A = \mathrm{End}\big(\bigoplus_{\lambda\in\Lambda} X(\lambda)\big)$ is a quasi-hereditary algebra with respect to Λ if and only if*

(i) $\mathrm{card}(\Lambda) = n$ *and*

(ii) $\mathrm{rad}\, X(\lambda) = \sum_{\lambda<\kappa} X(\kappa)$.

Let us add that under the conditions of the theorem, we can easily verify the following facts:

(a) as mentioned earlier, there is a duality functor on the category of A-modules which fixes the simple modules $S(\lambda)$, $\lambda \in \Lambda$;

(b) the algebra A is lean (see [ADL]) and every standard module $\Delta(\lambda)$ has a simple socle isomorphic to $S(\lambda_1)$;

(c) $[\Delta(\lambda) : S(\kappa)] \leq 1$ for all $\lambda, \kappa \in \Lambda$; in fact, $[\Delta(\lambda) : S(\kappa)] = 1$ if and only if $\kappa \leq \lambda$, and thus $\dim_K \Delta(\lambda) = \mathrm{card}\{\kappa \mid \kappa \leq \lambda\}$;

(d) $R/\mathrm{rad}\, R \simeq X(\lambda_n) \in \mathcal{X}$, $\dim_K P(\lambda_n) = n$ and generally

$$\dim_K P(\kappa) = \sum_{\lambda\leq\kappa} \dim_K \Delta(\lambda);$$

thus $\dim_K A = \sum_{\lambda\in\Lambda}\big(\dim_K \Delta(\lambda)\big)^2$;

(e) the dominant dimension of A is ≥ 2 (see [T]).

2. **Proof of sufficiency.** Let A be a finite dimensional (associative) algebra. Let $\{S(\lambda) \mid \lambda \in \Lambda\}$ be the set of all non-isomorphic (left) simple A-modules indexed by a partially ordered set Λ. For every λ, denote by $P(\lambda)$ the projective cover of $S(\lambda)$ and by $\Delta(\lambda)$ the corresponding standard module, *i.e.* the maximal factor module of $P(\lambda)$ with composition factors of the form $S(\kappa)$ for $\kappa \leq \lambda$.

We say that A is quasi-hereditary with respect to Λ if there is a linear order $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ on Λ refining the given partial order and satisfying the following conditions: for each $1 \leq i \leq n$,

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(i) the standard module defined above equals

$$\Delta(\lambda_i) = P(\lambda_i)/\operatorname{trace}\Big(\bigoplus_{j>i} P(\lambda_j) \longrightarrow P(\lambda_i)\Big),$$

(ii) the endomorphism algebra of $\Delta(\lambda_i)$ is a division algebra and

(iii) $P(\lambda_i)$ can be filtered by $\Delta(\lambda_j)$'s, $j \geq i$.

Here, $\operatorname{trace}(X \longrightarrow Y)$ denotes the submodule of Y generated by all homomorphic images of X in Y. The latter condition is equivalent to the fact that the factors

$$\operatorname{trace}\Big(\bigoplus_{j=k}^{n} P(\lambda_j) \longrightarrow P(\lambda_i)\Big) \Big/ \operatorname{trace}\Big(\bigoplus_{j=k+1}^{n} P(\lambda_j) \longrightarrow P(\lambda_i)\Big)$$

of the trace filtration of $P(\lambda_i)$ are direct sums of $\Delta(\lambda_k)$'s $(i \leq k \leq n)$ [D].

The endomorphisms of $X = \oplus_{\lambda\in\Lambda} X(\lambda)$ will operate on the module from the left; thus we shall deal with the left regular representation ${}_A A$ of the (basic) K-algebra $A = \operatorname{End}_R(X)$. Denote by e_λ the canonical idempotent $X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{m_\lambda} X$, $\lambda \in \Lambda$, and note that the set $\{S(\lambda) \mid \lambda \in \Lambda\}$ of all (left) simple A-modules is indexed by the partially ordered set Λ. Put, for every $\lambda \in \Lambda$, $\Lambda(\lambda) = \{\mu \in \Lambda \mid X(\mu) \subsetneq X(\lambda)\}$. Furthermore, since every $X(\lambda)$ is local, there is $x_\lambda \in R$ such that $X(\lambda) = x_\lambda R$.

Now, for the remaining portion of this section assume that conditions (i) and (ii) of the theorem hold. Let us remark that condition (ii) can be expressed in the form $\operatorname{rad} X(\lambda) = \sum_{\mu\in\Lambda(\lambda)} X(\mu)$, $\lambda \in \Lambda$. It follows that there is the largest element $\lambda_n \in \Lambda$ (*i.e.* $\lambda \leq \lambda_n$ for all $\lambda \in \Lambda$) and $X(\lambda_n)$ is the (unique) simple R-module.

First, establish the following three lemmas.

LEMMA 1. *The set $\{x_\lambda \mid \lambda \in \Lambda\}$ is a K-basis of the vector space R_K, and the set of all ideals $X(I) \subseteq R$ generated by $\{x_\lambda \mid \lambda \in I\}$, for every subset I of Λ, forms a distributive lattice with respect to addition and intersection.*

PROOF. In view of (ii), $\{x_\lambda \mid \lambda \in \Lambda\}$ generates the K-space R_K. Furthermore, (i) implies that this set is a K-basis. The rest then follows immediately.

LEMMA 2. *Every R-homomorphism $f: X(\lambda) \longrightarrow \sum_{\mu\in I} X(\mu) \subseteq R$ for some $I \subseteq \Lambda$, factors through the canonical (summation) map $p: \oplus_{\mu\in I} X(\mu) \longrightarrow \sum_{\mu\in I} X(\mu)$. In particular, every R-homomorphism $f: X(\lambda) \longrightarrow \operatorname{rad} X(\kappa)$ factors through the canonical map $\oplus_{\mu\in\Lambda(\kappa)} X(\mu) \longrightarrow \operatorname{rad} X(\kappa)$.*

PROOF. The R-homomorphism f is induced by multiplication; thus

$$f(x_\lambda) = x_\lambda r \in \Big[\sum_{\mu\in I} X(\mu)\Big] \cap X(\lambda) = \sum_{\mu\in I}[X(\mu) \cap X(\lambda)]$$

by Lemma 1. Hence $x_\lambda r = \sum_{\mu\in I} x_\lambda r_\mu$ with $x_\lambda r_\mu \in X(\mu) \cap X(\lambda)$. Consequently, $f = pg$, where $g: X(\lambda) \longrightarrow \oplus_{\mu\in I} X(\mu)$ is given by $g(x_\mu) = (x_\lambda r_\mu \mid \mu \in I)$, as required.

LEMMA 3. *For every $\lambda \in \Lambda$,*

$$(*) \qquad \{m_\kappa m_{\kappa\lambda} p_\lambda \mid X(\lambda) \subseteq X(\kappa)\},$$

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where $m_{\kappa\lambda}$ denotes the embedding $X(\lambda) \subseteq X(\kappa)$, is a K-basis for the (left) standard module $\Delta(\lambda)$. In fact,

$$(**) \qquad \Delta(\lambda) = P(\lambda)/\operatorname{trace}\Big(\bigoplus_{\mu\in\Lambda(\lambda)} P(\mu) \longrightarrow P(\lambda)\Big).$$

PROOF. By definition, $\Delta(\lambda) = P(\lambda)/\operatorname{trace}\big(\oplus_{X(\mu)\not\supseteq X(\lambda)} P(\mu) \longrightarrow P(\lambda)\big)$. Thus to prove $(**)$, it is sufficient to show that every R-homomorphism $f: X(\lambda) \longrightarrow X(\mu)$ with incomparable λ, μ can be factored through a direct sum $\oplus_{\rho\in\Lambda(\lambda)} X(\rho)$. However, this follows readily from Lemma 2, since f cannot be a monomorphism and thus f factors through $\operatorname{rad} X(\lambda) = \sum_{\rho\in\Lambda(\lambda)} X(\rho)$.

Now, since no monomorphism $f: X(\lambda) \longrightarrow X(\kappa)$ can be factored through $\oplus_{\rho\in\Lambda(\lambda)} X(\rho)$, $(*)$ can be seen easily to be a K-basis of $\Delta(\lambda)$.

REMARK. Let us point out that Lemma 3 describes the structure of the standard modules: the factorizations $m_{\kappa\lambda} = m_{\kappa\rho} m_{\rho\lambda}$ correspond to the embeddings $X(\lambda) \subseteq X(\rho) \subseteq X(\kappa)$. In particular, every standard module $\Delta(\lambda)$ has a simple socle generated by $m_{\lambda_1\lambda}$, and hence is isomorphic to $S(\lambda_1)$.

An immediate consequence of Lemma 3 is the fact that the standard modules $\Delta(\lambda)$ remain unchanged under any refinement of the partial order of Λ. We shall therefore consider $\Lambda = \{1, 2, \ldots, n\}$ with its natural order, keeping in mind that $X(j) \subset X(i)$ implies $i < j$. Hence, we shall deal with the complete sequence $(e_1, e_2, \ldots, e_n)$ of primitive orthogonal idempotents: $1_X = \sum_{i=1}^n e_i$. Write $\varepsilon_t = \sum_{i=t}^n e_i$ for $1 \leq t \leq n$ and $\varepsilon_{n+1} = 0$.

To complete the proof of sufficiency, we are going to show that $\operatorname{End}_A\big(\Delta(i)\big)$ is a division algebra, for $1 \leq i \leq n$, and that all factors $A\varepsilon_j Ae_i / A\varepsilon_{j+1} Ae_i$ of the trace filtration of Ae_i are direct sums of $\Delta(j)$'s, $i \leq j \leq n$. The first statement is an instant consequence of Lemma 3: the multiplicity $[\Delta(i) : S(k)] = 1$ if $X(i) \subseteq X(k)$ and $[\Delta(i) : S(k)] = 0$ otherwise. Hence, there is no non-zero map from $\Delta(i)$ into $\operatorname{rad}\Delta(i)$. The second statement is established in the following lemma.

LEMMA 4. *If $X(j) \subseteq X(i)$, then $A\varepsilon_j Ae_i / A\varepsilon_{j+1} Ae_i \simeq \Delta(j)$. If $X(j) \not\subseteq X(i)$, then $A\varepsilon_j Ae_i = A\varepsilon_{j+1} Ae_i$.*

PROOF. We know that $X(j) \subseteq X(i)$ if and only if there is a surjective R-homomorphism $f: X(i) \longrightarrow X(j)$. Since the elements of $A\varepsilon_j Ae_i$ are of the form $m\phi p_i$ with $\phi: X(i) \longrightarrow \oplus_{t=j}^n X(t)$ and $m: \oplus_{t=j}^n X(t) \longrightarrow X$, and those of $A\varepsilon_{j+1} Ae_i$ are of the same form, with ϕ satisfying the additional condition that $p_j m\phi: X(i) \longrightarrow X(j)$ is not surjective, it turns out immediately that $X(j) \not\subseteq X(i)$ yields $A\varepsilon_j Ae_i = A\varepsilon_{j+1} Ae_i$.

On the other hand, if $X(j) \subseteq X(i)$, denote by p_{ji} the surjective R-homomorphism from $X(i)$ to $X(j)$ which maps x_i into $x_j = x_i r$. Evidently, if $f: X(i) \longrightarrow X(j)$ is another surjective R-homomorphism, then there is an automorphism g of $X(j)$ such that $f = gp_{ji}$. Now, if $h: X(j) \longrightarrow X(k)$ is any monomorphism (for instance, m_{kj} of Lemma 3), then $hp_{ji}: X(i) \longrightarrow X(k)$ cannot be factored through $\oplus_{t=j+1}^n X(t)$ since it cannot be factored through $\sum_{t\in\Lambda(j)} X(t) = \operatorname{rad} X(j)$. Recall that such a factorization always exists if h is not a monomorphism. In view of Lemma 3, $A\varepsilon_j Ae_i / A\varepsilon_{j+1}Ae_i \simeq \Delta(j)$.

This completes the proof of sufficiency of the theorem.

3. **Proof of necessity.** We have $X = \bigoplus_{i=1}^n X(i)$, where the linear order of the index set is a refinement of the partial order given by mutual embeddings of the direct summands in R; thus $X(j) \subseteq X(i) \subseteq R$ implies $i \leq j$. Recall that $X(1) = R$ and $\Lambda(i) = \{j \mid X(j) \underset{\neq}{\subset} X(i)\}$.

We assume that $A = \operatorname{End}_R X$ is quasi-hereditary with respect to the complete sequence $(e_1, e_2, \ldots, e_n)$ of primitive orthogonal idempotents defined by the canonical projections p_i and embeddings m_i of the direct summands. Let us, however, point out that A is quasi-hereditary with respect to the original partial order in the sense that $\Delta(i)$ is the maximal factor module of $P(i) \simeq Ae_i$ whose composition factors are only of the form $S(k)$, where k satisfies the inclusion $X(i) \subseteq X(k) \subseteq R$. Thus $\operatorname{trace}\big(\bigoplus_{j=i+1}^n P(j) \longrightarrow P(i)\big) = \operatorname{trace}\big(\bigoplus_{X(i)\not\subseteq X(j)} P(j) \longrightarrow P(i)\big)$.

We are going to prove the necessity of conditions (i) and (ii) of the theorem in Lemmas 6 and 7. First, let us present an auxiliary result.

LEMMA 5. *Let $f\colon X(i) \longrightarrow X(k)$ be an R-homomorphism. If f is a monomorphism, then $m_k f p_i \notin A\varepsilon_{i+1}Ae_i$. If f is not a monomorphism, and A is quasi-hereditary, then $m_k f p_i \in A\varepsilon_{i+1}Ae_i$. Thus, if A is quasi-hereditary, then the multiplicity* $[\Delta(i) : S(k)] = 1$ *for* $X(i) \subseteq X(k)$ *and* $[\Delta(i) : S(k)] = 0$ *otherwise.*

PROOF. The image $\operatorname{Im} f$ of a homomorphism $f\colon X(i) \longrightarrow X(k)$ which factors through $\bigoplus_{j=i+1}^n X(j)$ is isomorphic to a submodule of $\operatorname{rad} X(i)$. However, if f is a monomorphism then $\operatorname{Im} f \simeq X(i)$, and thus $m_k f p_i \notin A\varepsilon_{i+1}Ae_i$. Now, if f is not a monomorphism, then it induces a non-invertible endomorphism of $X(i)$, and therefore, in the case that A is quasi-hereditary, $m_k f p_i$ must belong to $A\varepsilon_{i+1}Ae_i$. Consequently, $[\Delta(i) : S(k)] \neq 0$ if and only if $X(i) \subseteq X(k)$. In fact, in this case, $[\Delta(i) : S(k)] = 1$. Indeed, any two monomorphisms $f_1, f_2\colon X(i) \longrightarrow X(k)$ are induced by multiplication by invertible elements and thus $f_2 = \alpha f_1$, with $\alpha \in R$ invertible. Since $m_k(\beta f_1)p_i \in A\varepsilon_{i+1}Ae_i$ for every non-invertible $\beta \in R$, we can write $m_k f_2 p_i - m_k(\bar{\alpha} f_1)p_i \in A\varepsilon_{i+1}Ae_i$ with $\bar{\alpha} \in R/\operatorname{rad} R \simeq K$, and the lemma follows.

LEMMA 6. *If A is quasi-hereditary, then condition (i) of the theorem holds.*

PROOF. By Lemma 5, $[\Delta(i) : S(1)] = 1$ for all $1 \leq i \leq n$. In view of the duality $D\colon A$-mod $\longrightarrow A$-mod satisfying $D\big(S(i)\big) \simeq S(i)$, which has been mentioned in the Introduction, $D\big(\Delta(i)\big) = \nabla(i)$ satisfying $[\nabla(i) : S(1)] = 1$ for all $1 \leq i \leq n$. Hence, the Bernstein-Gelfand-Gelfand reciprocity law yields $[P(1) : \Delta(i)] = [\nabla(i) : S(1)] = 1$ for all $1 \leq i \leq n$. Consequently,

$$\dim_K R = [P(1) : S(1)] = \sum_{i=1}^n [P(1) : \Delta(i)][\Delta(i) : S(1)] = n.$$

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LEMMA 7. *If A is quasi-hereditary, then condition(ii) of the theorem holds.*

PROOF. Clearly, $\sum_{j\in\Lambda(i)} X(j) \subseteq \operatorname{rad} X(i)$ for $1 \leq i \leq n$. Recall that $X(i) = x_i R$. Thus, if $x \in \operatorname{rad} X(i)$, there is $r \in R$ such that $x_i r = x$. Now, multiplication by r induces a non-invertible endomorphism of $X(i)$ which must factor through $\bigoplus_{j\in\Lambda(i)} X(j)$, so $f = \sum_{j\in\Lambda(i)} f_j g_j$ with $g_j: X(i) \longrightarrow X(j)$ for all $j \in \Lambda(i)$, and thus $x \in \operatorname{Im} f \subseteq \sum_{j\in\Lambda(i)} X(j)$. We conclude that $\operatorname{rad} X(i) = \sum_{j\in\Lambda(i)} X(j)$.

This completes the proof of the theorem.

4. **Final comments.** Let us conclude the paper with a few observations and examples.

First, it is immediate to see that the (ordered) quiver Q_A of the algebra A is given by the monomorphisms and epimorphisms between the direct summands of X. To be more explicit, let $(1, 2, \ldots, n)$ be the sequence of the vertices of Q_A corresponding to a (linear) order of the direct summands $X(1) = R, X(2), \ldots, X(n) = R/\operatorname{rad} R$ of the module X (which refines the partial order Λ of the theorem). Then, for $i > j$, there is an arrow $i \longrightarrow j$ in Q_A if and only if $X(i) \subset X(j) \subseteq R$ and there is no $X(k)$ satisfying $X(i) \subset X(k) \subset X(j) \subseteq R$ for $k \neq i, j$. Furthermore, in that case, there is an arrow $i \longleftarrow j$ corresponding to an epimorphism $X(j) \longrightarrow X(i)$ which cannot be factored through any $X(k)$, $k \neq i, j$. Thus, Q_A is a connected quiver with single arrows which appear in pairs: either there are no arrows between two vertices i and j of Q_A or there is a pair of arrows, $i \rightleftarrows j$. From here, we can easily read the structure of the standard modules established earlier: each $\Delta(i)$ is given by the subquiver of Q_A consisting of all sequences of arrows

$$i = j_0 \longrightarrow j_1 \longrightarrow \cdots \longrightarrow j_{t-1} \longrightarrow j_t = j, \quad i = j_0 > j_1 > \cdots > j_{t-1} > j_t = j,$$

and the respective vertices.

Recall that the trace filtration of the projective-injective indecomposable module

$$P(1) = Ae_1 = A\varepsilon_1 Ae_1 \supset A\varepsilon_2 Ae_1 \supset \cdots \supset A\varepsilon_n Ae_1 \supset 0$$

has the property that $A\varepsilon_i Ae_1 / A\varepsilon_{i+1} Ae_1 \simeq \Delta(i)$ for every $1 \leq i \leq n$. Here, the extensions

$$0 \longrightarrow A\varepsilon_{i+1} Ae_1 \longrightarrow A\varepsilon_i Ae_1 \longrightarrow \Delta(i) \longrightarrow 0$$

are determined by the arrows of Q_A corresponding to the epimorphisms. Observe that there is a (unique) embedding of $P(i)$ in $P(1)$, for every $1 \leq i \leq n$.

The following examples should serve as simple illustrations of the theorem, as well as indications of its limitations.

1. $R = K[x]/\langle x^t \rangle$, $t \geq 1$. There is a unique choice of X (the direct sum of all indecomposable R-modules) and thus A is the respective Auslander algebra. The quiver Q_A is as follows:

$$1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows t-1 \rightleftarrows t.$$

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2. $R = K[x,y]/\langle xy, x^t - y^t\rangle$, $t \geq 2$. Here, for $t \geq 3$, we have several choices for X; for instance, we get the following forms of Q_A:

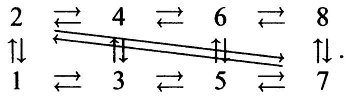
$$\begin{array}{ccccccccccccc} 2 & \rightleftarrows & 4 & \rightleftarrows\cdots\rightleftarrows & 2s & \rightleftarrows & 2s+2 & \rightleftarrows & 2s+4 & \rightleftarrows\cdots\rightleftarrows & 2t \\ \uparrow\downarrow & & \uparrow\downarrow & \cdots & \uparrow\downarrow & & & & & \cdots & \uparrow\downarrow \\ 1 & \rightleftarrows & 3 & \rightleftarrows\cdots\rightleftarrows & 2s-1 & \rightleftarrows & 2s+1 & \rightleftarrows & 2s+3 & \rightleftarrows\cdots\rightleftarrows & 2t-1 \end{array},$$

$1 \leq s \leq t$.

3. $R = K[x,y]/\langle x^2 - y^3, x^3 - y^4, x^4\rangle$. Here, the algebra is 8-dimensional. Write $\bar{p}$ for the canonical image of $p \in K[x,y]$ in R, and consider

$$X = R \oplus \bar{x}R \oplus \bar{y}R \oplus \overline{xy}R \oplus \overline{y^2}R \oplus \overline{xy^2}R \oplus \overline{x^2}R \oplus \overline{x^3}R$$

(in that linear order). Then $A = \mathrm{End}_R X$ is a 159-dimensional algebra whose quiver Q_A has the form



4. Consider again the 4-dimensional algebra $R = K[x,y]/\langle xy, x^2 - y^2\rangle$. Taking

$$X = R \oplus R/\overline{x^2}R \oplus \overline{x^2}R$$

(thus only 3 direct summands, not all local-colocal), or

$$X' = R \oplus (\bar{x}R \oplus R/\overline{x^2}R)/\langle \overline{x^2} - (\bar{y} + \overline{x^2}R)\rangle \oplus \bar{x}R \oplus \overline{x^2}R$$

(thus not all direct summands are local-colocal), the respective endomorphism algebras are still quasi-hereditary. The first one $A = \mathrm{End}_R X$ is a 19-dimensional algebra (without duality) whose quiver Q_A is

$$1 \overset{\rightleftarrows}{\leftarrow} 2 \overset{\rightleftarrows}{\leftarrow} 3.$$

The algebra $A' = \mathrm{End}_R X'$ is a 39-dimensional algebra with duality (and uniserial standard modules whose socles are isomorphic to $S(1)$, $[\Delta(4) : S(2)] = 2$) with $Q_{A'}$ of the form

$$\begin{array}{ccccc} 1 & \rightleftarrows & 2 & \rightleftarrows & 4 \\ & & \uparrow\downarrow & & \\ & & 3 & & \end{array}.$$

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