SOME PROPERTIES OF NON-COMMUTATIVE REGULAR GRADED RINGS

by THIERRY LEVASSEUR

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Introduction. Let A be a noetherian ring. When A is commutative (of finite Krull dimension), A is said to be Gorenstein if its injective dimension is finite. If A has finite global dimension, one says that A is regular. If A is arbitrary, these hypotheses are not sufficient to obtain similar results to those of the commutative case. To remedy this problem, M. Auslander has introduced a supplementary condition. Before stating it, we recall that the grade of a finitely generated (left or right) module is defined by

$$j_A(M) = \inf\{i \,/\, \mathrm{Ext}^i_A(M, A) \neq 0\} \in \mathbb{N} \cup \{+\infty\}.$$

The Auslander condition is: for all M as above, for all $i \geq 0$ and every A-submodule N of $\mathrm{Ext}^i_A(M, A)$, one has $j_A(N) \geq i$. A ring of finite (left and right) injective dimension (resp. finite homological global dimension) which satisfies the Auslander condition is called Auslander-Gorenstein (resp. Auslander-regular). On the other hand, assume that A is a graded algebra over a field k, of the form $A = k \oplus A_1 \oplus A_2 \oplus \ldots$. Then M. Artin and W. Schelter have introduced in **[1]** a definition of regularity for A. (See Section 6.1 for a precise definition.) We shall then say that A is AS-regular.

Let A be a commutative regular affine k-algebra, and assume for simplicity that A is a domain. One knows that A is a Cohen-Macaulay ring, which is equivalent to:

$$\dim M + j_A(M) = \dim A \quad \text{for all non-zero finitely generated } A\text{-modules } M, \qquad \text{(CM)}$$

where dim denotes the Krull dimension. When A is noncommutative, a natural substitute for the Krull dimension is the Gelfand-Kirillov dimension, denoted by GKdim. We then have many examples of Auslander-regular k-algebras which have the property (CM), where dim is replaced by GKdim. The main examples occur when A is a filtered ring with a regular commutative graded associated ring, see **[6]**. The work of **[3]**, **[4]** shows that the property (CM) plays an important role in the study of modules over AS-regular algebras.

The purpose of the present work is to compare the two notions of regularity introduced above and to study the property (CM) for graded noetherian k-algebras. The paper is organized as follows.

Sections 1 and 2 recall results about filtered and graded rings and Auslander-Gorenstein or regular rings.

Section 3 investigates the behaviour of the Auslander condition when we factor a graded k-algebra $A = \bigoplus_{n \geq 0} A_n$ by a normal homogeneous non-zero divisor of positive degree.

In Section 4, we show that over an Auslander-Gorenstein ring of injective dimension μ, $M \to \delta(M) = \mu - j_A(M)$ defines a dimension function in the sense of **[17]**. This is used to prove that an Auslander-regular k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ is a domain.

Section 5 studies the property (CM) over a graded k-algebra such that finitely generated graded modules have polynomial growth; we call this condition (PG). The main results are obtained in 5.10 and 5.13. Theorem 5.10 gives the following. Let $A = \bigoplus_{n \geq 0} A_n$ be a finitely generated positively graded k-algebra with $\dim_k A_0 < \infty$; suppose

$\Omega \in A_d$, $d > 0$, is a normal non-zero divisor in A such that $A/\Omega A$ is Auslander-Gorenstein and satisfies the condition (PG) and property (CM); then the same is true for A. In 5.13, a criterion is given for a graded k-algebra of injective dimension 2 to be an Auslander-Gorenstein algebra satisfying property (CM).

In Section 6, we compare the two notions of regularity given above. Theorem 6.3 shows that an Auslander-regular graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ such that $n \to \dim_k A_n$ has polynomial growth is AS-regular. In 6.6, we consider the algebras $B(X, \sigma, \mathscr{L})$ constructed in **[5]** from an elliptic curve X, an automorphism σ of X and an invertible sheaf $\mathscr{L}$ of degree at least 3 on X. Using 5.13, we show that $B(X, \sigma, \mathscr{L})$ is Auslander-Gorenstein of dimension 2 and has the property (CM). As a corollary (see 6.7), we deduce that the Sklyanin algebra as defined in **[18]** is Auslander-regular and satisfies the property (CM).

In **[15]**, this last fact is used to obtain detailed results about graded modules over the Sklyanin algebra.

1. Preliminaries.

1.1. For the rest of the paper, we fix a commutative field k. The dimension of a k-vector space is denoted by $\dim_k$.

Let A be a ring. We denote by $\mathrm{Mod}(A)$ (resp. $\mathrm{Mod}_f(A)$) the category of left or right A-modules (resp. finitely generated A-modules). We write ${}_AM$ or M_A to indicate that M is a left or right A-module. The ring A is called noetherian if it is left and right noetherian.

Let ${}_AM$ (resp. M_A) be in $\mathrm{Mod}(A)$ and $a \in A$. We say that a is a non-zero-divisor (n.z.d) in M if $ax = 0$, $x \in M$, implies $x = 0$ (resp. $xa = 0$ implies $x = 0$). If $a \in A$ is a n.z.d. in ${}_AA$ (resp. A_A), we say that a is a right (resp. left) n.z.d. in A. If a is a left and right n.z.d., we simply say that a is a n.z.d.

1.2. Let M be an A-module. We let $\mathrm{pd}_A(M)$ denote the projective dimension of M and $\mathrm{injdim}_A(M)$ the injective dimension of M. We say that A has finite global dimension (resp. finite injective dimension) if the left and right global dimensions of A are finite and equal (resp. the modules A_A and ${}_AA$ have finite injective dimensions which are equal). In that case we denote these numbers by $\mathrm{gldim}(A)$ (resp. $\mathrm{injdim}(A)$).

Definition. Let M be in $\mathrm{Mod}(A)$. The grade number of M is

$$j_A(M) = \inf\{i \,/\, \mathrm{Ext}^i_A(M, A) \neq (0)\} \in \mathbb{N} \cup \{+\infty\}.$$

If no confusion can arise, we write $j(M)$ for $j_A(M)$. Notice that $j_A((0)) = +\infty$. When A is noetherian, $j_A(M) \leq \mathrm{pd}_A(M)$, and if furthermore $\mathrm{injdim}(A) = \mu < \infty$, we have $j_A(M) \leq \mu$ for all non-zero $M \in \mathrm{Mod}_f(A)$ (see Remark 2.2(1)).

1.3. An (increasing) filtration $\{A(n)\}_{n \in \mathbb{Z}}$ on A is a sequence of additive subgroups of A such that: $1 \in A(0)$, $A(n)A(m) \subset A(n+m)$, $A = \bigcup_{n \in \mathbb{Z}} A(n)$. The associated graded ring, denoted by $G(A)$ or $\mathrm{gr}(A)$, is the direct sum $\bigoplus_{n \in \mathbb{Z}} A(n)/A(n-1)$. We have similar definitions for a decreasing filtration. When A is a k-algebra, we require that $k \subset A(0)$. If $\bigcap_{n \in \mathbb{Z}} A(n) = (0)$, the filtration is called separated. We refer to **[17]** for the notions of filtered modules, associated graded modules, etc.

1.4. Assume $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded ring. We say that A is positively graded if $A_n = (0)$ for all $n<0$. Then $\mathcal{M} = \bigoplus_{n>0} A_n$ is called the augmentation ideal and $A_0 \cong A/\mathcal{M}$ is a bimodule over A. If furthermore A is a k-algebra with $A_0 = k$ then we write $A = k \oplus A_1 \oplus A_2 \oplus \ldots$. We denote by $\mathrm{Mod}^g(A)$ (resp. $\mathrm{Mod}_f^g(A)$) the category of graded A-modules (resp. finitely generated graded A-modules). Let $M = \bigoplus_m M_m \in \mathrm{Mod}^g(A)$. For $p \in \mathbb{Z}$, the shifted module $M[p]$ is defined in $\mathrm{Mod}^g(A)$ by setting $M[p]_m = M_{p+m}$. Recall that if M, $N \in \mathrm{Mod}^g(A)$ then $\mathrm{HOM}_A(M,N)$ is the $\mathbb{Z}$-graded group such that $\mathrm{HOM}_A(M,N)_p = \{\phi \in \mathrm{Hom}_A(M,N)/\phi(M_m) \subset N_{m+p}$ for all $m\}$. When $M \in \mathrm{Mod}_f^g(A)$, one has $\mathrm{HOM}_A(M,N) = \mathrm{Hom}_A(M,N)$; thus the derived functors $\mathrm{EXT}_A^{\bullet}(M,—)$ coincide with the usual $\mathrm{Ext}_A^{\bullet}(M,—)$. When not otherwise specified, a map $M \to N$ between graded A-modules will be an element of $\mathrm{HOM}_A(M,N)_0$.

The graded ring A has finite graded injective dimension μ if ${}_AA$ and A_A are both of injective dimension μ in the category $\mathrm{Mod}^g(A)$. We then write $\mathrm{grinjdim}(A) = \mu$. Recall that $E \in \mathrm{Mod}^g(A)$ is injective in $\mathrm{Mod}^g(A)$ if and only if the functor $\mathrm{HOM}_A(—,E)$ is exact, i.e. $\mathrm{EXT}_A^i(—,E) = 0$ for all $i \geq 1$. It follows that $\mathrm{grinjdim}(A) = \mu$ is equivalent to $\mathrm{EXT}_A^i(—,{}_AA) = \mathrm{EXT}_A^i(—,A_A) = (0)$ for $i > \mu$, $\mathrm{EXT}_A^{\mu}(—,{}_AA) \neq (0)$ and $\mathrm{EXT}_A^{\mu}(—,A_A) \neq (0)$. Similarly we can define the graded global dimension of A, denoted by $\mathrm{grgldim}(A)$; details are left to the reader.

The following result is proved in [**7**, 2.18].

PROPOSITION. *Let A be a graded ring. Then A is noetherian if and only if it is graded noetherian, i.e. every left or right graded ideal is finitely generated.*

1.5. We recall here some well-known facts of Homological Algebra. Our reference is [**9**] to which the reader is referred for the details. We fix a noetherian ring A. Assume that to each ${}_AM \in \mathrm{Mod}_f(A)$ is attached a projective resolution $R.(M)$ by finitely generated projective A-modules. We set $E_A^*(M) = H^*(\mathrm{Hom}_A(R.(M),A))$. The abelian groups $E_A^*(M)$ are finitely generated right A-modules and are naturally identified with the groups $\mathrm{Ext}_A^*(M,A)$. A similar result holds for $M_A \in \mathrm{Mod}_f(A)$. We shall often write $E^*(M)$ in place of $E_A^*(M)$ when there is no possible ambiguity. Recall that if M', $M'' \in \mathrm{Mod}_f(A)$ and $f \in \mathrm{Hom}_A(M',M)$, $f' \in \mathrm{Hom}_A(M'',M')$, there exist morphisms $\alpha^*(f): E_A^*(M) \to E_A^*(M')$, $\alpha^*(f'): E_A^*(M') \to E_A^*(M'')$ such that $\alpha^*(f \circ f') = \alpha^*(f') \circ \alpha^*(f): E_A^*(M) \to E_A^*(M'')$.

REMARKS. (1) If $\alpha^n(f)$ and $\alpha^n(f')$ are injective for some $n \in \mathbb{N}$ then we get compatible injections $E_A^n(M) \subset E_A^n(M') \subset E_A^n(M'')$ given by the maps $\alpha^n(f)$, $\alpha^n(f')$, $\alpha^n(f \circ f')$.

(2) Assume that A is graded, $M \in \mathrm{Mod}_f^g(A)$ and that $R.(M)$ consists of finitely generated projective graded A-modules. Then the modules $E_A^*(M)$ are naturally graded and, for all $p \in \mathbb{Z}$, we have $E_A^*(M[p]) = E_A^*(M)[-p]$.

2. Auslander conditions. In this section, we summarize the results we shall use about Auslander-Gorenstein, or Auslander-regular, rings. The main references are [**6**], [**7**], [**8**], [**10**], [**14**], [**16**].

2.1. Definition. Let A be a noetherian ring.

(a) An A-module M satisfies the Auslander-condition if: $\forall q \geq 0$, $j_A(N) \geq q$ for all A-submodules N of $E_A^q(M)$.

(b) The ring A is said to be Auslander-Gorenstein (resp. Auslander-regular) of dimension μ if: $\operatorname{injdim}(A) = \mu < \infty$ (resp. $\operatorname{gldim}(A) = \mu < \infty$), and every $M \in \operatorname{Mod}_f(A)$ satisfies the Auslander-condition.

2.2. The principal tool for studying Auslander-Gorenstein rings is given by the following result whose proof can be found in [**14**].

Theorem. *Assume A is a noetherian ring with $\mu = \operatorname{injdim}(A) < \infty$. Let M be in* $\operatorname{Mod}_f(A)$.

(a) *There exists a convergent spectral sequence in* $\operatorname{Mod}_f(A)$:

$$E_2^{p,-q}(M) = E_A^p(E_A^q(M)) \Rightarrow \mathbb{H}^{p-q}(M)$$

with $\mathbb{H}^{p-q}(M) = 0$ if $p \neq q$ and $\mathbb{H}^0(M) = M$. The resulting finite filtration of M by A-submodules is called the b-filtration. It has the form:

$$(0) = F^{\mu+1}M \subset F^{\mu}M \subset \ldots \subset F^1M \subset F^0M = M.$$

(b) *When $j_A(E_A^q(M)) \geq q$ for all q, there are exact sequences*

$$0 \to \frac{F^pM}{F^{p+1}M} \to E_2^{p,-p}(M) \to Q(p) \to 0$$

for all $p \in \{0, \ldots, \mu\}$, where $Q(p)$ has a filtration by submodules such that each composition factor is a subquotient of some $E_2^{p+i+1,-p-i}(M)$ with $i \geq 1$.

Remarks. (1) From (a), it follows that $j_A(M) \leq \mu$ when $M \in \operatorname{Mod}_f(A) \backslash (0)$.

(2) Let M and M' be in $\operatorname{Mod}_f(A)$. Suppose we have fixed projective resolutions $R.(M)$ and $R.(M')$ as in 1.5. Let f be in $\operatorname{Hom}_A(M', M)$. By standard constructions in homological algebra, to each map of complexes $\{f : R.(M') \to R.(M)\}$ above f is associated a map $E.(f)$ between the associated spectral sequences $E_2^{\cdot,\cdot}(M')$, $E_2^{\cdot,\cdot}(M)$, and their invariants. It is easily seen that the induced map $\mathbb{H}^0(M') \to \mathbb{H}^0(M)$ coincides with f. The homomorphisms $E_2^{p,-q}(f) : E_2^{p,-q}(M') \to E_2^{p,-q}(M)$ come from the maps $\alpha^q(f) : E_A^q(M) \to E_A^q(M')$ described in 1.5. In particular they are of the form $\alpha^p(\alpha^q(f))$. It follows that if $\alpha^n(f)$ is an isomorphism then $E_2^{p,-n}(f)$ is an isomorphism for all p.

2.3. Theorem. *Let A be an Auslander-Gorenstein ring. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence in* $\operatorname{Mod}_f(A)$ *then we have* $j_A(M) = \inf\{j_A(M'), j_A(M'')\}$.

This theorem is proved in [**7**, 1.8]. By analogy with similar properties for dimensions, one can say that the grade number is exact (on short exact sequences). We shall discuss this property in Section 4.

2.4. Definition. Let A be a noetherian ring and M be in $\operatorname{Mod}_f(A) \backslash (0)$. Then M is called pure if $j_A(N) = j_A(M)$ for all non-zero submodules N of M. When $n = j_A(M)$ we say that M is n-pure.

For an Auslander-Gorenstein ring the b-filtration of a module has a nice interpretation.

THEOREM. *Assume A is Auslander-Gorenstein and let M be a non-zero finitely generated A-module. Then if $n = j_A(M)$ we have*:

(a) *$E^n_A(M)$ is n-pure and $E^p_A(E^p_A(M))$ is* (0) *or p-pure*;

(b) *F^pM is the largest submodule X of M such that $j_A(X) \geq p$*;

(c) $n = \mathrm{Max}\{p \mid M = F^pM\}$.

For the proof see [**7**, **14**, **16**, **8**].

REMARK. Obviously A is 0-pure, i.e. $L^* = E^0_A(L) \neq (0)$ for all non-zero left or right ideals L of A.

3. Graded Auslander-Gorenstein rings. In this section $A = \bigoplus_{m \in \mathbb{Z}} A_m$ is a graded ring.

3.1. Assume A is noetherian, let M be in $\mathrm{Mod}^g_f(A)$. As recalled in 1.5, the modules $E^p_A(M)$ are naturally graded. Therefore the spectral sequence $E_2^{\cdot\cdot}(M)$ of Theorem 2.2 is in $\mathrm{Mod}^g_f(A)$. Working with graded modules in Definition 2.1, one can obviously define the notion of a "graded-Auslander-Gorenstein, or regular, ring". Fortunately we have the following result.

THEOREM ([**10**, Theorem 0.1]). *The noetherian graded ring A is Auslander-Gorenstein* (*resp. regular*) *if and only if A is graded-Auslander-Gorenstein* (*resp. regular*).

3.2. Let B be a ring and $\{B(\nu)\}_\nu$ be a separated filtration on B with associated graded ring $G(B)$. The Rees-ring attached to $\{B(\nu)\}_\nu$ is $R(B) := \bigoplus_\nu B(\nu)$. Assume $R(B)$ is noetherian. Recall that a filtration $\Gamma = \{M(m)\}_m$ on $M \in \mathrm{Mod}(B)$ is called good if $R(M) := \bigoplus_m M(m)$ belongs to $\mathrm{Mod}_f(R(B))$. Any $M \in \mathrm{Mod}_f(B)$ can be equipped with a good filtration, the associated graded module $G(M)$ is in $\mathrm{Mod}^g_f(G(B))$. The filtration $\{B(\nu)\}_\nu$, or the ring B itself if there is no ambiguity, is called Zariskian if $R(B)$ is noetherian and good filtrations on elements of $\mathrm{Mod}_f(B)$ are separated. For instance if the filtration is discrete and $G(B)$ is noetherian then B is Zariskian. We recall the following result.

PROPOSITION ([**7**, 3.1, 3.4]). *Let B be a filtered ring and $M \in \mathrm{Mod}_f(B)$. Assume $R(B)$ is noetherian and let Γ be a good filtration on M with associated graded module $G(M)$.*

(i) *There exists a canonical good filtration on $E^p_B(M)$ such that $G(E^p_B(M))$ is a subquotient of $E^p_{G(B)}(G(M))$ in $\mathrm{Mod}^g_f(G(B))$. If B is Zariskian one has $j_B(M) \geq j_{G(B)}(G(M))$.*

(ii) *Suppose B is Zariskian and $G(B)$ is Auslander-Gorenstein. Then $j_B(M) = j_{G(B)}(G(M))$.*

3.3. We return to $A = \bigoplus_m A_m$ graded noetherian. In [**10**, Theorem 0.2], it is shown that $\mathrm{gldim}(A)$ (resp. $\mathrm{injdim}(A)$) is finite if and only if $\mathrm{grgldim}(A)$ (resp. $\mathrm{grinjdim}(A)$) is finite. Furthermore we have bounds:

$$\mathrm{grinjdim}(A) \leq \mathrm{injdim}(A) \leq \mathrm{grinjdim}(A) + 1.$$

When $A = \bigoplus_{m \geq 0} A_m$ is positively graded, we can filter A by setting $A(\nu) = \sum_{i=0}^{\nu} A_i$. It is clear that $G(A) = \bigoplus_\nu A(\nu)/A(\nu - 1)$ and A are equal. Thus $\{A(\nu)\}_\nu$ is Zariskian. Notice that if

$M = \bigoplus_{i=-d}^{+\infty} M_i$ is a left-limited graded module, the filtration $\left\{M(\nu) := \sum_{i=-d}^{\nu} M_i\right\}_\nu$ is good, the modules $G(M)$ and M coincide in $\mathrm{Mod}^g(A)$. In this situation we have the following (presumably well-known) result (we include a proof for completeness).

LEMMA. *Let $A = \bigoplus_{m \geq 0} A_m$ be a positively graded noetherian ring. Then* $\mathrm{injdim}(A) < \infty$ *if and only if* $\mathrm{grinjdim}(A) < \infty$, *in which case these two numbers are equal.*

Proof. We filter A by $\{A(\nu)\}_\nu$ as above. Suppose $\mu = \mathrm{injdim}(A) < \infty$ and let M be in $\mathrm{Mod}_f(A)$ such that $N := E_A^\mu(M) \neq (0)$. Choose any good filtration on M. By Proposition 3.2(i) and the previous remarks, we know that $G(N)$ is a non-zero subquotient of $E_{G(A)}^\mu(G(M)) = E_A^\mu(G(M))$ for some good filtration on N. Hence $E_A^\mu(G(M)) \neq (0)$. Since $G(M) \in \mathrm{Mod}_f^g(A)$, we get $\mathrm{grinjdim}(A) \geq \mu = \mathrm{injdim}(A)$. The lemma then follows from the results of **[10]** quoted above.

3.4. Let B be any ring and let Ω be a normal element of B, i.e. $\Omega B = B\Omega$. Assume Ω is a n.z.d. in B. Then we can define an automorphism σ of B by the rule: $b\Omega = \Omega\sigma(b)$. If ${}_BM \in \mathrm{Mod}(B)$, we shall denote by ${}^\sigma M \in \mathrm{Mod}(B)$ the abelian group M with left B-action given by $b.x = \sigma(b)x$. When $M_B \in \mathrm{Mod}(B)$ we make a similar definition to get $M^\sigma \in \mathrm{Mod}(B)$. The following assertions are easy to check.

(1) The left multiplication by Ω gives an element of $\mathrm{Hom}_B({}^\sigma M, M)$. $\mathrm{Ker}_M \Omega^p = \{x \in M \mid \Omega^p x = 0\}$ and $\Omega^p M$ are submodules of M for all $p \geq 0$.

(2) $\mathrm{Hom}_B({}^\sigma M, B) \cong \mathrm{Hom}_B(M, B)^\sigma$. The isomorphism is given by

$$f \rightarrow \{f^\sigma : x \rightarrow \sigma(f(x))\}.$$

(3) The functor $F : \mathrm{Mod}(B) \rightarrow \mathrm{Mod}(B)$, $F(M) = {}^\sigma M$, is an equivalence of categories and restricts to an equivalence of categories $F : \mathrm{Mod}_f(B) \rightarrow \mathrm{Mod}_f(B)$.

(4) $E_B^i({}^\sigma M) \cong E_B^i(M)^\sigma$.

We can filter B by the Ω-adic filtration $\{\Omega^\nu B\}_{\nu \geq 0}$ (it is a decreasing filtration). The following is well known.

PROPOSITION. (a) *The associated graded ring $G(B) = \bigoplus_\nu \Omega^\nu B / \Omega^{\nu+1} B$ is isomorphic to the graded ring $(B/\Omega B)\ [X, \sigma]$.*

(b) (Rees-Lemma) *If $M \in \mathrm{Mod}(B/\Omega B)$ and $p \geq 0$ we have: $E_B^{p+1}(M) \cong E_{B/\Omega B}^p(M)$.*

REMARKS. (1) In (a), the graded ring $(B/\Omega B)\ [X, \sigma]$ is the Ore-extension defined by $\bar{b}X = X\overline{\sigma(b)}$, where X is an indeterminate of degree 1.

(2) The Ω-adic filtration is not Zariskian in general (it is generally not separated).

(3) The Rees-Lemma shows that $j_{B/\Omega B}(M) = j_B(M) - 1$. It is very easy to deduce from it that, if B is Auslander-Gorenstein of dimension μ then $B/\Omega B$ is Auslander-Gorenstein of dimension at most $\mu - 1$.

3.5. Let $A = \bigoplus_{m \geq 0} A_m$ be positively graded. Assume $\Omega \in A_d$, $d > 0$, is a normal n.z.d. in A. Then it is not difficult to prove the following proposition.

PROPOSITION. (a) *A is noetherian if and only if $A/\Omega A$ is noetherian.*

(b) *A is a domain if $A/\Omega A$ is a domain.*

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The proof of (a) is based on $G(A)=(A/\Omega A)[X,\sigma]$, cf. 3.3, and the following: if L is a left or right graded ideal of A, one has $L=\bigcap_{q\geq 0}(L+\Omega^q A)$.

Assume A is noetherian. Even if the Ω-adic filtration is not Zariskian, one has the following well-known result (see [**16**, 4.6.7]).

LEMMA. *The Ω-adic filtration satisfies*:

(1) *the Rees ring $\bigoplus_{\nu}\Omega^{\nu}A$ is noetherian, in particular good filtrations induce good filtrations on submodules*;

(2) *good filtrations on elements of* $\mathrm{Mod}_f^g(A)$ *are separated.*

For the convenience of the reader we sketch a proof of this lemma.

(1) Put $F_\nu A=\Omega^{-\nu}A$ if $\nu\leq 0$ and $F_\nu A=A$ if $\nu\geq 0$. The Rees ring is $R=\bigoplus_{\nu\in\mathbb{Z}}F_\nu A$. Let X be an indeterminate. Define $\theta:R\to A[X,X^{-1}]$ by $\theta\left(\sum_\nu a_\nu\right)=\sum_\nu a_\nu X^\nu$. Then θ gives an isomorphism from R onto $A[\Omega X^{-1},X]$ and one easily sees that this ring is noetherian. The second assertion is obvious.

(2) Let M be an object of $\mathrm{Mod}_f^g(A)$. Assume that $\{F_nM\}_n$ is a good filtration on M. Using [**7**, 2.20], one obtains

$$\bigcap_n F_nM\subset\bigcap_{j\geq 1}\Omega^jM=\{m\in M\ /\ (1-a)m=0 \text{ for some } a\in\Omega A\}.$$

Since M is graded and Ω is an element of positive degree we get that $\bigcap_{j\geq 1}\Omega^jM=(0)$.

3.6. THEOREM. *Let $A=\bigoplus_{m\geq 0}A_m$ be positively graded, $\Omega\in A_d$, $d>0$, a normal n.z.d. in A, $G(A)=\bigoplus_m\Omega^mA/\Omega^{m+1}A$. Then*:

(1) *if A is noetherian and* $\mathrm{injdim}(A/\Omega A)<\infty$, *we have* $\mathrm{injdim}(A)=\mathrm{injdim}(G(A))=\mathrm{injdim}(A/\Omega A)+1$;

(2) *$A/\Omega A$ is Auslander-Gorenstein of dimension ν if and only if A is Auslander-Gorenstein of dimension $\nu+1$.*

Proof. (1) By the Rees-Lemma, we know $\mathrm{injdim}(A/\Omega A)\leq\mathrm{injdim}(A)-1$. Recall that, if R is any noetherian ring with $\mathrm{injdim}(R)<\infty$ and σ is an automorphism of R, we have that $\mathrm{injdim}(R[X,\sigma])=\mathrm{injdim}(R)+1$ (the proof for σ the identity automorphism can be used unchanged). It follows that $\mathrm{injdim}(G(A))=\mathrm{injdim}(A/\Omega A)+1\leq\mathrm{injdim}(A)$. We thus have to show that $\mathrm{injdim}(A)\leq\mathrm{injdim}(G(A))$. Suppose $M\in\mathrm{Mod}_f^g(A)$ and let Γ be any good filtration on M (with respect to the Ω-adic filtration). Since $R(A)$ is noetherian, Proposition 3.2 (i) implies that $G(E_A^p(M))$ is a subquotient of $E_{G(A)}^p(G(M))$. Since the module $E_A^p(M)$ is graded, the good filtration on $E_A^p(M)$ is separated. Therefore $G(E_A^p(M))=(0)$ implies $E_A^p(M)=(0)$. Notice in passing that we have proved the following result.

SUBLEMMA. *If $M\in\mathrm{Mod}_f^g(A)$ then $j_A(M)\geq j_{G(A)}(G(M))$.*

Put $\gamma=\mathrm{injdim}(G(A))$. By the above sublemma, $E_A^p(M)=(0)$ for all $p>\gamma$ and $M\in\mathrm{Mod}_f^g(A)$. Hence $\mathrm{grinjdim}(A)\leq\gamma$ and we get equality by Lemma 3.3.

(2) Suppose $A/\Omega A$ is Auslander-Gorenstein of dimension ν. By Proposition 3.5(a), we get A noetherian. From (1), we know $\text{injdim}(A) = \nu + 1$. By Theorem 3.1, we only have to show the "graded-Auslander-condition": if $M \in \text{Mod}_f^g(A)$ and N is a graded submodule of $E_A^p(M)$, we have $j_A(N) \geq p$.

Choose a good filtration on M. Then the canonical good filtration on $E_A^p(M)$ induces a good separated filtration on N, for which $G(N)$ is a submodule of $G(E_A^p(M))$. By the sublemma of part (1), we get $j_A(N) \geq j_{G(A)}(G(N))$. Since $A/\Omega A$ is Auslander-Gorenstein, $G(A) \cong (A/\Omega A)[X, \sigma]$ is also Auslander-Gorenstein by [**10**, Theorem 4.2]. Therefore

$$p \leq j_{G(A)}(E_{G(A)}^p(G(M))) \leq j_{G(A)}(G(E_A^p(M)))$$

(notice that $(G(E_A^p(M)))$ is a subquotient of $E_{G(A)}^p(G(M))$.

By these remarks, we have

$$j_{G(A)}(G(E_A^p(M))) \leq j_{G(A)}(G(N)) \leq j_A(N).$$

Hence $p \leq j_A(N)$, proving that A is Auslander-Gorenstein. We already noticed the converse in Remark 3.4 (3).

REMARK. The reader should compare Theorem 3.6 with [**16**, Theorem 4.6.15] and [**10**, Theorem 4.3], where similar results are proved, which have been at the origin of this theorem.

4. The dimension function for an Auslander-Gorenstein ring.

4.1. Let A be a noetherian ring. Assume we are given a decreasing chain $(M_i)_{i \in \mathbb{N}}$ of submodules of $M_0 = M \in \text{Mod}_f(A)$. We have natural projections

$$f_{i+j,i} : M/M_{i+j} \to M/M_i, \; g_i : M \to M/M_i$$

which satisfy $f_{i+2,i} = f_{i+1,i} \circ f_{i+2,i+1}, g_i = f_{i+1,i} \circ g_{i+1}$. Using the notation of 1.5, we deduce

$$\alpha^\bullet(f_{i+2,i}) = \alpha^\bullet(f_{i+2,i+1}) \circ \alpha^\bullet(f_{i+1,i}), \; \alpha^\bullet(g_i) = \alpha^\bullet(g_{i+1}) \circ \alpha^\bullet(f_{i+1,i}). \tag{4.1.1}$$

LEMMA. *Suppose $n \in \mathbb{N}$ is such that, for all $i \geq 0$, $E_A^{n-1}(M_i) = E_A^{n-1}(M_i/M_{i+1}) = (0)$. Then there exists an increasing chain of submodules of $E_A^n(M)$:*

$$E_A^n(M/M_1) \subset \ldots \subset E_A^n(M/M_i) \subset \ldots \subset E_A^n(M),$$

where each inclusion $E_A^n(M/M_i) \subset E_A^n(M/M_{i+1})$ is given by $\alpha^n(f_{i+1,i})$ and is the restriction of the injective map $\alpha^n(g_i) : E_A^n(M/M_i) \to E_A^n(M)$.

Proof. By [**9**, p. 92], we have connecting morphisms $\alpha^n(f_{i+1,i}) : E^n(M/M_i) \to E^n(M/M_{i+1})$ and $\alpha^n(g_i) : E^n(M/M_i) \to E^n(M)$. The hypothesis states that these maps are injective. The lemma then follows from 1.5 and (4.1.1).

Under the hypothesis of the previous lemma, we have the next corollary.

COROLLARY. (a) *There exists $q \in \mathbb{N}$ such that $\alpha^n(f_{q+j,q}) : E^n(M/M_q) \to E_A^n(M/M_{q+j})$ is an isomorphism for all $j \geq 0$.*

(b) *Assume $\mu = \text{injdim}(A) < \infty$. The maps $E_2^{n,-n}(f_{i+1,i})$ between the spectral sequences $E_2^{\bullet,\bullet}(M/M_{i+1})$ and $E_2^{\bullet,\bullet}(M/M_i)$, defined in 2.2, are isomorphisms for all $i \geq q$.*

Proof. (a) follows from $E_A^n(M) \in \mathrm{Mod}_f(A)$ and A noetherian.
(b) is a consequence of Remark 2.2(2).

4.2. THEOREM. *Assume A is an Auslander-Gorenstein ring. Let M be in $\mathrm{Mod}_f(A)$ with $j_A(M) \geq n$ and let $(M_i)_{i \in \mathbb{N}}$ be a decreasing chain of submodules of M. Then there exists $q \in \mathbb{N}$ such that $j_A(M_i/M_{i+1}) \geq n+1$ for all $i \geq q$.*

Proof. Recall that $j_A(X) \geq j_A(M)$ for any subquotient X of M (see Theorem 2.3). Hence we may assume $M = M_0$ and apply Lemma 4.1 to the chain $(M_i)_{i \in \mathbb{N}}$. Put $N_i = M/M_i$; since $j(N_i) \geq n$, we have $N_i = F^n N_i$ in the b-filtration of N_i (see Theorem 2.4(c)). The map $f_{i+1,i}: N_{i+1} \to N_i$ gives, by restriction, an A-module map between $F^p N_{i+1}$ and $F^p N_i$ for all p. (See Remark 2.2 (2) or Theorem 2.4 (b).)

Let $q \in \mathbb{N}$ be given by Corollary 4.1. If $i \geq q$, the morphism $\alpha := E_2^{n,-n}(f_{i+1,i})$ is an isomorphism and, by Remark 2.2 (2), we get a commutative diagram

$$\begin{array}{ccc} 0 \to \dfrac{F^n N_{i+1}}{F^{n+1} N_{i+1}} & \to & E_2^{n,-n}(N_{i+1}) \\ \downarrow \bar{f} & & \downarrow \alpha \\ 0 \to \dfrac{F^n N_i}{F^{n+1} N_i} & \to & E_2^{n,-n}(N_i) \end{array}$$

where the vertical map $\bar{f}$ is induced by $f_{i+1,i}$. Consider $X := (M_i/M_{i+1} + F^{n+1}N_{i+1})/F^{n+1}N_{i+1}$. It is a submodule of $N_{i+1}/F^{n+1}N_{i+1} = F^n N_{i+1}/F^{n+1}N_{i+1}$. Since $f_{i+1,i}(M_i/M_{i+1}) = (0)$, we have $\bar{f}(X) = 0$. But as α is an isomorphism, it follows that $X = (0)$; that is, $M_i/M_{i+1} \subset F^{n+1}N_{i+1}$. By Theorem 2.4(b), this means $j(M_i/M_{i+1}) \geqslant n+1$.

REMARK. Theorem 4.2 can be deduced from [**7**, 1.17], where it is shown that a chain $M = M_0 \supset M_1 \supset \ldots \supset M_p$ with $j_A(M_i/M_{i+1}) = j_A(M)$ for all i has at most $\varepsilon(M)$ terms for some fixed integer $\varepsilon(M)$. We shall come back to this number in (4.6.5).

4.3. It follows from [**7**, 1.19] that if M is a finitely generated module over an Auslander-Gorenstein ring A and if $\phi \in \mathrm{End}_A(M)$ is injective then $j_A(M/\phi(M)) \geq j_A(M) + 1$. We are going to reprove that result using 4.2.

THEOREM. *Let A be an Auslander-Gorenstein ring, σ an automorphism of A and $M \in \mathrm{Mod}_f(A)$. If $\phi \in \mathrm{Hom}_A({}^{\sigma}M, M)$ is injective then we have $j_A(M/\phi(M)) \geq j_A(M) + 1$.*

Proof. Recall that ${}^{\sigma}M$ has been defined in 3.4. Obviously we may assume $M \neq (0)$ and we know $j(M/\phi(M)) \geq j(M)$ by Theorem 2.3. Suppose $j(M/\phi(M)) = j(M)$ and set

$$\phi_0(M) := M,\ \phi_{i+1}(M) := \phi({}^{\sigma}\phi_i(M)) \text{ for all } i \geq 0.$$

Each $\phi_i(M)$ is an A-submodule of M and $\phi_1(M) = \phi(M)$. If $M \supsetneqq \phi_i(M)$, an easy induction gives $\phi_{i+1}(M) \subsetneqq \phi_i(M)$ for all i. Furthermore there exist isomorphisms in $\mathrm{Mod}_f(A)$

$$\varphi_i : {}^{\sigma^i}\!\left[\frac{M}{\phi_1(M)}\right] \to \frac{\phi_i(M)}{\phi_{i+1}(M)}.$$

Denoting by $\bar{x}$ the class of an element in a quotient module, the φ_i's are defined as

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follows. For $i=1$, we put $\varphi_i(\bar{x})=\overline{\phi(x)}$. Assuming φ_i has been defined, consider

$$\overline{\phi_{i+1}}:{}^{\sigma}\left[\frac{\phi_i(M)}{\phi_{i+1}(M)}\right]\to\frac{\phi_{i+1}(M)}{\phi_{i+2}(M)},\qquad \overline{\phi_{i+1}}(\bar{x})=\overline{\phi(x)}.$$

Then put $\varphi_{i+1}:=\overline{\phi_{i+1}}\circ\varphi_i$; with a slight abuse of notation, we have $\varphi_{i+1}(\bar{x})=\overline{\phi(\varphi_i(\bar{x}))}$. It is easily seen that the φ_i's are bijective A-linear maps. Using 3.4 (4), we get $E_A^{\cdot}(\phi_i(M)/\phi_{i+1}(M))\cong E_A^{\cdot}(M/\phi(M))$. Hence $j(\phi_i(M)/\phi_{i+1}(M))=j(M/\phi(M))=j(M)$ for all i. The chain $\{\phi_i(M)\}_{i\geq 0}$ contradicts Theorem 4.2, hence the result.

4.4. Corollary. *Let $A=\bigoplus_{m\geq 0}A_m$ be a positively graded Auslander-Gorenstein ring, $\Omega\in A_d$, $d>0$, a normal n.z.d. in A. Suppose $M\in\mathrm{Mod}_f^g(A)$ such that Ω is a n.z.d. in M. Then $j_A(M/\Omega M)=j_A(M)+1$.*

Proof. Define $\sigma\in\mathrm{Aut}(A)$ by $a\Omega=\Omega\sigma(a)$. Notice that $\sigma(A_m)=A_m$ for all m. Therefore the module ${}^{\sigma}M$ defined in 3.4 is graded with $({}^{\sigma}M)_m=M_m$. The hypothesis says that there is an exact sequence in $\mathrm{Mod}_f(A)$

$$0\to{}^{\sigma}M\xrightarrow{\Omega}M\to M/\Omega M\to 0. \qquad (*)$$

The multiplication by Ω is a graded map of degree $d>0$. It induces right multiplication by Ω from $E_A^p(M)$ to $E_A^p({}^{\sigma}M)$ for all p. Suppose $E_A^{p+1}(M/\Omega M)=(0)$, the long exact sequence coming from (*) gives $E_A^p({}^{\sigma}M)=E_A^p(M)\Omega$. By the remark above, $E_A^p(M)=E_A^p(M)^{\sigma}$ as graded groups; since $E_A^p(M)^{\sigma}=E_A^p({}^{\sigma}M)$, see 3.4 (5), the graded Nakayama's lemma forces $E_A^p(M)=(0)$. Thus $j(M/\Omega M)\leq j(M)+1$. To get the reverse inequality, apply Theorem 4.3 to $\Omega\in\mathrm{Hom}_A({}^{\sigma}M,M)$.

4.5. Definition. Let A be an Auslander-Gorenstein ring of dimension μ. Define the dimension of $M\in\mathrm{Mod}_f(A)$ by $\delta(M)=\mu-j(M)$.

Note that $j(M)\in\{0,\ldots,\mu\}\cup\{\infty\}$; hence $\delta(M)\in\{0,\ldots,\mu\}\cup\{-\infty\}$. The term dimension given to $\delta(M)$ is justified by the next result.

Proposition. *The function $\delta:\mathrm{Mod}_f(A)\to\{0,\ldots,\mu,-\infty\}$ is an exact dimension function. Furthermore it is finitely partitive. This means:*

(i) $\delta((0))=-\infty$,

(ii) *if $0\to M'\to M\to M''\to 0$ is exact then $\delta(M)=\mathrm{Max}\{\delta(M'),\delta(M'')\}$,*

(iii) *if $PM=(0)$ for some prime P and M is a torsion A/P-module then $\delta(M)\leq\delta(A/P)-1$ (similarly for M a right module),*

(iv) *if $M=M_0\supset M_1\supset\ldots\supset M_i\supset\ldots$ is a chain of submodules then $\delta(M_i/M_{i+1})\leq\delta(M)-1$ for $i\gg 0$.*

Proof. We follow the terminology of [**17**, 6.8.4 and 8.7.3]. The properties (i), (ii) and (iv) are clear from 1.2, 2.3, 4.2. To prove (iii), notice that it's enough to show $\delta(A/P+Aa)\leq\delta(A/P)-1$ for a non-zero-divisor $\bar{a}$ in A/P. This follows from Theorem 4.3 applied to $\phi:A/P\to A/P$, $\phi(\bar{x})=\overline{xa}$.

4.6. We assume A Auslander-Gorenstein of dimension μ. We are going to list consequences and properties coming from the dimension δ.

(4.6.1) $M \in \mathrm{Mod}_f(A)$ is called *homogeneous* if $\delta(N) = \delta(M)$ for all non-zero submodules N of M. If $\delta(M) = s$, we shall simply say M is s-homogeneous. Notice that M s-homogeneous is the same as M $(\mu - s)$-pure, see 2.4. In particular, by Remark 2.4, A is μ-homogeneous as a left or right module.

(4.6.2) $M \in \mathrm{Mod}_f(A)$ is said to be *critical* if $\delta(M/N) < \delta(M)$ for all non-zero submodules N of M. If $\delta(M) = s$, we say M is s-critical. This is equivalent to $j_A(M/N) > j_A(M) = s$ by definition.

(4.6.3) Any non-zero $M \in \mathrm{Mod}_f(A)$ has a critical submodule: use the proof of [**17**, 6.2.10].

(4.6.4) Let M be in $\mathrm{Mod}_f(A)$. A *critical composition series* for M is a chain $M = M_t \supset M_{t+1} \supset \ldots \supset M_0 = (0)$ such that each factor M_i/M_{i+1} is critical. Notice that $\delta(M) = \underset{1 \le i \le t}{\mathrm{Max}} \{\delta(M_i/M_{i-1})\}$. Any $M \in \mathrm{Mod}_f(A)$ has a critical composition series. One can find a critical series with $\delta(M_i/M_{i-1}) \ge \delta(M_{i-1}/M_{i-2})$ for all i. Furthermore, two such critical composition series have the same length. This follows from (4.6.3) together with properties of δ using standard arguments (see [**17**, 6.2.20, 6.2.21] with δ in place of K).

(4.6.5) Let $M \in \mathrm{Mod}_f(A)$ with $\delta(M) = s$. If (*) $M = M_t \supset M_{t-1} \supset \ldots \supset M_0 = (0)$ is a critical composition series of M, we set

$$\varepsilon_s(M) = \#\{s\text{-critical factors in } (*)\},\ \varepsilon_\mu(M) = \#\{\mu\text{-critical factors in } (*)\}.$$

Then the following are easy to prove:

the numbers $\varepsilon_s(M)$ and $\varepsilon_\mu(M)$ depend only on M;

the function $M \to \varepsilon_s(M)$ is additive on short exact sequences in $\{M \in \mathrm{Mod}_f(A) \;/\; \delta(M) = s\}$;

the function $M \to \varepsilon_\mu(M)$ is additive on short exact sequences in $\mathrm{Mod}_f(A)$.

The number $\varepsilon_\mu(M)$ will be called the *δ-critical length* of M.

(4.6.6) The b-filtration of $M \in \mathrm{Mod}_f(A)$ obtained in 2.3, 2.4 can be reinterpreted as follows. Put $n = j_A(M)$, $\delta(M) = s = \mu - n$. The submodule F^pM is the largest submodule X of M such that $\delta(X) \le \mu - p$. The module $E^n_A(M)$ is s-homogeneous. We have exact sequences

$$0 \to \frac{F^pM}{F^{p+1}M} \to E^p_A(E^p_A(M)) \to Q(p) \to 0,$$

where $E^p_A(E^p_A(M))$ is (0) or $(\mu - p)$-homogeneous and $\delta(Q(p)) \le \mu - p - 2$.

(4.6.7) It is not difficult to prove that the number $\varepsilon_s(M)$ defined in (4.6.4) is equal to the number $\varepsilon(M)$ obtained in [**7**, 1.17].

4.7. In this section, we suppose that $A = \bigoplus_{m \ge 0} A_m$ is a positively graded Auslander-Gorenstein ring of dimension μ. Let δ be its dimension as defined in 4.5. Let M be in $\mathrm{Mod}^g_f(A)$. We say that M is graded-critical (resp. graded-homogeneous) if $\delta(M/N) < \delta(M)$ (resp. $\delta(N) = \delta(M)$) for all non-zero graded submodules N of M. These definitions are not of great interest since we have the next lemma.

LEMMA. *The module $M \in \mathrm{Mod}^g_f(A)$ is graded-critical (resp. graded-homogeneous) if and only if it is critical (resp. homogeneous).*

Proof. Set $M = \bigoplus_m M_m$. Recall that $\left\{A(n) := \bigoplus_{j \le n} A_j\right\}_n$ is a filtration with associated graded ring equal to A. If Γ is a good filtration on $M' \in \mathrm{Mod}_f(A)$ (w.r.t to $\{A(n)\}_n$), we have $\delta(\mathrm{gr}_\Gamma(M')) = \delta(M')$ by Proposition 3.2 (ii). Filter the module M by

$$\Gamma = \left\{M(n) := \sum_{i \le n} M_i\right\}_n.$$

It is a good filtration and we obviously have $\mathrm{gr}_\Gamma(M) = M$. If N is any non-zero submodule of M, Γ induces good separated filtrations on N and M/N still denoted by Γ. Then the exact sequence $0 \to \mathrm{gr}_\Gamma(N) \to M = \mathrm{gr}_\Gamma(M) \to \mathrm{gr}_\Gamma(M/N) \to 0$ in $\mathrm{Mod}_f^g(A)$ shows that $\delta(N) = \delta(M)$ if M is graded-homogeneous and $\delta(M/N) < \delta(M)$ if M is graded-critical. The converse is obvious.

PROPOSITION. *Any non-zero $M \in \mathrm{Mod}_f^g(A)$ has a critical series composed of graded submodules.*

Proof. Notice first that M contains a graded-critical submodule (repeat the proof of [**17**, 6.2.10] with graded modules and δ in place of K). It follows as usual that M has a graded-critical composition series. Applying the lemma above, we conclude that it is a critical composition series in $\mathrm{Mod}_f(A)$.

4.8. THEOREM. *Let $A = \bigoplus_{m \ge 0} A_m$ be a positively graded Auslander-regular k-algebra with $A_0 = k$. Then A is a domain.*

Proof. We follow [**3**, Section 3]. Recall that A is μ-homogeneous, see (4.6.1). We first prove that $\varepsilon_\mu(A) = 1$. Any $M \in \mathrm{Mod}_f^g(A)$ has a finite resolution by graded free modules of the form $P_i = \bigoplus_{j=1}^{d_i} A[m_{i,j}]$, see [**17**, 12.2.9]. By additivity of $\varepsilon_\mu(-)$ (see (4.6.4)), we get $\varepsilon_\mu(M) = \sum_i (-1)^i \ \varepsilon_\mu(P_i) = \sum_i d_i \varepsilon_\mu(A) \in \mathbb{Z}\varepsilon_\mu(A)$. Now choose $M \in \mathrm{Mod}_f^g(A)$ μ-critical (for instance a graded-critical ideal of A). We have $\varepsilon_\mu(M) = 1 \in \mathbb{Z}\varepsilon_\mu(A)$ and therefore $\varepsilon_\mu(A) = 1$. Let $a \in A$, $a \ne 0$. We must show that a is a right n.z.d., the proof being identical on the left. Consider the exact sequence $0 \to Aa \to A \to (A/Aa) \to 0$. Since A is μ-homogeneous we have $\delta(Aa) = \mu$, hence $\varepsilon_\mu(Aa) \ge 1$. Since $\varepsilon_\mu(A) = 1 = \varepsilon_\mu(Aa) + \varepsilon_\mu(A/Aa)$ we have $\varepsilon_\mu(A/Aa) = 0$. Hence $\delta(A/Aa) \le \mu - 1$ which means $j(A/Aa) \ge 1$, i.e. $\mathrm{Hom}_A(A/Aa, A) \equiv \{b \in A \ / \ ab = 0\} = (0)$. Thus a is a right n.z.d.

5. The Cohen-Macaulay property. The rings in this section are k-algebras over the fixed field k. The material concerning Gelfand-Kirillov dimension comes from [**13**].

5.1. We first recall the definition of polynomial growth for a function $d: \mathbb{Z} \to \mathbb{N}$, $d \ne 0$. Suppose $d(n) = 0$ for $n \ll 0$, d increasing for $n \ge n_0$. Then d has polynomial growth of degree $\rho \in \mathbb{N}$ if there exist $K, m \in \mathbb{N}$ such that

$$d(n) \le Kn^\rho, \ n^\rho \le d(mn) \text{ for almost all } n.$$

We shall write $G(d) = P_\rho$ when this holds. We then have $\rho = \overline{\lim_n} \log_n d(n)$, $\log_n d(n) := \log d(n)/\log n$.

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5.2. We shall use the following properties of Gelfand-Kirillov dimension, denoted by GKdim. Let A be a k-algebra filtered by $\{A(i)\}_i$ (increasing) such that $\dim_k A(i) < \infty$ for all i (in particular it is discrete). Assume that $\mathrm{gr}(A) = \oplus A(i)/A(i-1)$ is a finitely generated k-algebra.

(5.2.1) Let $M \in \mathrm{Mod}_f(A)$, $\Gamma = \{M(i)\}_i$ a filtration on M such that $\dim_k M(i) < \infty$ and $\mathrm{gr}_\Gamma(M) \in \mathrm{Mod}_f^g(\mathrm{gr}(A))$. Then, setting $f_\Gamma(n) = \dim_k M(n)$, we have $\mathrm{GKdim}_A M = \mathrm{GKdim}_{\mathrm{gr}(A)} \mathrm{gr}_\Gamma(M) = \overline{\lim_n} \log_n f_\Gamma(n)$. See [13, 6.6].

(5.2.2) If $M \in \mathrm{Mod}_f(A)$, we have $\mathrm{GKdim}_A M = 0$ if and only if $\dim_k M < \infty$ and $\mathrm{GKdim}_A M \geq 1$ if $\dim_k M = +\infty$.

(5.2.3) Assume $\mathrm{gr}(A)$ noetherian. Then GKdim is exact in $\mathrm{Mod}_f(A)$: $\mathrm{GKdim}_A M = \mathrm{Max}\{\mathrm{GKdim}_A M', \mathrm{GKdim}_A M''\}$ for every exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathrm{Mod}_f(A)$; see [13, 6.14].

5.3. Assume B is a k-algebra. One defines an s-homogeneous or s-critical B-module (for GKdim) as in (4.6.1) or (4.6.2), with GKdim replacing δ. Combining 6.8.15 and 8.3.16 of [17], one obtains the following theorem.

THEOREM. *Suppose B is a noetherian k-algebra such that* GKdim *is exact and B is (left and right) homogeneous (for* GKdim*). Then B has a (left and right) Artinian quotient ring. Furthermore the set S of n.z.d. in B satisfies*

$$S = \{b \in B \,/\, \mathrm{GKdim}\, B \,/\, bB < \mathrm{GKdim}\, B\} = \{b \in B/\mathrm{GKdim}\, B/Bb < \mathrm{GKdim}\, B\}.$$

5.4. Let $A = \bigoplus_{n\geq 0} A_n$ be a finitely generated positively graded k-algebra with $\dim_k A_0 < \infty$. Then $\dim_k A_n < \infty$ for all n and $\left\{A(n) := \bigoplus_{i=0}^{n} A_i\right\}_{n\geq 0}$ is a positive filtration with $\mathrm{gr}(A) = A$. Hence the results recalled in 5.2 apply to $\mathrm{Mod}_f(A)$ (assuming A noetherian for (5.2.3)). We introduce a condition named (PG) for polynomial growth.

DEFINITION. The algebra A satisfies (PG) if, for every $M = \bigoplus_n M_n$ in $\mathrm{Mod}_f^g(A)\backslash(0)$, one has $G(f_M) = P_s$ for some $s \in \mathbb{N}$. Here $f_M(n) = \dim_k M(n)$, $M(n) = \sum_{i\leq n} \dim_k M_i$.

REMARKS. (1) $\Gamma := \{M(n)\}_n$ is a good filtration on M (with respect to the filtration $\{A(n)\}_n$). This is because $M \cong \mathrm{gr}_\Gamma(M)$ as a $\mathrm{gr}(A)$-module. Thus $\mathrm{GKdim}_A M = \overline{\lim_n} \log_n f_n(n) = s$ by (5.2.1).

(2) If I is a graded ideal of A then, if A satisfies (PG), A/I satisfies (PG).

5.5. There is an example of an algebra satisfying (PG) which is of particular importance for the applications. Suppose $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ is noetherian. If $M = \bigoplus_n M_n \in \mathrm{Mod}_f^g(A)$, the Hilbert series is defined by $h_M(t) := \sum_{n\in\mathbb{Z}} (\dim_k M_n) t^n$. Suppose that, for all $M \in \mathrm{Mod}_f^g(A)$, one has $h_M(t) := q_M(t)/p_A(t)$, $p_A(t) \in \mathbb{Z}[t]$, $q_M(t) \in \mathbb{Z}[t, t^{-1}]$, where the roots of $p_A(t)$ are roots of unity. Then [4, 2.21] shows that $\dim_k M_n = an^{s-1}/(s-1)! +$ (terms of degree less than $s-1$ in n), where $a \in \mathbb{N}$ and s is the order of the pole at $t = 1$ of $h_M(t)$. We have $(1-t^s)h_M(1) \in \mathbb{N}$, $s = \mathrm{GKdim}_A M$ and the algebra satisfies

(PG). When $p_A(t)=(1-t)^\mu$, $\mu \in \mathbb{N}$, $h_M(t)$ takes the form $f_M(t)/(1-t)^s$, $f_M(t) \in \mathbb{Z}[t,t^{-1}]$ with $e(M):=f_M(1) \in \mathbb{N}$.

5.6. We return to $A=\bigoplus_{n\geq 0} A_n$ as in 5.4. Assume that A is noetherian. The following result should be well known. We include a proof for completeness.

Proposition. *Let $\Omega \in A_d$, $d>0$, be a normal element. Then A satisfies* (PG) *if and only if $A/\Omega A$ satisfies* (PG).

Proof. By remark 5.4 (2), we only have to show one implication. So assume that $A / \Omega A$ satisfies (PG). Let $M=\bigoplus_n M_n \in \mathrm{Mod}_f^g(A)$, non-zero. We may assume $M_n=0$ for all $n<q_0\leq 0$ (we don't impose $M_{q_0}\neq(0)$). Let $p \in \mathbb{N}$ and recall that $\mathrm{Ker}_M\,\Omega^p=\{x \in M/\Omega^p x=0\}$ is a graded submodule of M. Since $\{\mathrm{Ker}_M\,\Omega^p\}_p$ is an increasing chain of submodules we can define $n(M)$ to be the smallest integer such that $\mathrm{Ker}_M\,\Omega^{n(M)}=\mathrm{Ker}_M\,\Omega^{n(M)+i}$ for all $i\geq 0$. We set $\bar{M}=M/\Omega M$, $M'=M/\mathrm{Ker}_M\,\Omega$. With obvious notations, it is easily seen that $x' \in \mathrm{Ker}_{M'}\,\Omega^p$ if and only if $x \in \mathrm{Ker}_M\,\Omega^{p+1}$. Hence $\mathrm{Ker}_{M'}\,\Omega^p=\mathrm{Ker}_M\,\Omega^{p+1}/\mathrm{Ker}_M\,\Omega$ and we have that $n(M')\leq n(M)-1$ if $n(M)\geq 1$. We shall prove by induction on $n(M)$ than $n \xrightarrow{f_n} \sum_{i\leq n} \dim_k M_i$ has polynomial growth.

1. Assume that $n(M)=0$, i.e. Ω is a n.z.d. in M. We have exact sequences

$$0\rightarrow M_{n-d}\xrightarrow{\Omega} M_n \rightarrow \bar{M}_n \rightarrow 0,$$

where $\bar{M}_n:=M_n/\Omega M_{n-d}$ gives the grading on $\bar{M}$. It is easy to deduce that $f_{\bar{M}}(n)=f_M(n)-f_M(n-d)$. Notice that $\bar{M}_n=(0)$ for $n<q_0$, $\bar{M}_{q_0}=M_{q_0}$ and set $E(n):=[(n-q_0)/d]$ (integral part). Then we have $f_M(n)=\sum_{i=0}^{E(n)} f_{\bar{M}}(n-id)$. Suppose $G(f_{\bar{M}})=P_\rho$, $\rho \in \mathbb{N}$, so that there exist $n_0 \in \mathbb{N}^*$, k, $m \in \mathbb{N}^*$ such that $f_{\bar{M}}(n)\leq Kn^\rho$ and $f_{\bar{M}}(mn)\geq n^\rho$ for all $n\geq n_0>0$. If $n\gg 0$, we get

$$f_M(n)\leq \sum_{q=q_0}^{n} f_{\bar{M}}(q)\leq C+K\sum_{q=n_0}^{n} q^\rho \leq C+K(n-n_0+1)n^\rho \leq K'n^{\rho+1},$$

where $C=\sum_{q=q_0}^{n_0-1} f_{\bar{M}}(q)$ and $K' \in \mathbb{N}$.

Let $n\gg 0$ and notice that $m(n-n_0)\leq mn-q_0/d$. We have

$$f_M(mnd)\geq \sum_{i=0}^{m(n-n_0)} f_{\bar{M}}(mnd-id)\geq \sum_{j=0}^{n-n_0} f_{\bar{M}}(mnd-mjd).$$

If $0\leq j\leq n-n_0$, one has $nd-jd\geq nd-nd+n_0 d\geq n_0$. Hence $f_{\bar{M}}(md(n-j))\geq d^\rho(n-j)^\rho$ by the choice of m and n_0. Therefore $f_M(mnd)\geq \sum_{j=0}^{n-n_0} d^\rho(n-j)^\rho \geq \sum_{j=0}^{n-n_0}(n-j)^\rho$. Since $\phi(n):=n_0^\rho+(n_0+1)^\rho+\ldots+n^\rho$ is a polynomial in n of degree $\rho+1$, we can find $m' \in \mathbb{N}^*$ such that $f_M(mm'nd)\geq \phi(m'n)\geq n^{\rho+1}$ if $n\gg 0$. This proves the case $n(M)=0$.

2. Assume $n(M)>0$. Set $K:=\mathrm{Ker}_M\,\Omega=\bigoplus_n K_n$; hence $K\neq(0)$ and we have $f_M(n)=f_K(n)+f_{M'}(n)$. Observe that $K\in\mathrm{Mod}_f^g(A/\Omega A)$ and that we can apply the induction to M'. Thus $G(f_K)=P_{\rho''}$ and $G(f_{M'})=P_{\rho'}$ for $\rho',\rho''\in\mathbb{N}$. Set $\rho=\mathrm{Max}\{\rho',\rho''\}$. It is easy to show that $G(f_M)=P_\rho$.

5.7. We continue with $A=\bigoplus_{n\geq 0}A_n$ as in 5.6 and we suppose that A satisfies (PG). Then we have the following result.

Lemma. *Let $\Omega\in A_d$, $d>0$, be a normal element. Assume Ω is a n.z.d in $M\in\mathrm{Mod}_f^g(A)\setminus(0)$. Then* $\mathrm{GKdim}_A\, M/\Omega M=\mathrm{GKdim}_A\, M-1$.

Proof. As in the proof of 5.6, we set $M=\bigoplus_{i\geq q_0}M_i$, $q_0\leq 0$, $\bar M=M/\Omega M=\bigoplus_i \bar M_i$. Hence $\bar M_i=(0)$ if $i<q_0$ and $\bar M_{q_0}=M_{q_0}$. Recall that $f_M(n)=\sum_{i=0}^{E(n)}f_{\bar M}(n-id)$, $E(n):=[(n-q_0)/d]$, (see part 1 in the proof of 5.6). Set $s=\mathrm{GKdim}\,M$. By [13, 5.1], we know $\mathrm{GKdim}\,\bar M\leq s-1$. Set $\rho=\mathrm{GKdim}\,\bar M$, so that $f_{\bar M}(n)\leq Kn^\rho$ for some $K\in\mathbb{N}$ and all $n\geq n_0>0$. Define $a(n)$ to be the greatest integer $q\leq E(n)$ such that $n-qd\geq n_0$, and put $c=\sum_{p=q_0}^{n_0-1}f_{\bar M}(p)$. Then if $n\gg 0$, we have

$$f_M(n)\leq c+\sum_{i=0}^{a(n)}f_{\bar M}(n-id)\leq K[a(n)+1]n^\rho+c.$$

But $a(n)\leq E(n)\leq n+c'$ for some $c'\in\mathbb{N}$ independent of n, hence $f_M(n)\leq Kn^{\rho+1}+c''$ for some constant c''. This implies $s=\overline{\lim_n}\,\log_n f_M(n)\leq\rho+1$.

5.8. Let $(R,\mathcal{M})$ be a noetherian commutative local ring. Recall that R is Cohen-Macaulay if and only if, for all non-zero $M\in\mathrm{Mod}_f(R)$, we have $\mathrm{Kdim}\,M+j_R(M)=\mathrm{Kdim}\,R$, where Kdim denotes the Krull dimension. Thus it is natural to make the following definition.

Definition. Let A be a noetherian k-algebra with $\mathrm{GKdim}\,A=\omega\in\mathbb{N}$. We say that A satisfies the Cohen-Macaulay property if $\mathrm{GKdim}_A\,M+j_A(M)=\omega$ for all

$$M\in\mathrm{Mod}_f(A)\setminus(0).$$

Remarks. (1) For short we shall say that A is CM.

(2) Notice that, if A is CM, we must have $j_A(M)<\infty$ and $\mathrm{GKdim}_A\,M\in\mathbb{N}$ for all $M\neq(0)$ in $\mathrm{Mod}_f(A)$.

(3) Recall that we have seen that over an Auslander-Gorenstein ring of dimension μ, $\delta(M)=\mu-j_A(M)$ defines a dimension function on $\mathrm{Mod}_f(A)$ (see 4.5). Thus, if A is also CM, we get $\delta(M)+(\omega-\mu)=\mathrm{GKdim}_A\,M$. See remark (5) below.

(4) Let $0\to M_1\to M\to M_2\to 0$ be exact in $\mathrm{Mod}_f(A)$. We always have $\mathrm{GKdim}\,M\geq\mathrm{Max}_i\{\mathrm{GKdim}\,M_i\}$ and $j_A(M)\geq\inf_i\{j_A(M_i)\}$. Therefore if A is CM we see that GKdim_A and $j_A(\ )$ are exact, i.e. we have equalities in these inequalities.

(5) Suppose $\mathrm{char}\,k=0$. Then the Weyl algebra $A_\mu(k)=k[p_1,\dots,p_\mu,q_1,\dots,q_\mu]$, $p_iq_j-q_jp_i=\delta_{ij}$, $[p_i,p_j]=[q_i,q_j]=0$, is Auslander-regular of dimension μ and satisfies

the Cohen-Macaulay property (here $\omega = 2\mu$). See [**6**, Chap. 2, Section 7.1], and [**14**, Théorème 4.4] for a generalization.

When A is graded, one can define a "graded CM property" by taking $M \in \mathrm{Mod}_f^g(A) \setminus (0)$ in Definition 5.8.

LEMMA. *Let $A = \bigoplus_{n \geq 0} A_n$ be a positively graded ring such that $\dim_k A_0 < \infty$. Assume A is Auslander-Gorenstein and satisfies the "graded CM-property". Then A is CM.*

Proof. Consider the Zariskian filtration $\left\{A(n) := \sum_{i \leq n} A_i\right\}_n$ on A. We recalled in Proposition 3.2(ii) and (5.2.1) that $j_A(N) = j_A(\mathrm{gr}_\Gamma(N))$, and $\mathrm{GKdim}_A N = \mathrm{GKdim}_A \mathrm{gr}_\Gamma(N)$ for any good filtration Γ on $N \in \mathrm{Mod}_f(A)$. This proves the lemma.

5.9. The following proposition shows that the CM-property makes an Auslander-Gorenstein algebra very close to a commutative one.

PROPOSITION. *Let A be a noetherian k-algebra with* $\mathrm{injdim}(A) = \mu < \infty$ *and* $\mathrm{GKdim}\, A = \omega \in \mathbb{N}$. *Assume A is CM. Then A is Auslander-Gorenstein if and only if $E_A^p(E_A^q(M)) = (0)$ for all $p < q$, and all $M \in \mathrm{Mod}_f(A)$. Furthermore when this holds we have*

(i) GKdim M *is exact and finitely partitive in* $\mathrm{Mod}_f(A)$,

(ii) *A is homogeneous for* GKdim *and has a left and right Artinian quotient ring.*

Proof. Assume $E^p(E^q(M)) = (0)$ if $p < q$. Let N be a submodule of $E^q(M)$. By the property CM we have $\mathrm{GKdim}\, E^q(M) \leq \omega - q$ and, since $\mathrm{GKdim}\, N \leq \mathrm{GKdim}\, E^q(M)$ always holds, we get $j(N) \geq q$. Then (i) and (ii) follow from Remark 5.8 (3), 4.5, (4.6.1) and 5.3.

REMARKS. (1) By means of (4.6.5) and $\delta(M) + (\omega - \mu) = \mathrm{GKdim}\, M$, the b-filtration can be interpreted in terms of GKdim. We leave the translation to the reader.

(2) Assume $A = \bigoplus_{n \geq 0} A_n$ is a positively graded noetherian ring with $\mathrm{injdim}(A) < \infty$ and A satisfies the "graded CM-property". Then the proof above shows that A is graded-Auslander-Gorenstein if and only if $E_A^p(E^q(M)) = (0)$ for all $p < q$ and $M \in \mathrm{Mod}_f^g(A)$. By Theorem 3.1 and Lemma 5.8, we deduce that A is Auslander-Gorenstein and CM, if $\dim_k A_0 < \infty$.

5.10. Let $(R, \mathcal{M})$ be a commutative noetherian local ring. It is well known that if $\Omega \in \mathcal{M}$ is a n.z.d. then R is Gorenstein or CM if and only if $R/\Omega R$ is such. Therefore the following result is not surprising.

THEOREM. *Let $A = \bigoplus_{n \geq 0} A_n$ be a finitely generated positively graded k-algebra with $\dim_k A_0 < \infty$. Suppose $\Omega \in A_d$, $d > 0$, is a normal n.z.d. in A. Then $B = A/\Omega A$ is Auslander-Gorenstein (of dimension ν), satisfies the property CM and condition* (PG) *if and only if A does so (with* $\mathrm{injdim}(A) = \nu + 1$).

Proof. Assume A is Auslander-Gorenstein, CM and satisfies (PG). Then by Remarks 3.4 (3) and 5.4 (2), B has the required properties.

To prove the converse we start with a general result.

Sublemma. *Let A be an Auslander-Gorenstein k-algebra for which* GKdim *is exact. Let $0 \to M_1 \to M \to M_2 \to 0$ be exact in* $\mathrm{Mod_f}(A)$ *with* $\mathrm{GKdim}_A M_i + j_A(M_i) = \omega \in \mathbb{N}$ *for* $i = 1, 2$. *Then* $\mathrm{GKdim}_A M + j_A(M) = \omega$.

Proof. We have

$$\mathrm{GKdim}\, M = \underset{i=1,2}{\mathrm{Max}} \{\mathrm{GKdim}\, M_i\} = \underset{i=1,2}{\mathrm{Max}} \{\omega - j_A(M_i)\} = \omega - \inf_{i=1,2} \{j_A(M_i)\} = \omega - j_A(M)$$

since A is Auslander-Gorenstein.

By Theorem 3.6, Proposition 5.6, Lemma 5.8, we only have to show that $\mathrm{GKdim}\, M + j(M) = \omega + 1$ for all $M \in \mathrm{Mod}_f^g(A)\backslash(0)$, where $\omega = \mathrm{GKdim}\, B$. We adopt the notation of the proof of Proposition 5.6: $\bar{M} = M/\Omega M$, $M' = M/\mathrm{Ker}_M \Omega$, where $\mathrm{Ker}_M \Omega^p = \{x \in M \mid \Omega^p x = 0\}$, $n(M)$ is the smallest integer q such that $\mathrm{Ker}_M \Omega^q = \mathrm{Ker}_M \Omega^{q+1}$. We noticed that $n(M') \leq n(M) - 1$ and we induct on $n(M)$. If $M = \mathrm{Ker}_M \Omega$, i.e. $M' = (0)$ then $M \in \mathrm{Mod}_f^g(B)$. Hence $\mathrm{GKdim}_B M + j_B(M) = \omega$, $\mathrm{GKdim}_B M = \mathrm{GKdim}_A M$, $j_B(M) = j_A(M) - 1$ (Rees Lemma, Proposition 3.4(b)). Thus $\mathrm{GKdim}_A M + j_A(M) = \omega + 1$. If $n(M) = 0$, Ω is a n.z.d in M. By Lemma 5.7, $\mathrm{GKdim}_A M/\Omega M = \mathrm{GKdim}_A M - 1$, $j_A(M/\Omega M) = j_B(M/\Omega M) + 1$ by the Rees Lemma and $j_A(M/\Omega M) = j_A(M) + 1$ by Corollary 4.4. Since $\mathrm{GKdim}_A M/\Omega M = \mathrm{GKdim}_B M/\Omega M = \omega - j_B(M/\Omega M)$ we obtain $\mathrm{GKdim}_A M + j_A(M) = \omega + 1$. If $n(M) \geq 1$, we may assume $M' \neq (0)$. Consider the exact sequence $0 \to \mathrm{Ker}_M \Omega \to M \to M' \to 0$. By the sublemma, induction and the results above, we obtain $\mathrm{GKdim}_A M + j_A(M) = \omega + 1$.

An easy induction yields the following corollary.

Corollary. *Let A be a finitely generated positively graded k-algebra with* $\dim_k A_0 < \infty$. *Suppose* $\underline{\Omega} = \{\Omega_1, \ldots, \Omega_l\}$ *is a regular normalizing sequence of homogeneous elements of positive degree in A. Put* $B := A/\underline{\Omega}A$ *and suppose that B is Auslander-Gorenstein of dimension ν, satisfies the CM-property and condition* (PG). *Then A is Auslander-Gorenstein of dimension* $\mu = \nu + l$, *satisfies the CM-property and condition* (PG).

Remarks. (1) $\underline{\Omega}$ regular normalizing means that, for all i, Ω_{i+1} is a normal n.z.d in $A/(\Omega_1, \ldots, \Omega_i)A$.

(2) Obviously if we suppose $\mathrm{gldim}(A) < \infty$ in the corollary, we obtain A Auslander-regular.

5.11. Let $(R, \mathcal{M})$ be a commutative noetherian local ring. To attempt to prove that R is Gorenstein one can proceed as follows. Find a regular sequence $(\Omega_1, \ldots, \Omega_l)$ contained in $\mathcal{M}$ with $l = \mathrm{Kdim}\, R$, then $R/(\Omega_1, \ldots, \Omega_l)$ is an Artinian ring and one has to prove that its socle is a 1-dimensional vector space. In the non-commutative setting it is in general impossible to find a regular normalizing sequence $(\Omega_1, \ldots, \Omega_l)$ such that $\mathrm{GKdim}\, A/(\Omega_1, \ldots, \Omega_l) = 0$. But, we shall see in Section 6 some examples where this factor ring has GKdim 2. For this reason we give in 5.13 a criterion to ensure that such an algebra is Auslander-Gorenstein. We begin with an easy lemma.

Lemma. *Let B be a noetherian k-algebra. Assume* GKdim *exact in* $\mathrm{Mod_f}(B)$, *B is ν-homogeneous, and the quotient ring Q of B is self-injective* (*i.e.* $\mathrm{injdim}(Q) = 0$). *Then for all* $M \in \mathrm{Mod_f}(B)$ *we have*:

(a) $\mathrm{GKdim}_B M \leq \nu - 1$ *if and only if* $M^* = E_B^0(M) = (0)$;

(b) *for all* $q \geq 1$, $\mathrm{GKdim}\, E_B^q(M) \leq \nu - 1$ *and* $E_B^0(E_B^q(M)) = (0)$.

Proof. (a) Denote by S the set of n.z.d. in B. Thus $Q = S^{-1}B = BS^{-1}$ by Theorem 5.3. Suppose $\operatorname{GKdim}_B M < \nu$. Let $\theta \in \operatorname{Hom}_B(M, B)$. Then $\operatorname{GKdim}_B \theta M \leq \operatorname{GKdim}_B M < \nu$, so $\theta M = 0$ since B is ν-homogeneous. Conversely if $M^* = (0)$ then $\operatorname{Hom}_Q(S^{-1}M, Q) \cong \operatorname{Hom}_B(M, B) \otimes_B Q = (0)$. Since Q is self-injective, we know $S^{-1}M \cong (S^{-1}M)^{**}$, the bidual over Q. Hence $S^{-1}M = (0)$ and $\operatorname{GKdim} M \leq \nu - 1$ follows easily.

(b) The first assertion is consequence of $\operatorname{injdim}(Q) = 0$ and $E^q_B(M) \otimes_B Q \cong E^q_Q(S^{-1}M)$. The second follows from (a) and the first.

5.12. In this section we assume that $B = k \oplus B_1 \oplus B_2 \oplus \ldots$ is a noetherian positively graded k-algebra. We set $\mathcal{M} = \bigoplus_{i \geq 1} B_i$ and consider $k = B/\mathcal{M}$ as a left or right graded B-module. We shall write $E^i_B({}_Bk) = \delta_{i,\nu} k_B$ or $E^i_B(k_B) = \delta_{i,\nu}\, {}_Bk$ to mean that this module is (0) for $i \neq \nu$ and isomorphic to k if $i = \nu$. Notice that $E^i_B(k) \in \operatorname{Mod}^g_f(B)$; hence we in fact have $E^\nu_B(k) \cong k[m]$ for some $m \in \mathbb{Z}$ (in $\operatorname{Mod}^g_f(B)$).

LEMMA. *Assume* $\nu = \operatorname{injdim}(B) < \infty$. *Then*

(a) *if* $E^i_B({}_Bk) = \delta_{i,\nu} k_B$ *and* ${}_BM \in \operatorname{Mod}^g_f(B)$ *is such that* $\dim_k M < \infty$, *we have* $E^i_B(M) = (0)$ *for* $i < \nu$ *and* $\dim_k E^\nu_B(M) = \dim_k M$;

(b) *if* $E^i_B({}_Bk) = \delta_{i,\nu} k_B$ *and* $E^i_B(k_B) = \delta_{i,\nu}\, {}_Bk$ *and we let* ${}_BN \in \operatorname{Mod}_f(B)$ *with* $\operatorname{pd}_B(N) < \infty$ *then we have* $\operatorname{Tor}^B_{\nu-i}(k_B, N) \cong \operatorname{Ext}^i_B({}_Bk, N)$ *for all* $i \geq 0$.

Proof. (a) We argue by induction on $\dim_k M$. It is true for $\dim_k M = 1$ by hypothesis. For any $M \in \operatorname{Mod}^g_f(B)$ there exists an exact sequence in $\operatorname{Mod}^g_f(B)$:

$$0 \to M' \to M \to {}_Bk[m] \to 0$$

for some $m \in \mathbb{Z}$. Then use induction and the long exact sequence in cohomology.

(b) We recall the following two facts.

(b.1) Let B be any noetherian ring with $\operatorname{injdim}(B) = \nu < \infty$ and let ${}_BN \in \operatorname{Mod}_f(B)$ with $\operatorname{pd}_B(N) < \infty$. Then $\operatorname{injdim}_B(N) \leq \nu$. This can be proved by using the spectral sequence

$$\operatorname{Tor}^B_p(\operatorname{Ext}^q_B(M, B), N) \Rightarrow \operatorname{Ext}^{q-p}_B(M, N), \text{ where } {}_BM \in \operatorname{Mod}_f(B) \text{ (see [12]).}$$

(b.2) There exists a convergent spectral sequence

$$E_2^{p,-q} = \operatorname{Ext}^p_B(\operatorname{Ext}^q_B(k_B, B), N) \Rightarrow \operatorname{Tor}_{-(p-q)}(k_B, N)$$

(see [12] and use (b.1)).

We now prove the assertion. Since $\operatorname{Ext}^q_B(k_B, B) \cong \delta_{q,\nu B} k$, the spectral sequence of (b.2) degenerates to isomorphisms $E_2^{p,-\nu} \cong \operatorname{Ext}^p_B({}_Bk, N) \cong E_\infty^{p,-\nu} = \operatorname{Tor}_{\nu-p}(k_B, N)$.

5.13. THEOREM. *Let* $B = k \oplus B_1 \oplus B_2 \oplus \ldots$ *be a noetherian graded algebra such that*:

(a) $\operatorname{GKdim}_B M \in \mathbb{N}$ *for all* $M \in \operatorname{Mod}^g_f(B)$;

(b) $\operatorname{injdim}(B) = \operatorname{GKdim} B = 2$, B *is 2-homogeneous with a self-injective quotient ring*;

(c) $E^i_B({}_Bk) = \delta_{i,2} k_B$ *and* $E^i_B({}_Bk) = \delta_{i,2} k_B$.

Then B is Auslander-Gorenstein and satisfies the property CM.

Proof. We are going to prove that $E^p(E^q(M)) = (0)$ for $p < q$ and $M \in \operatorname{Mod}_f(B)$, and that B is CM. The theorem will then follow from Proposition 5.9.

By Lemma 5.11 (b), we have $E^0(E^q(M)) = 0$ if $q \geq 1$. Since $\operatorname{injdim}(B) = 2$, it remains to show $E^1(E^2(M)) = (0)$. Consider the spectral sequence $E_2^{p,-q} = E^p(E^q(M)) \Rightarrow \mathbb{H}^{p-q}$ of

Theorem 2.2. We get $E_\infty^{1,-2}=(0)$ (because $\mathbb{H}^n=(0)$ if $n\neq 0$) and for all $r\geq 2$ we have maps

$$E_r^{1-r,r-3}\xrightarrow{d_r}E_r^{1,-2}\xrightarrow{d_r}E_r^{r+1,-1-r}.$$

But since $1-r<0$ and $1+r>2$ we have $E_r^{1-r,r+3}=E_r^{r+1,-1-r}=(0)$. Hence $E^1(E^2(M))=E_2^{1,-2}=E_3^{1,-2}=\ldots=E_\infty^{1,-2}=(0)$.

We shall break the proof of the property CM into four steps. First notice that one can reduce to the graded CM-property by Remark 5.9 (2).

1. By hypothesis GKdim takes values in $\mathbb{N}$ on $\mathrm{Mod}_f^g(B)$; hence $\mathrm{GKdim}_B M=0$, 1 or 2 for all $M\in\mathrm{Mod}_f^g(B)$. If $\mathrm{GKdim}_B M=0$ then $\dim_k M<\infty$ and $j_B(M)=2$ by Lemma 5.12. If $\mathrm{GKdim}_B M=2$, we have $E^0(M)=M^*\neq(0)$ by Lemma 5.11, i.e. $j_B(M)=0$. Therefore the case $\mathrm{GKdim}_B M=1$ is the only difficult case and all that follows is to deal with this case.

2. We recall the Auslander Approximation Theorem in $\mathrm{Mod}_f^g(B)$, see [**11**, Proposition 3.8]. The Auslander Approximation Theorem states the following. Let B be a graded noetherian ring. Let $M\in\mathrm{Mod}_f^g(B)$ and suppose there exists $r\in\mathbb{N}$ such that $E_B^p(E_B^q(M))=(0)$ for all $p<q\leq r$. Then there exists $M_1\in\mathrm{Mod}_f^g(B)$, $\mathrm{pd}_B(M_1)\leq r$, a graded morphism $f:M\to M_1$ such that $\mathrm{Ext}^i(f):E_B^i(M_1)\to E_B^i(M)$ is an isomorphism for all $i\in\{1,\ldots,r\}$.

The Auslander Approximation Theorem applies to any $M\in\mathrm{Mod}_f^g(B)$, B as in the theorem, since we showed $E^p(E^q(M))=(0)$ if $p<q\leq 2$. Hence we get $f:M\to M_1$, $\mathrm{pd}_B(M_1)\leq 2$, $\alpha^i(f)$: $E_B^i(M_1)\xrightarrow{\sim}E_B^i(M)$ isomorphisms for $i=1,2$.

3. When ${}_BM\in\mathrm{Mod}_f^g(B)$, we define the torsion submodule by $T(M):=\{x\in M\ /\ \exists i\geq 0,\ \mathcal{M}^ix=0\}$ and we put $\bar{M}=M/T(M)$. Recall that the socle of M is $\mathrm{Soc}(M)=\mathrm{Hom}_B({}_Bk,M)=\{x\in M\ /\ \mathcal{M}x=0\}$. We do the same for $M_B\in\mathrm{Mod}_f^g(B)$.

Claim. (i) $T(M)\neq(0)$ if and only if $\mathrm{Soc}(M)\neq(0)$ if and only if $E_B^2(M)\neq(0)$.

(ii) $E_B^i(M)\cong E_B^i(\bar{M})$, $i=0,1$.

(iii) $E_B^2(\bar{M})=(0)$ and $E_B^2(M)\cong E_B^2(T(M))$ is a finite dimensional vector space.

Proof of the claim. Clearly $T(M)\neq(0)$ if and only if $\mathrm{Soc}(M)\neq(0)$. Notice that $\dim_k T(M)<\infty$; hence $E^i(T(M))=(0)$ if $i=0,1$ and $\dim_k E^2(T(M))=\dim_k T(M)$ by Lemma 5.12(a). The long exact sequence

$$\ldots\ldots\to E^{i-1}(T(M))\to E^i(\bar{M})\to E^i(M)\to E^i(T(M))\to\ldots\ldots$$

proves (ii). Since $T(M)=(0)$, (i) implies (iii). If $T(M)\neq(0)$, we have $E^2(T(M))\neq(0)$ and hence $E^2(M)\neq(0)$ follows from the surjection $E^2(M)\to E^2(T(M))$. So it remains only to show that $E^2(M)\neq(0)$ implies $T(M)\neq(0)$, i.e. $\mathrm{Soc}(M)\neq(0)$. Let $f:M\to M_1$ be given by step 2. Remark that $\mathrm{pd}_B(M_1)=2$ (because $E^2(M_1)\cong E^2(M)\neq(0)$) and, since it is a graded module, we have $\mathrm{Tor}_2^B(k_B,M_1)\neq(0)$. By Lemma 5.12(b), we have $\mathrm{Tor}_{2-i}^B(k_B,M_1)\cong\mathrm{Ext}_B^i({}_Bk,M_1)$ for all i. In particular $\mathrm{Soc}(M_1)=\mathrm{Hom}_B({}_Bk,M_1)\cong\mathrm{Tor}_2^B(k_B,M_1)\neq(0)$. Recall that f induces a map between the spectral sequences $E_2^{p,-q}(M_1)$, $E_2^{p,-q}(M)$ and their invariants (see Remark 2.1 (2)). Hence we obtain a commutative diagram:

$$\begin{array}{ccccccc} F^3M=(0)\to & F^2M & \to & F^1M & \to & F^0M=M \\ & \downarrow f & & \downarrow f & & \downarrow f \\ F^3M_1=(0)\to & F^2M_1 & \to & F^1M_1 & \to & F^0M_1=M_1 \end{array}$$

Since $\alpha^i(f):E^i(M_1)\xrightarrow{\sim}E^i(M)$, $i=1,2$, we have isomorphisms $E_2^{i,-i}(f):E^i(E^i(M))\to E^i(E^i(M_1))$ (Remark 2.1 (2)).

By Theorem 2.2 (b) and $\text{injdim}(B)=2$, we conclude that f induces isomorphisms

$$f: F^2M \cong E^2(E^2(M)) \xrightarrow{\sim} F^2M_1 \cong E^2(E^2(M_1))$$

and

$$\bar{f}: \frac{F^1M}{F^2M_1} \cong E^1(E^1(M)) \xrightarrow{\sim} \frac{F^1M_1}{F^2M_1} \cong E^1(E^1(M_1)).$$

It follows that f is an isomorphism from F^1M onto F^1M_1. Since $E^2(M) \neq (0)$ and $E^1(E^2(M))=(0)$, $i=0,1$, we see that $E^2(E^2(M)) \neq (0)$, and therefore $F^2M \cong E^2(E^2(M)) \neq (0)$. Recall that we have $0 \to M_1/F^1M_1 \to E^0(E^0(M_1)) = M_1^{**}$. Hence $\text{Soc}(M_1/F^1M_1) \subset \text{Soc}(M_1^{**}) = (0)$ since M_1^{**} is a submodule of a free B-module and $\text{Soc}(B)=(0)$. It follows that $\text{Soc}(F^1M_1) = \text{Soc}(M_1)$. Thus $\text{Soc}(M) \supset \text{Soc}(F^1M) \cong \text{Soc}(F^1M_1) = \text{Soc}(M_1) \neq (0)$, proving $\text{Soc}(M) \neq (0)$.

4. We now finish the proof of the property CM for B. Assume $\text{GKdim}_B M = 1$. Then $E^0(M) = M^* = (0)$ by Lemma 5.11. Suppose $E^1(M)=(0)$; we get $E_2^{p,-1}(M) = E^p(E^1(M)) = (0)$ for all p. Hence $E_2^{p,-q}(M)=(0)$ except for $(p,q)=(2,2)$, and the spectral sequence degenerates to $M \cong E^2(E^2(M))$. By step 3, this implies $\dim_k M < \infty$ and a contradiction. Thus $j_B(M)=1$. If $\text{GKdim}_B M = 2$, we have $E^0(M) = M^* \neq (0)$ by Lemma 5.11, i.e. $j_B(M)=0$.

Remarks. (1) The condition on the quotient ring of B is in particular satisfied when B is semi-prime.

(2) Let $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ be a finitely generated graded algebra containing $\underline{\Omega} = \{\Omega_1, \ldots, \Omega_l\}$, a regular normalizing sequence of homogeneous elements of positive degree in A. Assume that $B = A/\underline{\Omega}A$ satisfies the hypothesis of the theorem above and condition (PG). We deduce from Corollary 5.10 that: A is Auslander-Gorenstein, satisfies the property CM and the condition (PG), $\text{injdim}(A) = l+2$.

6. Artin-Schelter regular algebras.

6.1. In **[1]** is introduced a notion of "regularity" for graded algebras. To avoid any confusion we have chosen to call them AS-regular algebras. We shall compare this definition with Auslander-regularity. For technical reasons it is natural to enlarge the class of AS-regular algebras to AS-Gorenstein algebras.

Definition. *Let $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ be a finitely generated k-algebra. We say that A is AS-regular (resp. AS-Gorenstein) of dimension μ if*:

(i) *$\text{gldim}(A) = \mu < \infty$ (resp. $\text{injdim}(A) = \mu < \infty$)*;

(ii) *the function $n \to \dim_k A_n$ has polynomial growth of degree $d-1 \in \mathbb{N} \cup \{-1\}$*;

(iii) *A is Gorenstein in the following sense: $E_A^i({}_Ak) = \delta_{i,\mu} k_A$ and $E_A^i(k_A) = \delta_{i,\mu}\, {}_Ak$.*

Remarks. (1) Assume A is a noetherian AS-regular algebra. Then A satisfies the condition (PG) defined in 5.4. In fact the Hilbert series of every $M \in \text{Mod}_f^g(A)$ has the form $h_M(t) = q_M(t)/p_A(t)$ as in 5.5.

(2) We have $d = \text{GKdim}\, A$.

(3) By the Rees lemma, if $\{\Omega_1, \ldots, \Omega_l\}$ is a regular normalizing sequence of homogeneous elements of positive degree in some AS-Gorenstein algebra then $B = A/(\Omega_1, \ldots, \Omega_l)$ is also AS-Gorenstein.

(4) The definition of an AS-regular algebra given in **[3]** requires that ${}_Ak$ has a free resolution in $\text{Mod}_f^g(A)$.

6.2. In [1] and [3], [4], the following natural questions are raised about AS-regular algebras.

(6.2.1) Is A noetherian?

(6.2.2) Is A a domain?

(6.2.3) Is it always true that $\mathrm{gldim}(A) = \mathrm{GKdim}\, A$ (i.e. $\mu = d$)?

When A is AS-Gorenstein, one can ask similar questions, replacing (6.2.2) and (6.2.3) by

(6.2.4). Is A μ-homogeneous for GKdim?

(6.2.5) Is it true that $\mathrm{injdim}(A) = \mathrm{GKdim}\, A$?

The results from [1] and [3], [4] give (partial) answers to these questions; we summarize in the following theorem.

THEOREM. *Let A be an AS-regular algebra of dimension μ.*

(i) *If $\mu \leq 3$ and A is generated by A_1 then A is a noetherian domain and* $\mathrm{GKdim}\, A = \mu$.

(ii) *If A is noetherian and* $\mathrm{GKdim}\, A = \mu \leq 4$ *then A is a domain.*

(iii) *If* $\mathrm{GKdim}\, A = \mu \leq 3$ *and A is noetherian then A satisfies the property CM and the condition*: $E^i_A(E^j_A(M)) = (0)$ *for all $i < j$ and* $M \in \mathrm{Mod}^{\mathrm{g}}_{\mathrm{f}}(A)$.

Using the results of 5.9, we deduce the next corollary.

COROLLARY. *Let A be an AS-regular algebra as in* (iii) *of the previous theorem. Then A is Auslander-regular and satisfies the property CM.*

We shall recover (particular) cases of this corollary in 6.7 and prove a similar result for some algebras of $\mathrm{gldim}(A) \leq 4$.

6.3. In 4.8, we showed that a positively graded Auslander-regular k-algebra with $A_0 = k$ is a domain. In view of Corollary 6.2, we may ask the following questions.

(6.3.1) Let $A = k \oplus A_1 \oplus A_2 \oplus \dots$ be finitely generated over k. Do we have: A AS-regular (resp. AS-Gorenstein) if and only if A is Auslander-regular (resp. Auslander-Gorenstein) and A satisfies condition (PG)? Furthermore does A satisfy the property CM with $\mathrm{GKdim}\, A = \mathrm{gldim}(A)$ (resp. $\mathrm{GKdim}\, A = \mathrm{injdim}\, A$)?

Notice that a positive answer to (6.3.1) gives positive answers to (6.2.j), $j = 1, \dots, 5$. One of the implications is easy, namely we have the following result.

THEOREM. *Let $A = k \oplus A_1 \oplus A_2 \oplus \dots$ be an Auslander-Gorenstein k-algebra of dimension μ. Then $E^i_A({}_Ak) = \delta_{i,\mu}\, k_A$. In particular if $n \to \dim_k A_n$ has polynomial growth A is AS-Gorenstein.*

Proof. Put $n = j({}_Ak)$, $m = j(k_A)$. We first have to show $n = m$. Set $N_A := E^n_A({}_Ak) \in \mathrm{Mod}^{\mathrm{g}}_{\mathrm{f}}(A)$. Since there exists a surjective graded morphism $N_A \to k_A[m] \to 0$, for some $m \in \mathbb{Z}$, we have $j(N) \leq j(k_A) = m$. But $j(N) = n$ by Theorem 2.4. Hence $n \leq m$ and, by symmetry, $n = m$. We now prove $n = \mu$. Let $S_A \in \mathrm{Mod}^{\mathrm{g}}_{\mathrm{f}}(A)$ be such that ${}_AM = E^\mu(S) \neq (0)$. S exists by Lemma 3.3. Since $E^i(E^\mu(S)) = (0)$ if $i < \mu$, we have $j(M) = \mu$. There exists an exact sequence $0 \to M' \to M \to {}_Ak[m] \to 0$ in $\mathrm{Mod}^{\mathrm{g}}_{\mathrm{f}}(A)$. It follows that $\mu = j(M) = \inf\{j({}_Ak),\, j(M')\}$; thus $n = j({}_Ak) = \mu$. It remains to show that $\dim_k E^\mu({}_Ak) = 1$ (same proof for k_A). The b-filtration for ${}_Ak$ takes the form: $(0) \subset F^\mu{}_Ak = \dots = {}_Ak$; hence ${}_Ak = F^\mu{}_Ak \cong E^\mu(E^\mu({}_Ak))$ by Theorem 2.2. Put $N_A = E^\mu({}_Ak)$, so that $j(N) = \mu$, and suppose that $\dim_k N > 1$. Consider an exact sequence in $\mathrm{Mod}^{\mathrm{g}}_{\mathrm{f}}(A)$: $0 \to N' \to N \to$

$k_A[m] \to 0$. We get $j(N') \geq j(N) = \mu$; hence $j(N') = \mu$. Therefore we have an exact sequence $0 \to E^{\mu}(k_A[m]) \to E^{\mu}(N) \to E^{\mu}(N') \to 0$. From $E^{\mu}(N') \neq (0)$ and $E^{\mu}(N) \cong {}_A k$, it follows that $E^{\mu}(k_A[m]) = E^{\mu}(k_A)[-m] = (0)$, contradicting the fact that $j(k_A) = \mu$.

6.4. We recall here results from [**5**] and [**3**], [**4**] which will be used to give an example of an Auslander-Gorenstein ring in the next section.

For simplicity we assume that k is algebraically closed and we denote by X an irreducible projective variety over k. The category of quasi-coherent (resp. coherent) $\mathcal{O}_X$-modules is denoted by $\mathcal{O}_X$-Mod (resp. $\mathcal{O}_X - \text{Mod}_{\text{coh}}$). Let σ be a k-automorphism of X. When $\mathcal{M} \in \mathcal{O}_X$-Mod we set $\mathcal{M}^{\sigma} := \sigma^* \mathcal{M}$. An invertible sheaf is called σ-ample if: for all $\mathcal{F} \in \mathcal{O}_X\text{-Mod}_{\text{coh}}$, all $q \geq 1$, $H^q(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}}) = 0$ if $n \gg 0$. From now on we fix a σ-ample invertible sheaf $\mathcal{L}$ and, as in [**5**], we define

$$\mathcal{B}_0 := \mathcal{O}_X, \ \mathcal{B}_n := \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \ldots \otimes \mathcal{L}^{\sigma^{n-1}} \quad \text{if} \quad n > 0,$$
$$\mathcal{B}_n := \mathcal{L}^{-\sigma^{-1}} \otimes \ldots \otimes \mathcal{L}^{-\sigma^{-n}} \quad \text{if} \quad n < 0,$$
$$\mathcal{B} = \bigoplus_{n \in \mathbb{Z}} \mathcal{B}_n, \quad B_n = H^0(X, \mathcal{B}_n), \quad B := B(X, \sigma, \mathcal{L}) := \bigoplus_{n \geq 0} B_n.$$

Then $\mathcal{B}$ is a sheaf of graded algebras and B is a graded domain. The main theorem of [**5**] proves the next theorem.

THEOREM. *The algebra* $B = k \oplus B_1 \oplus B_2 \oplus \ldots$ *is noetherian and there is an equivalence of categories*:

$$\text{Mod}^{g}(B)/\text{Tors} \simeq \mathcal{O}_X\text{-Mod},$$

where Tors *is the full subcategory of* $\text{Mod}^{g}(B)$ *consisting of modules which are direct limits of finite dimensional modules.*

REMARK. The equivalence from $\mathcal{O}_X$-Mod onto $\text{Mod}^{g}(B)/\text{Tors}$ is induced by the functor $\Gamma_* : \mathcal{O}_X\text{-Mod} \to \text{Mod}^{g}(B)$ defined by $\Gamma_*(\mathcal{M}) := \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{B}_n)$. It takes coherent $\mathcal{O}_X$-modules to finitely generated B-modules.

6.5. We assume that the variety X of 6.4 is a smooth elliptic curve and we fix $\sigma \in \text{Aut}_k(X)$. Then any ample invertible sheaf $\mathcal{L}$ is σ-ample and we have GKdim $B = 2$, see [**5**, 1.5] and Lemma 6.6.

The following proposition is a reformulation of results by A. Yekutieli, see [**20**], [**21**].

PROPOSITION. *The algebra* B *satisfies*: $\text{injdim}(B) = 2$, $E^i_B(k_B) = \delta_{i,2\,B} k$, $E^i_B({}_B k) = \delta_{i,2}\, k_B$.

Proof. By [**20**, 4.3.1] and the fact that the canonical sheaf on X is isomorphic to $\mathcal{O}_X$, we have a resolution by injectives in $\mathcal{O}_X\text{-Mod}: 0 \to \mathcal{O}_X \xrightarrow{\delta} \mathcal{K}^{-1}_X \xrightarrow{\delta} \mathcal{K}^0_X \to 0$. Set $J^0 = \Gamma_*(\mathcal{K}^{-1}_X)$, $J^1 = \Gamma_*(\mathcal{K}^0_X)$, $J^2 = \Gamma_*(\mathcal{K}^0_X)/\delta(\Gamma_*(\mathcal{K}^{-1}_X))$. Then, by the proof of [**20**, 4.5.3], one obtains: J^i is a graded bimodule over B injective on both sides, J^0 and J^1 are torsion free and $J^2 \cong \text{HOM}_k(B, k)$ is the injective hull of ${}_B k$, or k_B, in $\text{Mod}^{g}(B)$. Thus, applying Γ_* to the resolution above, we obtain an injective resolution $0 \to B \to J^0 \to J^1 \to J^2 \to 0$ in $\text{Mod}^{g}(B)$. This proves $\text{grinjdim}(B) = \text{injdim}(B) \leq 2$. By symmetry, the proposition will follow from $E^i_B({}_B k) = \delta_{i,2} k_B$. Since J^0, J^1 are torsion free and $J^2 \cong \text{HOM}_k(B, k)$, we have: $\text{Hom}_B({}_B k, J^i) = (0)$ if $i = 0, 1$ and $\text{Hom}_B({}_B k, J^2) = \text{Hom}_k(B, k)_0 \cong k_B$. Since $E^i_B({}_B k) = H^i(\text{HOM}_B(k, J^{\bullet})) = H^i(\text{Hom}_B(k, J^{\bullet}))$, the proof is complete.

6.6. Assume $(X, \sigma, \mathscr{L})$ is as in 6.5 and suppose furthermore that $\mathscr{L}$ is very ample, i.e. $\deg \mathscr{L} \geq 3$. Denote by C the algebra $B(X, \mathrm{Id}_X, \mathscr{L})$. Hence $C = \bigoplus_{n \geq 0} C_n$, where $C_n = H^0(X, \mathscr{L}^{\otimes n})$, is the homogeneous coordinate ring of the projective embedding $X \to \mathbb{P}(H^0(X, \mathscr{L})^*)$.

Lemma. *Let $M = \bigoplus_m M_m$ be in $\mathrm{Mod}_f^g(B)$. Then the function $m \to \dim_k M_m$ is given by a polynomial for $n \gg 0$. In particular B satisfies the condition* (PG).

Proof. Let $\mathscr{M}$ be a coherent $\mathcal{O}_X$-module. We first notice that, if $m \gg 0$, $\dim_k H^0(X, \mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{B}_m) = \dim_k H^0(X, \mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes m})$. This can be proved as follows. Taking a finite locally free resolution of $\mathscr{M}$, one reduces to the case where $\mathscr{M}$ is locally free. Then [**2**, Lemma 15] gives the result since $\mathscr{B}_m$ and $\mathscr{L}^{\otimes m}$ have the same degree. Now, by Theorem 6.4, we have equivalences

$$\mathrm{Mod}^g(B)/\mathrm{Tors} \cong \mathcal{O}_X\text{-Mod} \cong \mathrm{Mod}^g(C)/\mathrm{Tors}.$$

If $\dim_k M < \infty$, the lemma is obvious. Therefore we assume that M is torsion free and denote by $\mathscr{M} \in \mathcal{O}_X\text{-Mod}_{\mathrm{coh}}$ the corresponding $\mathcal{O}_X$-module. See $N = \Gamma_*(\mathscr{M}) \in \mathrm{Mod}_f^g(C)$. By the previous remark, $\dim_k M_m = \dim_k N_m$. Then it is well known that $m \to \dim_k N_m$ is given by a polynomial of degree $\mathrm{GKdim}\, N - 1$ for $n \gg 0$. Hence the lemma is proved.

Remark. The Hilbert series of $M \in \mathrm{Mod}_f^g(B)$ is of the form $h_M(t) = q_M(t)/(1-t)^2$, $q_M(t) \in \mathbb{Z}[t, t^{-1}]$. This is a consequence of the proof of the lemma and the truth of this assertion in $\mathrm{Mod}_f^g(C)$. We have $h_B(t) = (1 + (r-2)t + t^2)/(1-t)^2$ if $r = \deg \mathscr{L}$.

Theorem. *Let $(X, \sigma, \mathscr{L})$ be a triple as above. Then the algebra $B(X, \sigma, \mathscr{L})$ is Auslander-Gorenstein of dimension* 2, *satisfies the CM-property and the condition* (PG).

Proof. Notice that $\mathrm{GKdim}\, B = 2$ and apply the lemma, Proposition 6.5 and Theorem 5.13.

6.7. We deduce, from Remark 5.13 (2), the following result.

Corollary. *Let $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ be a finitely generated graded k-algebra. Assume $\Omega = \{\Omega_1, \ldots, \Omega_l\}$ is a regular normalizing sequence of homogeneous elements of positive degree in A such that $A/\Omega A \cong B(X, \sigma, \mathscr{L})$ for some triple $(X, \sigma, \mathscr{L})$ as in* 6.6. *Then A is Auslander-Gorenstein of dimension $l+2$ and satisfies the CM-property and condition* (PG).

Applications. (1) Let A be an AS-regular algebra of dimension 3 and type A as in [**3**]. Then A is Auslander-regular and satisfies the CM-property. This follows from the corollary since $A/\Omega A \cong B(X, \sigma, \mathscr{L})$ for a central element of degree 3 and a triple $(X, \sigma, \mathscr{L})$ as in 6.6 (with $\deg \mathscr{L} = 3$). In particular, we recover [**4**, Theorem 4.1] in this case.

(2) Let S be the Sklyanin algebra as in [**18**, Theorem 5.4]. Then S is Auslander-regular of dimension 4 and satisfies the CM-property. The corollary applies to S since $S/(\Omega_1, \Omega_2) \cong B(X, \sigma, \mathscr{L})$ for a regular centralizing sequence $\{\Omega_1, \Omega_2\}$ of quadratic elements and a triple $(X, \sigma, \mathscr{L})$ as in 6.6 (with $\deg \mathscr{L} = 4$). Since $\mathrm{gldim}(S) = 4$ by [**18**, Theorem 0.3], we have that S is Auslander-regular.

(3) In [**19**], the construction of the Sklyanin algebra is generalized to obtain AS-regular algebras A of dimension 4 such that $A/(\Omega_1, \Omega_2) \cong B(X, \sigma, \mathscr{L})$ for a regular normalizing sequence $\{\Omega_1, \Omega_2\}$ of quadratic elements and a triple $(X, \sigma, \mathscr{L})$ as in 6.6 (with $\deg \mathscr{L} = 4$). Thus the corollary can be applied to these algebras.

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Département de Mathématiques
Université de Bretagne Occidentale
6, Avenue Victor Le Gorgeu
29287 Brest Cedex
France