A note on flasque sheaves

J. M. Campbell

We give an elementary and self-contained proof of the result that the sheaf $\tilde{M}$ determined by an injective module M over a commutative noetherian ring is flasque.

Grothendieck shows in [1] (Corollary 2.7) that

the sheaf $\tilde{M}$ determined by an injective module M over a commutative noetherian ring A is flasque.

The proofs he gives use cohomology or relatively deep results on the structure of injective modules over commutative noetherian rings. The purpose of this note is to give an elementary and self-contained proof. We use the same notation and terminology as Macdonald [2]; in particular, the identity element of A is denoted by 1 .

Let U be an open subset of $\text{Spec}(A)$. Then U is the union of finitely many basic open subsets:

$$U = \bigcup_{i=1}^{n} D(f_i) \ .$$

A section of $\tilde{M}$ over U is then a family of elements

$$s_i = \frac{m_i}{f_i^{k_i}} \in M_{f_i} \qquad (m_i \in M , \ f_i \in A ; \ i = 1, \ldots, n)$$

with the property that there exist integers $l_{ij} \geq 0$ such that

$$(1) \qquad (f_i f_j)^{l_{ij}} \left(f_i^{k_i} m_j - f_j^{k_j} m_i \right) = 0 \qquad (i, j = 1, \ldots, n) \ .$$

Evidently, we may suppose that

$$k_i = k \ , \quad l_{ij} = l \qquad (i,\, j = 1,\, \ldots,\, n) \ ;$$

then (1) becomes

(2) $$\left(f_i f_j\right)^l \left[f_i{}^k m_j - f_j{}^k m_i\right] = 0 \qquad (i,\, j = 1,\, \ldots,\, n) \ .$$

To prove that $\tilde{M}$ is flasque we must show that there exists an element m of M and an integer $s \geq 0$ such that

(3) $$f_i{}^s \left[f_i{}^k m - m_i\right] = 0 \qquad (i = 1,\, \ldots,\, n) \ .$$

Consider, for $s \geq 0$, the system of equations

(4) $$f_i{}^{s+k} x = f_i{}^s m_i \qquad (i = 1,\, \ldots,\, n)$$

in one unknown x . A solution m in M to (4) clearly satisfies (3).

If (4) were compatible, that is, if

$$\sum_{i=1}^n a_i f_i{}^{s+k} = 0 \quad \text{implied} \quad \sum_{i=1}^n a_i f_i{}^s m_i = 0 \quad \text{for all} \quad a_i \in A \ ,$$

then the assignment

$$f_i{}^{s+k} \mapsto f_i{}^s m_i \qquad (i = 1,\, \ldots,\, n)$$

would define a homomorphism $\theta : L \to M$, where L is the ideal in A generated by $f_i{}^{s+k}$ $(i = 1,\, \ldots,\, n)$. But then, since M is injective, θ could be continued to a homomorphism

$$\bar{\theta} : A \to M \ ,$$

and the element $\bar{\theta}(1)$ would be a solution to (4).

It therefore suffices to prove that we can choose s to make (4) a compatible system. We use induction on n , and suppose initially that $n = 1$.

For $s \geq 0$, let J_s be the ideal in A consisting of all elements a such that

$$af_1{}^s m_1 = 0 \ .$$

Then $J_s \subseteq J_{s+1}$, and since A is noetherian, we may choose s so that

$$J_q = J_s \quad \text{for all} \quad q \geq s \ .$$

If $af_1{}^{s+k} = 0$ then $a \in J_{s+k} = J_s$, giving $af_1{}^s m_1 = 0$. Thus (4) is compatible for this choice of s .

Now suppose that $n > 1$. By induction, we may choose $s \geq 0$ so that the system

$$f_i{}^{s+k} x = f_i{}^s m_i \qquad (i = 2, \ldots, n)$$

is compatible. Then M contains an element m_0 such that

$$(5) \qquad\qquad f_i{}^{s+k} m_0 = f_i{}^s m_i \qquad (i = 2, \ldots, n) \ .$$

Let K_s be the ideal in A consisting of all elements a such that

$$af_1{}^s \left(m_1 - f_1{}^k m_0 \right) = 0 \ .$$

Then $K_s \subseteq K_{s+1}$, and we may evidently choose s so that (5) remains valid, $K_q = K_s$ for all $q \geq s$, and $s \geq l$.

We claim that (4) is a compatible system for this choice of s . In fact, let

$$(6) \qquad\qquad \sum_{i=1}^{n} a_i f_i{}^{s+k} = 0 \ .$$

Then $a_1 f_1{}^{s+k+l} m_1 + \sum_{i=2}^{n} a_i f_i{}^{s+k} f_1{}^l m_1 = 0$. Using (2), (5), (6) and the fact that $s \geq l$, we obtain

$$a_1 f_1{}^{s+k+l} \left(m_1 - f_1{}^k m_0 \right) = 0 \ .$$

But then $a_1 \in K_{s+k+l} = K_s$, so that

$$a_1 f_1{}^s \left(m_1 - f_1{}^k m_0 \right) = 0 \ .$$

Using (5) and (6), we obtain

$$\sum_{i=1}^{n} a_i f_i{}^s m_i = 0 \ ,$$

as required for compatibility. This completes the proof.

References

[1]   A. Grothendieck, *Local cohomology* (Lecture Notes in Mathematics 41, Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[2]   I.G. Macdonald, *Algebraic geometry* (W.A. Benjamin, New York, Amsterdam, 1968).

Canberra College of Advanced Education,
Canberra, ACT.