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ON THE REDUCTION OF POSITIVE QUATERNARY QUADRATIC FORMS

Dedicated to George Szekeres on his 65th birthday

E. S. BARNES and M. J. COHN

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Abstract

A fundamental region for the reduction of positive quaternary quadratic forms is exhibited. It is a convex polyhedral cone with twelve edges in the 10-dimensional space of quaternary quadratic forms.

1. Introduction

Two positive definite n-ary quadratic forms $f(\mathbf{x}) = \mathbf{x}'A\mathbf{x}$ and $g(\mathbf{x}) = \mathbf{x}'B\mathbf{x}$, where A, B are symmetric, are said to be equivalent (written $f \sim g$) if there exists an integral unimodular matrix T for which $B = T'AT$. The basic problem of the reduction theory of positive quadratic forms is to specify a region D (called a fundamental region) in the $\frac{1}{2}n(n+1)$-dimensional coefficient space of forms satisfying

(i) for every positive definite n-ary quadratic form f, there exists a form $f_0 \in D$ with $f_0 \sim f$

and

(ii) if f, g are distinct forms in the interior of D, then $f \not\sim g$.

It is desirable if, as for the well-known reduction methods for $n = 2$ and $n = 3$, D is a convex polyhedral cone. Minkowski (1905) and Venkov (1940) established methods for the construction of such fundamental regions for all n; however, when $n \geqq 4$, it is difficult to describe these explicitly, and they have

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very large numbers of facets and edges. For $n = 4$, Štogrin (1974) produced a fundamental region which is the union of three cones, but which is not convex; here we specify a fundamental region which is a convex cone with only twelve edges.

We show:

A fundamental region of positive quaternary quadratic forms is given by the convex cone of forms which is the set of non-negative linear combinations of the following twelve forms:

$$x_1^2 + x_2^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2,$$

$$(x_2 - x_3)^2 + (x_2 - x_4)^2, (x_2 - x_4)^2, x_3^2 + x_4^2 + (x_3 - x_4)^2,$$

$$\omega_0(\boldsymbol{x}) = \sum_{0 \leq i < j \leq 4} (x_i - x_j)^2 \qquad (\textit{where } x_0 \equiv 0),$$

$$\omega_1(\boldsymbol{x}) = 4\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4\}$$

and

$$\alpha(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2$$
$$+ (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_1 + x_2 - x_3 - x_4)^2.$$

We prove this by a refinement of the reduction into the cones $R(\varphi_0)$ and $R(\varphi_1)$ of the first reduction method of Voronoi (1907). We note that ω_0 and ω_1 are respectively (multiples of) the forms adjoint to the perfect forms

$$\varphi_0(\boldsymbol{x}) = \sum_{1 \leq i \leq j \leq 4} x_i x_j, \quad \varphi_1(\boldsymbol{x}) = \varphi_0(\boldsymbol{x}) - x_1 x_2.$$

2. Some preliminaries

For the proof we require some properties of Voronoi reduction. Consider the transformations

$$\mathcal{T} : A \mapsto T'AT \tag{2.1}$$

where T is an integral unimodular matrix. We note that such a transformation is linear and therefore continuous. Voronoi's reduction method partitions the space of forms into cones $R(\varphi)$, where φ is perfect. A transformation $\mathcal{T}$ either leaves a cone $R(\varphi)$ invariant, or transforms it into a cone $R(\varphi')$ which has no interior form in common with $R(\varphi)$. So, if two distinct forms $f, g \in \operatorname{int} R(\varphi)$ are equivalent, then there exists an automorphism of $R(\varphi)$ transforming f to g.

It is frequently convenient to identify the transformations of the space of forms with the corresponding transformations of n-dimensional space. However, since both T and $-T$ correspond to the same transformation $\mathcal{T}$ in (2.1), it is sometimes convenient to remove the factor $\{\pm I\}$ in specifying the group $\mathcal{A}(\varphi)$ of automorphisms of a region $R(\varphi)$.

For $n = 4$, any positive definite quadratic form is equivalent to a form of $R(\varphi_0)$ or $R(\varphi_1)$. The automorphism groups of φ_0 and φ_1 are described by Coxeter (1951), where these forms are denoted by A_4 and B_4.

3. Reduction of $R(\varphi_0)$

The cone $R(\varphi_0)$ of Voronoi's reduction contains precisely those forms f of the type

$$f(x) = \sum_{0 \leqq i < j \leqq 4} \rho_{ij}(x_i - x_j)^2 \tag{3.1}$$

where $x_0 \equiv 0$ and $\rho_{ij} \geqq 0$ for all i, j. The group of automorphisms of $R(\varphi_0)$ (after removing the factor $\{\pm I\}$) may be identified with the group of permutations of $x_0, x_1, \cdots, x_n$.

Since two forms f, g in the interior of $R(\varphi_0)$ are equivalent only if there exists an automorphism of $R(\varphi_0)$ transforming f to g, we use the group of automorphisms of $R(\varphi_0)$ to obtain a fundamental region for those forms in $R(\varphi_0)$. Firstly, by a suitable permutation of $x_0, x_1, \cdots, x_n$, we may insist that

$$\rho_{12} = \min_{0 \leqq i < j \leqq 4} \rho_{ij}. \tag{3.2}$$

This divides the variables into the two sets $\{x_1, x_2\}$ and $\{x_0, x_3, x_4\}$. To distinguish x_0 from x_3 and x_4, by permutation of x_0, x_3 and x_4, we may insist further that

$$\rho_{34} = \min\{\rho_{03}, \rho_{04}, \rho_{34}\}. \tag{3.3}$$

The variables x_1 and x_2 are distinguished by arranging that

$$\rho_{01} = \min\{\rho_{01}, \rho_{02}\}. \tag{3.4}$$

The only variables not fully distinguished yet are x_3 and x_4. Therefore we stipulate that

$$\rho_{23} = \min\{\rho_{23}, \rho_{24}\}. \tag{3.5}$$

The conditions (3.2), (3.3), (3.4) and (3.5) define a convex polyhedral cone in the 10-dimensional coefficient space of forms with edge-forms

$$x_1^2 + x_2^2,\ x_2^2,\ x_3^2,\ x_4^2,\ (x_1 - x_3)^2,\ (x_1 - x_4)^2,$$

$$(x_2 - x_3)^2 + (x_2 - x_4)^2,\ (x_2 - x_4)^2,\ x_3^2 + x_4^2 + (x_3 - x_4)^2$$

and

$$\omega_0(\boldsymbol{x}) = \sum_{0 \leq i < j \leq 4} (x_i - x_j)^2 \qquad (\text{where } x_0 \equiv 0).$$

Let us denote this region by F. Clearly any form of the type (3.1) is equivalent to a form of F. Furthermore, a form f is in the interior of F if and only if each of (3.2), (3.3), (3.4) and (3.5) distinguish the relevant x_i uniquely, and so F must be a fundamental region for forms equivalent to those in $R(\varphi_0)$.

4. Reduction of $R(\varphi_1)$

The set $R(\varphi_1)$ is the convex polyhedral cone with edge-forms

$$x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2,$$
$$(x_2 - x_4)^2, (x_3 - x_4)^2, (x_1 + x_2 - x_3)^2, (x_1 + x_2 - x_4)^2 \quad \text{and} \quad (x_1 + x_2 - x_3 - x_4)^2.$$

The cone is more easily examined after making the transformation

$$2X_1 = x_1 + x_2$$
$$2X_2 = x_1 - x_2$$
$$2X_3 = -x_1 - x_2 + 2x_3$$
$$2X_4 = -x_1 - x_2 + 2x_4,$$

so that

$$\begin{aligned} X_1 - X_2 &= x_2 & \qquad X_1 + X_2 &= x_1 \\ X_1 - X_3 &= x_1 + x_2 - x_3 & X_1 + X_3 &= x_3 \\ X_1 - X_4 &= x_1 + x_2 - x_4 & X_1 + X_4 &= x_4 \\ X_2 - X_3 &= x_1 - x_3 & X_2 + X_3 &= -x_2 + x_3 \\ X_2 - X_4 &= x_1 - x_4 & X_2 + X_4 &= -x_2 + x_4 \\ X_3 - X_4 &= x_3 - x_4 & X_3 + X_4 &= -x_1 - x_2 + x_3 + x_4. \end{aligned}$$

Then the forms of $R(\varphi_1)$ are precisely those forms f with

$$f(\boldsymbol{x}) = \sum_{1 \leq i < j \leq 4} \sigma_{ij}(X_i - X_j)^2 + \sum_{1 \leq i < j \leq 4} \tau_{ij}(X_i + X_j)^2 \tag{4.1}$$

where σ_{ij}, $\tau_{ij} \geqq 0$ for all i, j. In particular, the adjoint form of φ_1 is now

$$\omega_1(x) = \sum_{i=1}^{4} X_i^2 = \frac{1}{6}\left\{ \sum_{1 \leq i < j \leq 4} (X_i - X_j)^2 + \sum_{1 \leq i < j \leq 4} (X_i + X_j)^2 \right\}. \tag{4.2}$$

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Similarly to the reduction of $R(\varphi_0)$, we make use here of the automorphism group $\mathscr{A}(\varphi_1)$ of $R(\varphi_1)$. Coxeter (1951) showed that this group (including $\pm I$) has order $3 \cdot 2^4 \cdot 4! = 1152$; it is generated by

(i) the permutations of X_1, X_2, X_3, X_4,

(ii) arbitrary changes of sign of the X_i

and

(iii) the transformation

$$U: 2X_i \mapsto 2X_i - \sum_{j=1} X_j \quad (i = 1,2,3,4),$$

for which

$$X_i - X_j \mapsto X_i - X_j$$

and

$$X_i + X_j \mapsto -(X_k + X_l),$$

where i, j, k, l is any arrangement of 1,2,3,4.

Now suppose f is a given form of $R(\varphi_1)$, expressed in the form (4.1). By applying permutations and changes of sign to the X_i, we may arrange that

$$\sigma_{13} = \min_{1 \leqq i < j \leqq 4} \{\sigma_{ij}, \tau_{ij}\}.$$

Further, we may require that σ_{14} is the least coefficient with one subscript 1 or 3, that is,

$$\sigma_{14} = \min\{\sigma_{12}, \tau_{12}, \sigma_{14}, \tau_{14}, \sigma_{23}, \tau_{23}, \sigma_{34}, \tau_{34}\}.$$

These two conditions give

$$\begin{array}{ll}
\sigma_{12} = \sigma_{13}' + \sigma_{14}' + \sigma_{12}' & \tau_{12} = \sigma_{13}' + \sigma_{14}' + \tau_{12}' \\
\sigma_{13} = \sigma_{13}' & \tau_{13} = \sigma_{13}' + \tau_{13}' \\
\sigma_{14} = \sigma_{13}' + \sigma_{14}' & \tau_{14} = \sigma_{13}' + \sigma_{14}' + \tau_{14}' \\
\sigma_{23} = \sigma_{13}' + \sigma_{14}' + \sigma_{23}' & \tau_{23} = \sigma_{13}' + \sigma_{14}' + \tau_{23}' \\
\sigma_{24} = \sigma_{13}' + \sigma_{24}' & \tau_{24} = \sigma_{13}' + \tau_{24}' \\
\sigma_{34} = \sigma_{13}' + \sigma_{14}' + \sigma_{34}' & \tau_{34} = \sigma_{13}' + \sigma_{14}' + \tau_{34}'
\end{array}$$

where $\sigma_{ij}', \tau_{ij}' \geqq 0$ for all i, j. Hence

$$f(\mathbf{x}) = (6\sigma_{13}' + 4\sigma_{14}') \sum_{i=1}^{4} X_i^2 + \sum_{\substack{1 \leqq i < j \leqq 4 \\ (i,j) \neq (1,3),(1,4)}} \sigma_{ij}'(X_i - X_j)^2 + \sum_{1 \leqq i < j \leqq 4} \tau_{ij}'(X_i + X_j)^2,$$

and so, by (4.2), we may write

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$$f(\boldsymbol{x}) = \mu\omega_1(\boldsymbol{x}) + \sum_{1 \leqq i < j \leqq 4} \sigma_{ij}(X_i - X_j)^2 + \sum_{1 \leqq i < j \leqq 4} \tau_{ij}(X_i + X_j)^2 \tag{4.3}$$

where $\mu \geqq 0$; $\sigma_{ij}, \tau_{ij} \geqq 0$ for all i, j; $\sigma_{13} = \sigma_{14} = 0$.

A change in the sign of X_2 and the transformation U both transform f to another form of the type (4.3) with $\sigma_{13} = \sigma_{14} = 0$. Furthermore, the three edge-forms $(X_1 - X_2)^2$, $(X_1 + X_2)^2$ and $(X_3 + X_4)^2$ are equivalent under these two automorphisms. Hence, we may take f in the form (4.3) with $\sigma_{13} = \sigma_{14} = 0$ and

$$\tau_{34} = \min\{\sigma_{12}, \tau_{12}, \tau_{34}\}.$$

We now split cases:

I. Suppose $\tau_{34} \leqq \sigma_{34}$. Then $\tau_{34} = \min\{\sigma_{12}, \tau_{12}, \sigma_{34}, \tau_{34}\}$ and we may subtract the term

$$\tau_{34}\{(X_1 - X_2)^2 + (X_1 + X_2)^2 + (X_3 - X_4)^2 + (X_3 + X_4)^2\} = 2\tau_{34} \sum_{i=1}^{4} X_i^2$$
$$= 2\tau_{34}\omega_1(\boldsymbol{x})$$

and get f in the form (4.3) with $\sigma_{13} = \sigma_{14} = \tau_{34} = 0$.

II. Suppose $\sigma_{34} \leqq \tau_{34}$. Then $\sigma_{34} = \min\{\sigma_{12}, \tau_{12}, \sigma_{34}, \tau_{34}\}$ and we may subtract the term

$$\sigma_{34}\{(X_1 - X_2)^2 + (X_1 + X_2)^2 + (X_3 - X_4)^2 + (X_3 + X_4)^2\} = 2\sigma_{34}\omega_1(\boldsymbol{x})$$

and get f in the form (4.3) with $\sigma_{13} = \sigma_{14} = \sigma_{34} = 0$.

The two cases correspond precisely to forms f belonging to the cones Δ' and Δ'' respectively of the second reduction method of Voronoi (1908, 1909); the above is a simpler method of obtaining the same reduction. Before we proceed to specify fundamental subregions of Δ' and Δ'', we calculate the orders of the automorphism groups of Δ' and Δ''.

Voronoi (1907, §§34–38 and §43) examined the facets of $R(\varphi_1)$, that is, the 9-dimensional faces of $R(\varphi_1)$ in the 10-dimensional coefficient space of quaternary quadratic forms. He proved the existence of exactly 64 facets. Of these, 48 are equivalent under automorphisms of $R(\varphi_1)$ to the set of forms f of $R(\varphi_1)$ in the form (4.1) with $\sigma_{13} = \sigma_{14} = \tau_{34} = 0$, and the remaining 16 equivalent to the set of forms with $\sigma_{13} = \sigma_{14} = \sigma_{34} = 0$. The latter 16 facets are not equivalent to the other 48; for the difference between any two linear forms corresponding to zero coefficients in the expansion of f for a form of the second facet is of the type

$$(X_a - X_b) - (X_a - X_c) = -(X_b - X_c)$$

which is a linear form corresponding to a zero coefficient, whereas for a form f of the first facet

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$$(X_1 - X_3) - (X_1 - X_4) = -(X_3 - X_4)$$

which is not a linear form corresponding to a zero coefficient. So the 64 facets fall into two equivalence classes, the first containing 48 facets and the second 16 facets.

Now, the region Δ' is the set of forms f as in (4.3) with $\sigma_{13} = \sigma_{14} = \tau_{34} = 0$. So it is the convex cone spanned by the form $\omega_1(\boldsymbol{x})$ and a facet of $R(\varphi_1)$ of the first kind. Since there are 48 such facets, the group $\mathscr{A}(\Delta')$ of automorphisms of Δ' (including $\pm I$) is a subgroup of $\mathscr{A}(\varphi_1)$ of index 48 and so has order $3 \cdot 2^4 \cdot 4!/48 = 4!$. Similarly, the region Δ'' is the cone spanned by $\omega_1(\boldsymbol{x})$ and a facet of the second kind. Hence the group $\mathscr{A}(\Delta'')$ of automorphisms of Δ'' is a subgroup of $\mathscr{A}(\varphi_1)$ of index 16 and so has order $3 \cdot 2^4 \cdot 4!/16 = 3 \cdot 4!$.

We now consider the two cases separately.

I. *Reduction of* Δ'. Since the group $\mathscr{A}(\Delta')$ of automorphisms of Δ' is the group of automorphisms of $R(\varphi_1)$ which preserve the property $\sigma_{13} = \sigma_{14} = \tau_{34} = 0$, clearly $\mathscr{A}(\Delta')$ contains

(i) the permutations of $-X_1, X_3$ and X_4

(ii) $X_2 \mapsto -X_2$

and

(iii) $(X_1, X_3, X_4) \mapsto (-X_1, -X_3, -X_4)$.

The group generated by these automorphisms has order $3! \cdot 2 \cdot 2 = 4!$, which is precisely the order of $\mathscr{A}(\Delta')$; hence the group generated is $\mathscr{A}(\Delta')$.

Let f be a form in Δ'. By permuting $-X_1, X_3$ and X_4, we may arrange that

$$\sigma_{34} = \min\{\tau_{13}, \tau_{14}, \sigma_{34}\}.$$

We may still apply a permutation of X_3 and X_4, and the sign changes in (iii); so we insist further that

$$\tau_{12} = \min\{\sigma_{12}, \tau_{12}\}$$

and

$$\tau_{23} = \min\{\tau_{23}, \tau_{24}\}.$$

Thus each form in Δ' is equivalent to a form f as in (4.3) with

$$\mu \geqq 0; \ \sigma_{ij}, \tau_{ij} \geqq 0 \text{ for all } i, j$$

$$\sigma_{13} = \sigma_{14} = \tau_{34} = 0$$

$$\sigma_{34} = \min\{\tau_{13}, \tau_{14}, \sigma_{34}\}$$

$$\tau_{12} = \min\{\sigma_{12}, \tau_{12}\}$$

$$\tau_{23} = \min\{\tau_{23}, \tau_{24}\}.$$

By its method of specification, it is evident that the set of such forms is a fundamental region for forms of Δ'; we denote it by F'. It is convex polyhedral cone with edge-forms

$$\begin{aligned}
&\omega_l(\boldsymbol{x}) &&= \omega_l(\boldsymbol{x})\\
&(X_1 - X_2)^2 + (X_1 + X_2)^2 &&= x_1^2 + x_2^2\\
&(X_1 - X_2)^2 &&= x_2^2\\
&(X_1 + X_3)^2 &&= x_3^2\\
&(X_1 + X_4)^2 &&= x_4^2\\
&(X_2 - X_3)^2 &&= (x_1 - x_3)^2\\
&(X_2 - X_4)^2 &&= (x_1 - x_4)^2\\
&(X_2 + X_3)^2 + (X_2 + X_4)^2 &&= (x_2 - x_3)^2 + (x_2 - x_4)^2\\
&(X_2 + X_4)^2 &&= (x_2 - x_4)^2\\
&(X_1 + X_3)^2 + (X_1 + X_4)^2 + (X_3 - X_4)^2 &&= x_3^2 + x_4^2 + (x_3 - x_4)^2.
\end{aligned}$$

Clearly, F and F' have a common facet.

II. *Reduction of* Δ''. We recall that Δ'' is the set of forms f as in (4.3) with $\sigma_{13} = \sigma_{14} = \sigma_{34} = 0$. The group $\mathscr{A}(\Delta'')$ of automorphisms of Δ'' contains

(i) the permutations of X_1, X_3 and X_4
(ii) $X_2 \mapsto -X_2$
(iii) $(X_1, X_3, X_4) \mapsto (-X_1, -X_3, -X_4)$

and

(iv) the automorphism T_0 which is obtained by applying firstly the automorphism

$$U: 2X_i \mapsto 2X_i - \sum_{j=1}^{4} X_j \qquad \text{for } i = 1,2,3,4,$$

and then the automorphism

$$(X_1, X_2, X_3, X_4) \mapsto (X_4, -X_2, X_1, X_3).$$

Since T_0 may be shown to have order three, it is easily verified that the group generated by these automorphisms has order $3! \cdot 2 \cdot 2 \cdot 3 = 3 \cdot 4!$ which is the order of $\mathscr{A}(\Delta'')$ (including $\pm I$). So the group generated is $\mathscr{A}(\Delta'')$.

Under T_0, the forms $(X_i \pm X_j)^2$ fall into four equivalence classes of three; the corresponding coefficients are

$$\{\sigma_{13}, \sigma_{14}, \sigma_{34}\}, \{\sigma_{12}, \tau_{13}, \tau_{23}\}, \{\tau_{12}, \tau_{14}, \sigma_{24}\} \quad \text{and} \quad \{\sigma_{23}, \tau_{24}, \tau_{34}\}.$$

So we may arrange that the least coefficient of those in the last three classes is one of τ_{13}, τ_{14} and τ_{34}. Then, by permuting X_1, X_3 and X_4, we may insist that τ_{34} is the least such coefficient, that is,

$$\tau_{34} = \min\{\sigma_{12}, \tau_{12}, \tau_{13}, \tau_{14}, \sigma_{23}, \tau_{23}, \sigma_{24}, \tau_{24}, \tau_{34}\}. \tag{4.4}$$

Since $\mathscr{A}(\Delta'')$ includes both $\pm I$, we may arbitrarily fix the sign of X_1, and so by (iii) of X_3 and X_4. Using (ii), we may now specify that

$$\tau_{12} = \min\{\sigma_{12}, \tau_{12}\}. \tag{4.5}$$

The only automorphism not yet used is the interchange of X_3 and X_4; so we may now insist that

$$\tau_{23} = \min\{\tau_{23}, \tau_{24}\}. \tag{4.6}$$

It is evident that the set of forms f as in (4.3) with $\sigma_{13} = \sigma_{14} = \sigma_{34} = 0$ satisfying (4.4), (4.5) and (4.6) is a fundamental region for those forms in Δ''; let us denote it by F''. It is a convex polyhedral cone with edge-forms

$$\begin{aligned}
&\omega_1(\boldsymbol{x}) && = \omega_1(\boldsymbol{x})\\
&(X_1 - X_2)^2 + (X_1 + X_2)^2 && = x_1^2 + x_2^2\\
&(X_1 - X_2)^2 && = x_2^2\\
&(X_1 + X_3)^2 && = x_3^2\\
&(X_1 + X_4)^2 && = x_4^2\\
&(X_2 - X_3)^2 && = (x_1 - x_3)^2\\
&(X_2 - X_4)^2 && = (x_1 - x_4)^2\\
&(X_2 + X_3)^2 + (X_2 + X_4)^2 && = (x_2 - x_3)^2 + (x_2 - x_4)^2\\
&(X_2 + X_4)^2 && = (x_2 - x_4)^2\\
&(X_1 - X_2)^2 + (X_1 + X_2)^2 + (X_1 + X_3)^2 && = x_1^2 + x_2^2 + x_3^2 + x_4^2\\
&\qquad + (X_1 + X_4)^2 + (X_2 - X_3)^2 && \qquad + (x_1 - x_3)^2 + (x_1 - x_4)^2\\
&\qquad + (X_2 + X_3)^2 + (X_2 - X_4)^2 && \qquad + (x_2 - x_3)^2 + (x_2 - x_4)^2\\
&\qquad + (X_2 + X_4)^2 + (X_3 + X_4)^2 && \qquad + (x_1 + x_2 - x_3 - x_4)^2.
\end{aligned}$$

By comparison, we see that F' and F'' have a facet in common, while F and F'' have a common 8-dimensional face.

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5. Union of the reduced regions

The region of forms given in §1 to be proved fundamental is the convex hull H of $F \cup F' \cup F''$. Since every positive definite quaternary quadratic form is equivalent to a form of $R(\varphi_0)$ or $R(\varphi_1)$, every such form is equivalent to one of $F \cup F' \cup F''$ and hence of H. To show that H is fundamental, we show that $F \cup F' \cup F''$ is precisely H and that the region H has no two equivalent forms in its interior.

Firstly, we show that $F \cup F' \cup F''$ is H. Since $F \cup F' \cup F''$ is a subset of H, it is sufficient to show that H is a subset of $F \cup F' \cup F''$. Now H is a convex cone with at most twelve edges. Eight edges are common to F, F' and F''; the other four are generated by the forms

$$x_3^2 + x_4^2 + (x_3 - x_4)^2, \quad \omega_0(\boldsymbol{x}),\ \omega_1(\boldsymbol{x})$$

and

$$\alpha(\boldsymbol{x}) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_1 + x_2 - x_3 - x_4)^2.$$

It is easily established that

$$2\omega_0(\boldsymbol{x}) + \omega_1(\boldsymbol{x}) \in F \cap F',$$

$$\{x_3^2 + x_4^2 + (x_3 - x_4)^2\} + \alpha(\boldsymbol{x}) \in F' \cap F''$$

and

$$\omega_0(\boldsymbol{x}) + \alpha(\boldsymbol{x}) \in F \cap F' \cap F''.$$

(The argument also shows that no edge-form of H is redundant, and so H has precisely twelve edges.) Hence, since any form of H may be expressed as a nonnegative linear combination of the twelve edge-forms of H, by considering the relative magnitudes of the coefficients corresponding to the above four edge-forms, we may deduce that a form of H belongs to one of F, F' and F'', and so to $F \cup F' \cup F''$.

Finally, we prove that no two distinct forms in the interior of H are equivalent. For suppose f, g are distinct interior forms of H with $f \sim g$. Then f, g are equivalent by a continuous transformation of the space of forms. So there exist equivalent forms f_0, g_0 arbitrarily close to f, g respectively for which $f_0 \neq g_0$ and either

(i) both f_0 and g_0, belong to the interior of the same one of F, F' and F''

or

(ii) f_0 and g_0 belong to the interiors of different sets of F, F' and F''.

Both situations are impossible, since F, F' and F'' are fundamental regions for forms of $\Delta(= R(\varphi_0))$, Δ' and Δ'' respectively.

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University of Adelaide
Adelaide
South Australia 5001.