SOME GENERATING-FUNCTION EQUIVALENCES†

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A generalization is given of a theorem of F. Brafman [1] on the equivalence of generating relations for a certain sequence of functions. The main result, contained in Theorem 2 below, may be applied to several special functions including the classical orthogonal polynomials such as Hermite, Jacobi (and, of course, Legendre and ultraspherical), and Laguerre polynomials.

1. Let a sequence of functions $f_n(x)$, $n = 0, 1, 2, \ldots$, be defined by the Rodrigues formula

$$f_n(x) = \frac{1}{n!} D_x^n\{(ax+b)^n F(x)\}, \; D_x = d/dx, \tag{1}$$

where a and b are constants, not both zero, and $F(x)$ is independent of n and differentiable an arbitrary number of times. The following result is due to F. Brafman [1].

THEOREM 1. *If a generating function*

$$\sum_{n=0}^{\infty} a_n f_n(x) t^n \tag{2}$$

is known for either

$$a_n = {}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} \; y \right] \tag{3}$$

or

$$a_n = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}, \tag{4}$$

then it is automatically known for the other.

Also a further result holds connecting a generating function of the set $f_n(x)$ with one of the set $f_{2n}(x)$.

Brafman's proof of Theorem 1 involves contour integration and makes use of Cauchy's integral formula and two known generating relations for certain hypergeometric polynomials (cf. [1], pp. 156–158). It may be of interest to observe that a substantially more general generating-function equivalence than what is contained in Theorem 1 would follow fairly easily from Lagrange's theorem [3, p. 133]

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$$\frac{f(z)}{1-t\phi'(z)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n \{[\phi(x)]^n f(x)\}, \; z = x+t\phi(z). \tag{5}$$

Indeed we first obtain the following

LEMMA. *For every sequence $\{f_n(x)\}$ defined by* (1), *the generating relation*

$$\sum_{n=0}^{\infty} \binom{m+n}{n} f_{m+n}(x)t^n = (1-at)^{-m-1} f_m\left(\frac{x+bt}{1-at}\right) \tag{6}$$

holds when $m = 0, 1, 2, \ldots$.

To prove this lemma we notice from (1) and (5) that

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{m+n}{n} f_{m+n}(x)t^n &= \frac{1}{m!} D_x^m \sum_{n=0}^{\infty} \frac{t^n}{n!} D_x^n\{(ax+b)^{m+n}F(x)\} \\ &= \frac{(1-at)^{-1}}{m!} D_x^m\{(az+b)^m F(z)\} \\ &= \frac{(1-at)^{-m-1}}{m!} D_z^m\{(az+b)^m F(z)\},\end{aligned}$$

since $z = (x+bt)/(1-at)$, and the generating relation (6) follows by appealing to the definition (1) once again.

We now state our main result given by

THEOREM 2. *For arbitrary coefficients* λ_n, $n = 0, 1, 2, \ldots$, *and integer* $N \geqq 1$, *if we let a generating function*

$$\sum_{n=0}^{\infty} A_n f_{rn}(x)t^n \tag{7}$$

be known for either

$$A_n = \sum_{k=0}^{[n/N]} \binom{n}{Nk} \lambda_k y^k \quad \text{and} \quad r = 1 \tag{8}$$

or

$$A_n = \lambda_n \quad \text{and} \quad r = N,$$

then it is automatically known for the other.

Proof. From (7) and (8) we have

$$\begin{aligned}\Omega \equiv \sum_{n=0}^{\infty} A_n f_n(x)t^n &= \sum_{n=0}^{\infty} f_n(x)t^n \sum_{k=0}^{[n/N]} \binom{n}{Nk} \lambda_k y^k \\ &= \sum_{k=0}^{\infty} \lambda_k (yt^N)^k \sum_{n=0}^{\infty} \binom{n+Nk}{Nk} f_{n+Nk}(x)t^n,\end{aligned}$$

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by inverting the order of summations. This process may be justified when the series involved converge absolutely. However, the series need not converge, and in such cases a divergent generating function is formally obtained.

Now apply the lemma with $m = Nk$, where N-1, $k = 0, 1, 2, \ldots$, in order to sum the inner series, and we find that

$$\Omega = (1-at)^{-1} \sum_{k=0}^{\infty} \lambda_k f_{Nk}\left(\frac{x+bt}{1-at}\right)\left[\frac{yt^N}{(1-at)^N}\right]^k, \tag{10}$$

whose second member involves a generating function of the type given by (7) and (9).

This evidently completes the proof of Theorem 2, which provides an elegant connection between a generating function of the set $f_n(x)$ with one of the set $f_{Nn}(x)$, where N is an arbitrary positive integer.

Alternatively, to prove Theorem 2, the principle illustrated in [**1**] may be applied *mutatis mutandis*. Indeed, if C denotes a simple closed contour about $z = x$, then from (1) and Cauchy's integral formula we readily have

$$f_n(x) = \frac{1}{2\pi i}\int_C \frac{(az+b)^n F(z)}{(z-x)^{n+1}}\, dz, \qquad n = 0, 1, 2, \ldots, \tag{11}$$

whence

$$\begin{aligned}\Omega &= \sum_{n=0}^{\infty} A_n f_n(x) t^n \\ &= \frac{1}{2\pi i}\int_C \frac{F(z)}{z-x}\sum_{n=0}^{\infty}\left[\frac{(az+b)t}{z-x}\right]^n \sum_{k=0}^{[n/N]}\binom{n}{Nk}\lambda_k y^k\, dz \\ &= \frac{1}{2\pi i}\int_C \frac{F(z)}{z-x}\sum_{k=0}^{\infty}\lambda_k y^k\left[\frac{(az+b)t}{z-x}\right]^{Nk}\sum_{n=0}^{\infty}\binom{n+Nk}{Nk}\left[\frac{(az+b)t}{z-x}\right]^n dz \\ &= (1-at)^{-1}\sum_{k=0}^{\infty}\left[\frac{yt^N}{(1-at)^N}\right]^k \frac{1}{2\pi i}\int_C \frac{(az+b)^{Nk}F(z)\, dz}{\{z-(x+bt)/(1-at)\}^{Nk+1}},\end{aligned}$$

by the familiar binomial expansion, and the generating-function equivalence (10) follows, since the pole at $z = (x+bt)/(1-at)$ can always be placed inside C by taking t sufficiently small and then the result extended by analytic continuation on t.

We remark in passing that Theorem 2 will yield the generating-function equivalences contained in Theorem 1 when the arbitrary coefficients λ_n are specialized by

$$\lambda_n = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}, \qquad n = 0, 1, 2, \ldots, \tag{12}$$

y is replaced by $-y$, and the arbitrary positive integer N is set equal to 1 or 2.

2. Applications. Since a fairly large number of special functions satisfy a Rodrigues formula of type (1), the generating-function equivalences given by Theorem 2 are widely applicable. We content ourselves by noting the familiar instances

$$f_n(x) = \frac{(-1)^n e^{-x^2}}{n!} H_n(x) = \frac{1}{n!} D_x^n \{e^{-x^2}\}, \tag{13}$$

$${}_n(x) = x^\alpha e^{-x} L_n^{(\alpha)}(x) = \frac{1}{n!} D_x^n \{x^{n+\alpha} e^{-x}\}, \tag{14}$$

$$f_n(x) = x^{\alpha-n} e^{-x} L_n^{(\alpha-n)}(x) = \frac{1}{n!} D_x^n \{x^\alpha e^{-x}\}, \tag{15}$$

$$\begin{aligned} f_n(x) &= 2^n (x-1)^\alpha (x+1)^{\beta-n} P_n^{(\alpha,\, \beta-n)}(x) \\ &= \frac{1}{n!} D_x^n \{(x-1)^{n+\alpha} (x+1)^\beta\}, \end{aligned} \tag{16}$$

$$\begin{aligned} f_n(x) &= 2^n (x-1)^{\alpha-n} (x+1)^\beta P_n^{(\alpha-n,\, \beta)}(x) \\ &= \frac{1}{n!} D_x^n \{(x+1)^{n+\beta} (x-1)^\alpha\}, \end{aligned} \tag{17}$$

$$\begin{aligned} f_n(x) &= 2^n (x-1)^{\alpha-n} (x+1)^{\beta-n} P_n^{(\alpha-n,\, \beta-n)}(x) \\ &= \frac{1}{n!} D_x^n \{(x-1)^\alpha (x+1)^\beta\}, \end{aligned} \tag{18}$$

involving the classical orthogonal polynomials of Hermite, Laguerre, and Jacobi. The results of the preceding section would apply also to the ultraspherical polynomials $P_n^\alpha(x)$, the Legendre polynomials $P_n(x)$, and the Bessel polynomials $y_n(x, \alpha-n, \beta)$, since we have

$$\begin{aligned} f_n(x) &= (-1)^n (x^2-1)^{-\alpha-n/2} P_n^\alpha(x/\sqrt{(x^2-1)}) \\ &= \frac{1}{n!} D_x^n \{(x^2-1)^{-\alpha}\}, \end{aligned} \tag{19}$$

$$P_n(x) = P_n^{\frac{1}{2}}(x), \tag{20}$$

and (cf. [2], p. 111, Equation (47))

$$\begin{aligned} f_n(x) &= \frac{\beta^n x^{\alpha-n-2} e^{-\beta/x}}{n!} y_n(x, \alpha-n, \beta) \\ &= \frac{1}{n!} D_x^n \{x^{n+\alpha-2} e^{-\beta/x}\}. \end{aligned} \tag{21}$$

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Thus, in each of the aforementioned cases, one can easily apply the lemma as well as Theorem 2 to derive a (known) generating relation of type (6) and a new class of generating-function equivalences of type (10). We omit the details.

On the other hand, for the sequence of hypergeometric functions

$$g_n^{(\alpha,\beta)}(x) = \frac{(\alpha)_n}{n!}\,{}_2F_1[\tfrac{1}{2}n+\tfrac{1}{2}\alpha, \tfrac{1}{2}n+\tfrac{1}{2}\alpha+\tfrac{1}{2}; \beta; x], \qquad n = 0, 1, 2, \ldots, \tag{22}$$

it is easily verified that

$$\sum_{n=0}^{\infty} \binom{m+n}{n} g_{m+n}^{(\alpha,\beta)}(x)t^n = (1-t)^{-\alpha-m} g_m^{(\alpha,\beta)}\left(\frac{x}{(1-t)^2}\right), \qquad m = 0, 1, 2, \ldots, \tag{23}$$

which is of type (6). Thus, as an analogue of the generating-function equivalence (10), we have

$$\sum_{n=0}^{\infty} A_n g_n^{(\alpha,\beta)}(x)t^n = (1-t)^{-\alpha} \sum_{n=0}^{\infty} \lambda_n g_{Nn}^{(\alpha,\beta)}(x/(1-t)^2)[yt^N/(1-t)^N]^n, \tag{24}$$

where the A_n are given by (8).

In view of the familiar Gaussian transformation

$${}_2F_1[\alpha, \beta; \gamma; z] = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1[\gamma-\alpha, \gamma-\beta; \gamma; z], \qquad |z| < 1, \tag{25}$$

it follows from (22) that

$$g_n^{(2\alpha+k,\alpha+1/2)}((x^2-1)/x^2) = \frac{(n+k)!\,x^{n+k+2\alpha}}{n!\,(2\alpha)_k} P_{n+k}^{\alpha}(x), \qquad n, k = 0, 1, 2, \ldots, \tag{26}$$

since $P_n^{\alpha}(x)$ may be defined by

$$P_n^{\alpha}(x) = \frac{(2\alpha)_n x^n}{n!}\,{}_2F_1[-\tfrac{1}{2}n, -\tfrac{1}{2}n+\tfrac{1}{2}; \alpha+\tfrac{1}{2}; (x^2-1)/x^2]. \tag{27}$$

Now the A_n given by (8) can evidently be reduced in terms of the ultraspherical polynomials $P_n^{\beta}(x)$ if in (8) we set $N = 2$, $\lambda_n = (\tfrac{1}{2})_n/(\beta+\tfrac{1}{2})_n$, and replace y by $(y^2-1)/y^2$. Hence, by interpreting the first member of (24) with the help of (26), and the second member by means of (22), we shall arrive at the bilinear generating relation

$$\sum_{n=0}^{\infty} \frac{(n+k)!}{(2\beta)_n} P_{n+k}^{\alpha}(x) P_n^{\beta}(y) t^n = (2\alpha)_k (x-yt)^{-2\alpha-k} \cdot F_4\left[\alpha+\tfrac{1}{2}k, \alpha+\tfrac{1}{2}k+\tfrac{1}{2}; \alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}; \frac{x^2-1}{(x-yt)^2}, \frac{(y^2-1)t^2}{(x-yt)^2}\right], \tag{28}$$

where $k = 0, 1, 2, \ldots$, and F_4 denotes the fourth type of Appell's functions defined by

$$F_4[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{29}$$

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REFERENCES

1. F. Brafman, Generating functions and associated Legendre polynomials, *Quart. J. Math.* (Oxford) (2) **10** (1959), 156–160.

2. H. L. Krall and O. Frink, A new class of orthogonal polynomials: The Bessel polynomials, *Trans. Amer. Math. Soc.* **65** (1949), 100–115.

3. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth edition (Cambridge, 1963).

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