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SOME PROPERTIES OF SEMI-ABELIAN p-GROUPS

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Dedicated to Professor Ming-Yao Xu for his early work on finite p-groups

Abstract

We prove a cohomological property for a class of finite p-groups introduced earlier by Xu, which we call semi-abelian p-groups. This result implies that a semi-abelian p-group has noninner automorphisms of order p, which settles a long-standing problem for this class. We answer also, independetly, an old question posed by Xu about the power structure of semi-abelian p-groups.

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1. Introduction

Let G be a finite p-group. Following Xu (see [14]), we say that G is *strongly semi-p-abelian* if the following property holds in G:

$$(xy^{-1})^{p^n} = 1 \Leftrightarrow x^{p^n} = y^{p^n} \text{ for any postive integer } n.$$

For brevity, we shall use the term *semi-abelian* for such a group. It is easy to see that such a group satisfies the properties:

(i) the exponent of $\Omega_n(G)$ divides p^n;
(ii) $|G : G^{\{p^n\}}| = |\Omega_n(G)|$, and so $|G : G^{p^n}| \leq |\Omega_n(G)|$.

Hence, semi-abelian p-groups share some nice properties with the regular p-groups introduced by Hall (see [3] for their theory). It is not difficult to show that every regular p-group is semi-abelian; however, the class of semi-abelian p-groups is much larger, and in fact every finite p-group can occur as a quotient of a semi-abelian p-group (see Section 3).

Let G be a regular p-group, and $1 < N \lhd G$ such that G/N is not cyclic. Schmid showed in [12] that the Tate cohomology groups $\hat{H}^n(G/N, Z(N))$ are all nontrivial, where $Z(N)$ is considered as a G/N-module with the action induced by conjugation in G. Our first purpose is to show that Schmid's result holds in a more general context.

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Theorem 1.1. *Let G be a semi-abelian p-group, and $1 < N \lhd G$ such that G/N is neither cyclic nor a generalised quaternion group. Then $\hat{H}^n(G/N, \mathrm{Z}(N)) \neq 0$, for all integers n.*

Note that Schmid conjectured that Theorem 1.1 holds for an arbitrary finite p-group G if one takes $N = \Phi(G)$. This conjecture has been refuted by Abdollahi in [2]. Nevertheless, it is interesting to find other classes of p-groups which satisfy the conclusion of Theorem 1.1 (see [1, Question 1.2]).

The above theorem is intimately related to the study of noninner automorphisms of finite p-groups. Let A be an abelian normal subgroup of a group G, considered as a G-module via conjugation. Each element δ of $\mathrm{Der}(G, A)$, the group of crossed homomorphisms or derivations from G into A, determines an endomorphism ϕ_δ of G, given by $\phi_\delta(x) = x\delta(x)$, $x \in G$. This map ϕ defines a bijection between the two sets $\mathrm{Der}(G, A)$ and $\mathrm{End}_A(G)$, where

$$\mathrm{End}_A(G) = \{\theta \in \mathrm{End}(G) \mid x^{-1}\theta(x) \in A, \text{ for all } x \in G\}.$$

If we consider only the set $\mathrm{Der}(G/\mathrm{C}_G(A), A)$ of derivations $\delta : G \to A$, that are trivial on $\mathrm{C}_G(A)$, then the map ϕ induces an isomorphism between $\mathrm{Der}(G/\mathrm{C}_G(A), A)$ and the group $\widetilde{\mathrm{C}}(A)$ of the automorphisms of G acting trivially on $\mathrm{C}_G(A)$ and G/A.

It is straightforward to see that this isomorphism maps the inner derivations, $\mathrm{Ider}(G/\mathrm{C}_G(A), A)$, into a group of inner automorphisms lying in $\widetilde{\mathrm{C}}(A)$, though an inner automorphism lying in $\widetilde{\mathrm{C}}(A)$ need not necessarily be induced by an inner derivation. This case can be avoided by assuming that $\mathrm{C}_G(\mathrm{C}_G(A)) = A$; indeed, if $\phi_\delta(x) = x^g$ for some $g \in G$ and all $x \in G$, then $g \in \mathrm{C}_G(\mathrm{C}_G(A))$, so g lies in A and δ is the inner derivation induced by g^{-1}. We have established the following proposition.

Proposition 1.2. *Under the above assumption, there is an isomorphism between $\mathrm{Der}(G/\mathrm{C}_G(A), A)$ and $\widetilde{\mathrm{C}}(A)$, which maps $\mathrm{Ider}(G/\mathrm{C}_G(A), A)$ exactly to the inner automorphisms lying in $\widetilde{\mathrm{C}}(A)$.*

This well-known fact in the literature (see [8]) permits us to reduce the problem of existence of noninner automorphisms of some group to a cohomological problem. For instance, this allowed W. Gaschütz to prove that any nonsimple finite p-group has noninner automorphisms of p-power order.

It is conjectured by Berkovich that a more refined version of Gaschütz's result holds, more precisely that a nonsimple finite p-group has noninner automorphisms of order p (see [11, Problem 4.13]). While it is not clear that a positive answer has deep implications for our understanding of finite p-groups, this problem has attracted wide interest, and its difficulty may stimulate further development of new techniques in finite p-group theory. The reader may find more information about this problem and relevant references in [1].

Our second result settles this problem in the class of semi-abelian p-groups.

Theorem 1.3. *Let G be a semi-abelian finite p-group. Then G has a noninner automorphism of order p.*

Our notation is standard in the literature. Let S be a group. For a positive integer n, we denote by $S^{\{p^n\}}$ the set of the p^nth powers of all the elements of S and by S^{p^n}

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the subgroup generated by $S^{\{p^n\}}$. The subgroup generated by the elements of order dividing p^n is denoted by $\Omega_n(S)$. We denote by $\lambda_n^p(S)$ the terms of the lower p-series of S which are defined inductively by

$$\lambda_1^p(S) = S \quad \text{and} \quad \lambda_{n+1}^p(S) = [\lambda_n^p(S), S]\lambda_n^p(S)^p.$$

If A is an S-module, then A_S denotes the subgroup of fixed elements in A under the action of S.

The rest of this paper is divided into two sections. In Section 2 we prove Theorem 1.1 and Theorem 1.3; and in Section 3 we answer an old question posed by Xu (see [14, Problem 3]) about the power structure of semi-abelian p-groups. This result follows quickly from a result of Bubboloni and Corsi Tani (see [4]), but it seems that this link has not been noted before.

2. Proofs

Let Q be a finite p-group, and A be a Q-module of p-power order. Recall that A is said to be *cohomologically trivial* if $\hat{H}^k(S, A) = 0$ for all $S \le Q$ and all integers k.

W. Gaschütz and K. Ushida proved (independently) that $\hat{H}^1(Q, A) = 0$ implies that $\hat{H}^k(S, A) = 0$ for all $S \le Q$ and all integers $k \ge 1$ (see [8, Lemma 2, Section 7.5]). This statement can be slightly improved (using dimension-shifting) as noted in [9].

Proposition 2.1. *Let Q be a finite p-group, and A be a Q-module which is also a finite p-group. If $\hat{H}^n(Q, A) = 0$ for some integer n, then A is cohomologically trivial.*

We shall use Proposition 2.1 to reduce the proof of Theorem 1.1 to the nonvanishing of the Tate cohomology groups in dimension 0, which is easier to handle.

We need the following result due to Schmid (see [12, Proposition 1]).

Proposition 2.2. *Let Q be a finite p-group, and $A \neq 1$ be a Q-module which is also a finite p-group. If A is cohomologically trivial, then $C_Q(A_K) = K$, for every $K \le Q$.*

We also need to prove the following lemma.

Lemma 2.3. *Under the assumption of Theorem 1.1, set $A = Z(N)$ and let S/N be a subgroup of exponent p of G/N. Then $A^p \le A_{S/N}$, so that $C_{S/N}(A^p) = S/N$.*

Proof. Let be $x \in S$ and $a \in A$. We have $x^p \in N$, hence

$$x^p = (x^p)^a = (x^a)^p = (x[x, a])^p.$$

As G is semi-abelian, we have $[x, a]^p = 1$. It follows that $(a^{-1}a[a, x])^p = [a, x]^p = 1$, and again since G is semi-abelian,

$$a^p = (a[a, x])^p = (a^x)^p = (a^p)^x.$$

This shows that A^p is centralised by every element of S/N. □

Proof of Theorem 1.1. Assume for a contradiction that $\hat{H}^n(G/N, A) = 0$ for some integer n, where A denotes $Z(N)$. As G/N is not cyclic and different from the generalised quaternion groups Q_{2^m}, there is in G/N a subgroup S/N of exponent p

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and order at least p^2. It follows from Proposition 2.1 that $\hat{H}^n(S/N, A) = 0$, so A is a cohomologically trivial S/N-module. Let $K/N \leq S/N$ be a subgroup of order p. Proposition 2.1 implies that $\hat{H}^0(K/N, A) = 0$. We have $\hat{H}^0(K/N, A) = A_{K/N}/A^\tau = 0$, where A^τ is the image of A under the trace homomorphism $\tau : A \to A$ induced by K/N. As K/N is cyclic of order p, our trace map is given by

$$a^\tau = aa^x \cdots a^{x^{p-1}} \quad \text{for } a \in A \text{ and any fixed } x \in K - N,$$

from which it follows that

$$a^\tau = (ax^{-1})^p x^p.$$

Now as G is semi-abelian, $a \in \ker\tau$ if and only if $a^p = 1$; that is, $\ker\tau = \Omega_1(A)$. This implies that $|A^\tau| = |A^p|$. As $A_{K/N} = A^\tau$, and $A^p \leq A_{K/N}$ by Lemma 2.3, we have $A^p = A_{K/N}$. By Proposition 2.2, $\mathrm{C}_{S/N}(A^p) = \mathrm{C}_{S/N}(A_{K/N}) = K/N$. However, Lemma 2.3 implies that $S/N = K/N$, a contradiction. □

Remark 2.4. Assume that G is regular and N is maximal in G. Schmid (see [12, Proposition 2]) showed that the trace map τ of the G/N-module $\mathrm{Z}(N)$ is given by $\tau(a) = a^p$ for $a \in \mathrm{Z}(N)$. We may ask if this remains true if G is semi-abelian.

Let $A = \langle a_1 \rangle \oplus \langle a_2 \rangle$, with $a_1^4 = a_2^2 = 1$, and let G be the semi-direct product $A\langle t \rangle$, where $t^4 = 1$ and t acts on A as follows:

$$a_1{}^t = a_1 a_2 \quad \text{and} \quad a_2{}^t = a_2.$$

It is straightforward to see that $\Omega_1(G)$ is central and has order 8, and so we have a well-defined surjective map:

$$G/\Omega_1(G) \longrightarrow G^{\{2\}}$$
$$x\Omega_1(G) \mapsto x^2.$$

On the other hand, an elementary calculation yields $G^{\{2\}} = \{1, a_1{}^2, t^2, a_1{}^2 a_2 t^2\}$, so that $G/\Omega_1(G)$ and $G^{\{2\}}$ have the same order. It follows that the above map is bijective, thus G is semi-abelian.

Take $N = A\langle t^2 \rangle$, so N is abelian and maximal in G. We have

$$\tau(a_1) = a_1 a_1{}^t = a_1{}^2 a_2 \neq a_1{}^2.$$

Thus Schmid's result does not hold for semi-abelian p-groups in general.

Before proving Theorem 1.3, we need the following reduction from [5].

Proposition 2.5. *Let G be a finite p-group such that $\mathrm{C}_G(\mathrm{Z}(\Phi(G)) \neq \Phi(G)$. Then G has a noninner automorphism of order p.*

Note that Ghoraishi improved [7, Proposition 2.5], where he reduced the problem of Berkovich to the p-groups G satisfying $H \leq \mathrm{C}_G(H) = \Phi(G)$, where H is the inverse image of $\Omega_1(\mathrm{Z}(G/\mathrm{Z}(G)))$ in G. A family of examples which satisfy the condition $\mathrm{C}_G(\mathrm{Z}(\Phi(G)) = \Phi(G)$ and do not satisfy Ghoraishi's condition can be found in the same paper.

Proof of Theorem 1.3. Assume for a contradiction that every automorphism of G of order p is inner. Let $A = \mathrm{Z}(\Phi(G))$. By Proposition 2.5, we have $\mathrm{C}_G(A) = \Phi(G)$ and so $\mathrm{C}_G(\mathrm{C}_G(A)) = A$. If we prove that $\mathrm{Der}(G/\mathrm{C}_G(A), A) = \mathrm{Der}(G/\Phi(G), \mathrm{Z}(\Phi(G))$

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has exponent p, then our first assumption together with Proposition 1.2 implies that $\hat{H}^1(G/\Phi(G), Z(\Phi(G))) = 0$, which contradicts Theorem 1.1. So we only need to prove, for any derivation $\delta \in \mathrm{Der}(G, Z(\Phi(G))$ which is trivial on $\Phi(G)$, that $\delta(x)^p = 1$, for all $x \in G$. Indeed,

$$\delta(x^p) = \delta(x)\delta(x)^x \cdots \delta(x)^{x^{p-1}} = (\delta(x)x^{-1})^p x^p.$$

As δ is trivial on $\Phi(G)$, we have $\delta(x^p) = (\delta(x)x^{-1})^p x^p = 1$, and since G is semi-abelian it follows that $\delta(x)^p = 1$. □

3. Remarks on a particular class of semi-abelian *p*-groups

Xu proved in [13] that any finite p-group G with p odd, which satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$, is semi-abelian. Here $\gamma_{p-1}(G)$ is the $(p-1)$th term of the lower central series of G. He asked if such a group must be power closed (see [14, Problem 3]), that is, every element of G^{p^n} is a p^nth power.

A negative answer to this question will follow from the following important result of Bubboloni and Corsi Tani (see [4]).

Recall that a p-central group is a group in which every element of order p is central. Bubboloni and Corsi Tani used the term TH-group instead of p-central group, where TH-group refers to Thompson, who seems to have been the first to observe the importance of p-central groups (see [10, Hilfssatz III.12.2]).

Theorem 3.1 [4, Theorem 3.3]. *Let d and n be two positive integers, p an odd prime, and F the free group on d generators. Then $G_n = F/\lambda_{n+1}^p(F)$ is a p-central p-group. More precisely, we have $\Omega_1(G_n) = \lambda_n^p(F)/\lambda_{n+1}^p(F)$.*

Corollary 3.2. *A finite p-group with p odd, which satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$, need not necessarily be power closed.*

Proof. Obviously a p-central p-group satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$. Assume for a contradiction that the result is false. So the p-groups G_n are all power closed. Now every finite p-group is a quotient of some G_n (for appropriate n and d), and the property of being power closed is inherited by quotients. It follows that every finite p-group is power closed, which is a contradiction. □

Let us briefly mention another consequence of Theorem 3.1. It is widely believed that finite p-central p-groups are dual (in a sense) to powerful p-groups. Since the inverse limits of powerful p-groups have roughly a uniform structure (see [6, Section 3]), which qualifies them to play a central role in studying analytic pro-p-groups (see [6]), it is natural to ask if there is a restriction on the structure of a *pro-p-central p-group*, that is, an inverse limit of finite p-central p-groups.

Let F be the free group on a finite number of generators. As every normal subgroup of F of p-power index contains a subgroup $\lambda_n^p(F)$ for some n, it follows that

$$\widehat{F}_p \cong \varprojlim F/\lambda_n^p(F)$$

where $\widehat{F}_p$ is the pro-p-completion of F. This shows that the free pro-p-group $\widehat{F}_p$ is pro-p-central, so there is no reasonable restriction on the structure of a (finitely generated) pro-p-central p-group.

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