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# LETTER TO THE EDITOR

# A SAMPLE PATH APPROACH TO MEAN BUSY PERIODS FOR MARKOV-MODULATED QUEUES AND FLUIDS

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**Abstract**

The mean busy period of a Markov-modulated queue or fluid model is computed by an extension of the time-reversal argument connecting the steady-state distribution and the maximum of a related Markov additive process.

FLUID FLOW; MARKOV-MODULATION; TIME REVERSAL
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## 1. Introduction

For many queuing processes, one can identify the stationary distribution with the distribution of the maximum of the time-reversed net input process. This idea can be traced back at least to Lindley [9] and has turned out to be a fruitful tool in the study of queues exploited by many authors. In particular, an early classical paper is Loynes [11] which deals with a general stationary set-up; some references relevant for the present paper are the studies [2], [6], [3], [5] by the authors which consider various Markov-modulated models.

The purpose of this note is to point out a less established extension of the time reversal relation, and to show how it leads to a simple way of calculating mean busy periods in terms of steady-state quantities, a problem which is easy for most simple queues but non-trivial in a Markov-modulated setting.

The details will be given for a fluid queueing model arising in the study of recent ATM (asynchronous transfer mode) technology in telecommunications (the study of traditional queueing models is similar and the formulas are stated at the end of the paper).

## 2. Preliminaries

The process $\{(J_t, V_t)\}_{t\geqq 0}$ under study is defined by $\{J_t\}$ being an irreducible Markov process with finite state space $E$, and $\{V_t\}$ having piecewise linear paths with slope $r_i$ on intervals where $J_t = i$, $V_t > 0$, and reflection at 0. We can represent $\{V_t\}$ as the reflected version

$$V_t = S_t - \min_{0\leqq v\leqq t} S_v \tag{1}$$

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of the net input process

(2) $$S_t = \int_0^t r_{J_v}\, dv$$

(thus $\{S_t\}$ is a continuous-time Markov additive process defined on $\{J_t\}$). The stability condition ensuring the existence of a limiting steady state is

$$\sum_{i \in E} \pi_i r_i < 0,$$

where $\boldsymbol{\pi} = (\pi_i)_{i \in E}$ is the stationary distribution of $\{J_t\}$, and we let $(J, V)$ denote a pair of random variables having the limiting stationary distribution of $(J_t, V_t)$. Let $\boldsymbol{\Lambda} = (\lambda_{ij})_{i,j \in E}$ denote the intensity matrix for $\{J_t\}$. The time-reversed version $\{\tilde{J}_t, \tilde{S}_t\}$ of the Markov additive process $\{J_t, S_t\}$ is defined by letting the intensity matrix $\tilde{\boldsymbol{\Lambda}}$ *of* $\{\tilde{J}_t\}$ have elements $\tilde{\lambda}_{ij} = \pi_j \lambda_{ji} / \pi_i$, and letting $\{\tilde{S}_t\}$ be defined as $\{S_t\}$, with the same rates $r_i$ but $\{J_t\}$ replaced by $\{\tilde{J}_t\}$. Further let $\tilde{M}_t = \sup_{0 \leq u \leq t} \tilde{S}_u$, $\tilde{M} = \sup_{t \geq 0} \tilde{S}_t$,

$$\omega(T) = \inf\{t \leqq T : \tilde{M}_t = \tilde{M}_T\}, \qquad \omega = \inf\{t > 0 : \tilde{M}_t = \tilde{M}\}.$$

The classical time-reversal argument [3] then yields

(3) $$\mathbb{P}(V \in A, J = j) = \pi_j \mathbb{P}_j(\tilde{M} \in A).$$

Clearly, $\tilde{M}$ is the lifetime of the terminating Markov process $\{m(x)\}_{x \geq 0}$ that we obtain by observing $\{\tilde{J}_t\}$ when $\{\tilde{S}_t\}$ is at maximum values,

$$m(x) = \tilde{J}_{\tau(x)} \quad \text{where} \quad \tau(x) = \inf\{t > 0 : \tilde{S}_t = x\}.$$

The state space is $E_+ = \{i \in E : r_i > 0\}$ (similarly, we write $E_- = \{i \in E : r_i < 0\}$; for simplicity, it is assumed that $r_i \neq 0$ for all $i$ though this assumption is not crucial, cf. [3]). It follows by standard facts on phase-type distributions [14] that

(4) $$\mathbb{P}(V \in dx, J = j) = \pi_j \boldsymbol{\alpha}_j e^{\boldsymbol{U}x} \boldsymbol{u}, \quad x > 0, \qquad \mathbb{P}(V = 0, J = j) = \pi_j (1 - \boldsymbol{\alpha}_j \boldsymbol{e})$$

where $\boldsymbol{\alpha}_j = (\alpha_{jk})_{k \in E_+}$ is the row vector of initial $\mathbb{P}_j$-probabilities for $m(0)$, $\boldsymbol{U} = (u_{ij})_{i,j \in E_+}$ the intensity matrix of $\{m(x)\}$ and $\boldsymbol{u} = (u_i)_{i \in E_+} = -\boldsymbol{Ue}$ with $\boldsymbol{e}$ the column vector of ones.

Algorithms for computing $\boldsymbol{U}$ and the $\boldsymbol{\alpha}_j$ are given in Asmussen [3] and Rogers [15] (cf. also London et al. [10] and Barlow et al. [7]), whereas other approaches to the computation of the distribution of $(J, V)$ are discussed in a number of papers, see e.g. Anick et al. [1], Gaver and Lehoczky [8] for some early studies and [3], [15] for a more recent set of references.

For later use, we quote the following formula ([3] Theorem 3.1) connecting $\boldsymbol{U}$ and the $\boldsymbol{\alpha}_j$:

$$u_{ij} = \frac{1}{r_i \pi_i} \left\{ \pi_j \lambda_{ji} + \sum_{k \in E_-} \pi_k \lambda_{ki} \alpha_{kj} \right\}.$$

From this it follows by summing over $j$ and using the stationary equation $\sum_{j \in E_+ \cup E_-} \pi_j \lambda_{ji} = 0$ that

(5) $$u_i = -\sum_{j \in E_+} u_{ij} = \frac{1}{r_i \pi_i} \left\{ \sum_{k \in E_-} \pi_k \lambda_{ki} (1 - \boldsymbol{\alpha}_k \boldsymbol{e}) \right\}.$$

## 3. The mean busy period

A busy period of length $P_i = \inf\{t > 0 : V_t = 0\}$ and type $i \in E_+$ starts from $V_0 = 0$ and $J_0 = i$, and ends at the time $P_i$ when the process returns to 0. At any time $T$, we define $I_T$ as the type of the busy cycle in progress. That is, $I_T = J_{\kappa(T)}$ where $\kappa(T) = \sup\{t \leqq T : V_t = 0\}$. We shall use the following generalization of (3), which contains the additional information on the joint steady-state distribution of $(V, J, I)$.

*Proposition* 3.1. $\mathbb{P}(V \in A, J = j, I = i) = \pi_j \mathbb{P}_j(\tilde{M} \in A, \tilde{J}_{\omega-} = i)$.

*Proof.* Let $T < \infty$ be fixed. Considering stationary versions, we may assume $\tilde{J}_t = J_{(T-t)-}$ (left limit), $\tilde{S}_t = S_T - S_{T-t}$ for $0 \leqq t \leqq T$. Then

$$\kappa(T) = \sup \left\{ t : 0 \leqq t \leqq T, \sup_{0 \leqq u \leqq t} (S_t - S_u) = 0 \right\}$$

$$= \sup \left\{ t : 0 \leqq t \leqq T, \sup_{0 \leqq u \leqq t} (\tilde{S}_{T-u} - \tilde{S}_{T-t}) = 0 \right\},$$

$$T - \kappa(T) = \inf \left\{ t : 0 \leqq t \leqq T, \sup_{T \geqq v \geqq t} (\tilde{S}_v - \tilde{S}_t) = 0 \right\} = \omega(T).$$

Similarly, $\{V_T \in A\} = \{\tilde{M}_T \in A\}$ (this is the classical time-reversal relation), and thus

$$\mathbb{P}_\pi(V_T \in A, J_0 = k, J_T = j, I_T = i) = \mathbb{P}_\pi(V_T \in A, J_0 = k, J_T = j, J_{T-\omega(T)} = i)$$

$$= \mathbb{P}_\pi(\tilde{M}_T \in A, \tilde{J}_0 = j, \tilde{J}_T = k, \tilde{J}_{\omega(T)-} = i)$$

$$= \pi_j \mathbb{P}_j(\tilde{M}_T \in A, \tilde{J}_T = k, \tilde{J}_{\omega(T)-} = i).$$

Since $\tilde{M}$ is attained at the finite time $\omega$, $\tilde{J}_T$ is asymptotically independent of $\tilde{M}$, $\omega$, and hence letting $T \to \infty$ we get

$$\mathbb{P}_\pi(V_T \in A, J_0 = k, J_T = j, I_T = i) \approx \pi_j \mathbb{P}_j(\tilde{M} \in A, \tilde{J}_{\omega-} = i) \mathbb{P}(\tilde{J}_T = k)$$

$$\approx \pi_j \pi_k \mathbb{P}_j(\tilde{M} \in A, \tilde{J}_{\omega-} = i).$$

On the other hand, clearly

$$\mathbb{P}_\pi(V_T \in A, J_0 = k, J_T = j, I_T = i) = \pi_k \mathbb{P}_k(V_T \in A, J_T = j, I_T = i)$$

$$\to \pi_k \mathbb{P}(V \in A, J = j, I = i).$$

Identifying the two limits yields the result.

Note that $\tilde{M} \in dx$, $\tilde{J}_{\omega-} = i$ means that $\{m(x)\}$ has lifelength $x$ and dies (exits) in state $i$. From the interpretation of $\boldsymbol{u}$ as the exit rate vector [14], it thus follows that

$$\mathbb{P}_j(\tilde{M} \in dx, \tilde{J}_{\omega-} = i) = \boldsymbol{\alpha}_j e^{Ux} \boldsymbol{u}_i, \tag{6}$$

where $\boldsymbol{u}_i$ is the vector obtained from $\boldsymbol{u}$ by replacing all entries except the $i$th by zeros. Let further $\boldsymbol{e}_i$ be the column vector with 1 at the $i$th entry and 0 at the others.

*Proposition* 3.2. *Let* $\bar{\boldsymbol{\alpha}} = \sum_{j \in E} \pi_j \boldsymbol{\alpha}_j$. *Then*

$$\mathbb{E}_i P_i = \frac{-\bar{\boldsymbol{\alpha}} \boldsymbol{U}^{-1} e_i}{\pi_i r_i}.$$

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*Proof.* By Proposition 3.1 and (6),

$$\mathbb{P}(V>0, I=i)=\sum_{j\in E}\int_0^\infty \mathbb{P}(V\in dx, J=j, I=i)\,dx$$
$$=\sum_{j\in E}\int_0^\infty \pi_j\boldsymbol{\alpha}_j e^{Ux}\boldsymbol{u}_i\,dx$$
$$=-\sum_{j\in E}\pi_j\boldsymbol{\alpha}_j\boldsymbol{U}^{-1}\boldsymbol{u}_i=-\bar{\boldsymbol{\alpha}}\boldsymbol{U}^{-1}\boldsymbol{u}_i.$$

On the other hand,

$$\mathbb{P}(V>0, I=i)=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T I(V_t>0, I_t=i)\,dt$$
$$=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^\infty I(V_t>0, I_t=i, \kappa(t)\leqq T)\,dt$$

(the last identity uses the fact that the mean residual length of the busy period in progress at time $T$ is $o(T)$). Now when $V=0$, $J=k\in E_-$, a busy period of type $i$ is initiated at rate $\lambda_{ki}$ and has mean duration $\mathbb{E}_iP_i$. Hence

$$\mathbb{E}\int_0^\infty I(V_t>0, I_t=i, \kappa(t)\leqq T)\,dt=\mathbb{E}\int_0^T\sum_{k\in E_-}I(V_{t-}=0, J_{t-}=k)\lambda_{ki}\mathbb{E}_iP_i\,dt,$$

$$\mathbb{P}(V>0, I=i)=\sum_{k\in E_-}\mathbb{P}(V=0, J=k)\lambda_{ki}\mathbb{E}_iP_i$$
$$=\sum_{k\in E_-}\pi_k\lambda_{ki}(1-\boldsymbol{\alpha}_k\boldsymbol{e})\mathbb{E}_iP_i.$$

Putting the two expressions equal yields

$$\mathbb{E}_iP_i=\frac{-\bar{\boldsymbol{\alpha}}\boldsymbol{U}^{-1}\boldsymbol{u}_i}{\sum_{k\in E_-}\pi_k\lambda_{ki}(1-\boldsymbol{\alpha}_k\boldsymbol{e})}. \tag{7}$$

Now just use (5).

The treatment of fluid models carries over with minor changes to queues with Neuts' Markovian arrival process [13]. Here there is still an environmental Markov process $\{J_t\}$, but we write now the intensity matrix $\boldsymbol{\Lambda}$ as $\boldsymbol{C}+\boldsymbol{D}$, such that $d_{kl}$ is the intensity of a state transition from $k$ to $l$ accompanied by an arrival ($k=l$ is here allowed) and $c_{kl}$ ($k\neq l$) is the intensity of $J_t$ changing from state $k$ to $l$ without an arrival (the $c_{kk}$ are determined by $(\boldsymbol{C}+\boldsymbol{D})\boldsymbol{e}=0$). The service time distribution of a customer arriving at a transition from $k$ to $l$ is $B_{kl}$ (say). Thus the types of the busy periods are indexed by $kl\in E^2$ rather than by $i\in E$. If all $B_{kl}$ are phase-type, there is an analogue of the Markov process $\{m(x)\}$ as discussed in Asmussen and Perry [6], and the proof of Proposition 3.2 carries immediately over to give the formula

$$\mathbb{E}_{kl}P_{kl}=\frac{-\bar{\boldsymbol{\alpha}}\boldsymbol{U}^{-1}\boldsymbol{u}_{kl}}{\sum_{k,i\in E}\pi_k\,d_{ki}(1-\boldsymbol{\alpha}_{ki},\boldsymbol{e})}$$

(in obvious notation) which is simpler than that of [6] or earlier references like Machihari [12]. Asmussen and Bladt [5] discuss the related problem of computing the mean regenerative cycle by using a queueing version of Proposition 3.1 in a somewhat different manner.

Also for fluid models, Proposition 3.2 is simpler than the computational schemes suggested in Asmussen [4].

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