On the pluriclosed flow on Oeljeklaus–Toma manifolds

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Abstract. We investigate the pluriclosed flow on Oeljeklaus–Toma manifolds. We parameterize left-invariant pluriclosed metrics on Oeljeklaus–Toma manifolds, and we classify the ones which lift to an algebraic soliton of the pluriclosed flow on the universal covering. We further show that the pluriclosed flow starting from a left-invariant pluriclosed metric has a long-time solution ω_t which once normalized collapses to a torus in the Gromov–Hausdorff sense. Moreover, the lift of $\frac{1}{1+t}\omega_t$ to the universal covering of the manifold converges in the Cheeger–Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$, where $\tilde{\omega}_\infty$ is an algebraic soliton.

1 Introduction

Oeljeklaus–Toma manifolds [300pt]are a very interesting class of complex manifolds introduced and first studied in [17]. These manifolds are defined as compact quotients of the type

$$M = \frac{\mathbb{H}^r \times \mathbb{C}^s}{U \ltimes \mathcal{O}_\mathbb{K}},$$

where $\mathbb{H} \subseteq \mathbb{C}$ is the upper half-plane, $\mathcal{O}_\mathbb{K}$ is the ring of algebraic integers of an algebraic extension $\mathbb{K}$ of $\mathbb{Q}$ satisfing $[\mathbb{K} : \mathbb{Q}] = r + 2s$, and U is a free subgroup of rank r of $\mathcal{O}_\mathbb{K}^{*,+}$ satisfying some compatible conditions. The action of $U \ltimes \mathcal{O}_\mathbb{K}$ on $\mathbb{H}^r \times \mathbb{C}^s$ is defined via some embeddings of $\mathbb{K}$ in $\mathbb{R}$ and $\mathbb{C}$. Oeljeklaus–Toma manifolds have a rich geometric structure. For instance, they have a natural structure of $\mathbb{T}^{r+2s}$-torus bundle over a $\mathbb{T}^r$ and a structure of solvmanifold [13], i.e., they are always compact quotients of a solvable Lie group by a lattice. The Poincaré metric[1] $\omega_{\mathbb{H}^r} = \sqrt{-1}\sum_{a=1}^r \frac{dz_a \wedge d\bar{z}_a}{4(\mathfrak{Im} z_a)^2}$ induces a degenerate metric ω_∞ on M which has a central role in the study of geometric flows on these manifolds. The pair (r, s) is called the *type* of the manifold. The case of type $(r, s) = (1, 1)$ corresponds to the Inoue–Bombieri surfaces [11].

In [2, 7, 28, 32], the Chern–Ricci flow [10, 29] on Oeljeklaus–Toma manifolds M of type $(r, 1)$ is studied. According to the results in [2, 7, 28, 32], under some assumptions on the initial Hermitian metric, the flow has a long-time solution ω_t such that $(M, \frac{\omega_t}{1+t})$ converges in the Gromov–Hausdorff sense to an r-dimensional torus $\mathbb{T}^r$ as $t \to \infty$. The result can be adapted to Oeljeklaus–Toma manifolds of arbitrary type by assuming

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[1]In the whole paper, we identify a Hermitian metric with its fundamental form.

the initial metric to be left-invariant with respect to the structure of solvmanifold. Moreover, a result of Lauret in [15, 16] allows us to give a characterization of left-invariant Hermitian metrics on an Oeljeklaus–Toma manifold which lift to an algebraic soliton of the Chern–Ricci flow on the universal covering of the manifold (see Proposition 4.1).

Following the same approach, we focus on the pluriclosed flow on Oeljeklaus–Toma manifolds when the initial pluriclosed Hermitian metric is left-invariant. The pluriclosed flow is a geometric flow of pluriclosed metrics, i.e. of Hermitian metrics having the fundamental form $\partial\bar{\partial}$-closed, introduced by Streets and Tian in [25]. The flow belongs to the family of the Hermitian curvature flows [26] and evolves an initial pluriclosed metric along the $(1,1)$-component of the Bismut–Ricci form. Namely, on a Hermitian manifold (M, ω), there always exists a unique metric connection ∇^B, called the *Bismut connection* [4], preserving the complex structure and such that

$$\omega(T^B(\cdot, \cdot), J\cdot) \quad \text{is a 3-form,}$$

where T^B is the torsion of ∇^B. The *Bismut–Ricci form* of ω is then defined as

$$\rho_B(X, Y) := \sqrt{-1} \sum_{i=1}^n R_B(X, Y, X_i, \bar{X}_i),$$

where R_B is the curvature tensor of ∇^B and $\{X_i\}$ is a unitary frame with respect to ω. ρ_B is always a closed real form. Given a pluriclosed Hermitian metric ω on M, the *pluriclosed flow* is then defined as the geometric flow of pluriclosed metrics governed by the equation

$$\partial_t \omega_t = -\rho_B^{1,1}(\omega_t), \quad \omega_{|t=0} = \omega.$$

The pluriclosed flow was deeply studied in literature (see, for instance, [3, 5, 6, 9, 12, 19–24, 27] and the references therein).

Our main result is the following theorem.

Theorem 1.1 *Let ω be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus–Toma manifold M. Then the pluriclosed flow starting from ω has a long-time solution ω_t such that $(M, \frac{\omega_t}{1+t})$ converges in the Gromov–Hausdorff sense to $(\mathbb{T}^s, d)$. Moreover, ω lifts to an expanding algebraic soliton on the universal covering of M if and only if it is diagonal and the first s diagonal components coincide. Finally, $(\mathbb{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger–Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$, where $\tilde{\omega}_\infty$ is an algebraic soliton.*

Here, we recall that a left-invariant Hermitian metric ω on a Lie group G with a left-invariant complex structure is an *algebraic soliton* for a geometric flow of left-invariant Hermitian metrics if $\omega_t = c_t \varphi_t^*(\omega)$ solves the flow, where $\{c_t\}$ is a positive scaling and $\{\varphi_t\}$ is a family of automorphims of G preserving the complex structure. Moreover, the distance d in the statement is the distance induced by $3\omega_\infty$ on the torus base of M. Now, we describe the condition *diagonal* appearing in the statement of Theorem 1.1. The existence of a pluriclosed metric on an Oeljeklaus–Toma manifold imposes some restrictions (see [1, Corollary 3]). In particular, the manifold has type (s, s) and

admits a left-invariant $(1,0)$-coframe $\{\omega^1, \ldots, \omega^s, \gamma^1, \ldots, \gamma^s\}$ satisfying

$$\begin{cases} d\omega^k = \frac{\sqrt{-1}}{2}\omega^k \wedge \bar{\omega}^k, & k = 1, \ldots, s, \\ d\gamma^i = \sum_{k=1}^s \lambda_{ki}\, \omega^k \wedge \gamma^i - \sum_{k=1}^s \lambda_{ki}\bar{\omega}^k \wedge \gamma^i, & i = 1, \ldots, s, \end{cases}$$

with

$$\mathfrak{Im}\,\lambda_{ki} = -\frac{1}{4}\,\delta_{ik}.$$

By ω *diagonal*, we mean that it takes a diagonal form with respect to such a coframe. The first part of Theorem 1.1 in the case of the Inoue–Bombieri surfaces is proved in [5, Corollary 3.18].

Theorem 1.1 provides a description of the long-time behavior of the solution ω_t to the pluriclosed flow as $t \to \infty$. For the definition of the convergence in the Gromov–Hausdorff sense, we refer to Section 3, whereas here we briefly recall the definition of convergence in the Cheeger–Gromov sense: a sequence of pointed Riemannian manifolds (M_k, g_k, p_k) *converges in the Cheeger–Gromov sense to a pointed Riemannian manifold* (M, g, p) if there exists a sequence of open subsets A_k of M so that every compact subset of M eventually lies in some A_k, and a sequence of smooth maps $\phi_k \colon A_k \to M_k$ which are diffeomorphisms onto some open set of M_k which satisfy $\phi_k(p_k) = p$, such that

$$\phi_k^*(g_k) \to g \quad \text{smoothly on every compact subset, as } k \to \infty.$$

See [14, Section 6] for a deep analysis of Cheeger–Gromov convergence both in the general case and in the homogeneous one and [15, Section 5.1] for the case of Hermitian Lie groups.

2 Definition of Oeljeklaus–Toma manifolds

We briefly recall the construction of Oeljeklaus–Toma manifolds [17].

Let $\mathbb{Q} \subseteq \mathbb{K}$ be an algebraic number field with $[\mathbb{K} : \mathbb{Q}] = r + 2s$ and $r, s \geq 1$. Let $\sigma_1, \ldots, \sigma_r \colon \mathbb{K} \to \mathbb{R}$ be the real embeddings of $\mathbb{K}$ and $\sigma_{r+1}, \ldots, \sigma_{r+2s} \colon \mathbb{K} \to \mathbb{C}$ be the complex embeddings of $\mathbb{K}$ satisfying $\sigma_{r+s+i} = \bar{\sigma}_{r+i}$, for every $i = 1, \ldots, s$. We denote by $\mathcal{O}_\mathbb{K}$ the ring of algebraic integers of $\mathbb{K}$ and by $\mathcal{O}_\mathbb{K}^*$ the group of units of $\mathcal{O}_\mathbb{K}$. Let

$$\mathcal{O}_\mathbb{K}^{*,+} = \{u \in \mathcal{O}_\mathbb{K}^* \mid \sigma_i(u) > 0, \quad \text{for every } i = 1, \ldots, r\}$$

be the group of totally positive units of $\mathcal{O}_\mathbb{K}$. The groups $\mathcal{O}_\mathbb{K}$ and $\mathcal{O}_\mathbb{K}^{*,+}$ act on $\mathbb{H}^r \times \mathbb{C}^s$ as

$$a \cdot (z_1, \ldots, z_r, w_1, \ldots, w_s)$$
$$= (z_1 + \sigma_1(a), \ldots, z_r + \sigma_r(a), w_1 + \sigma_{r+1}(a), \ldots, w_s + \sigma_{r+s}(a)), \quad \text{for all } a \in \mathcal{O}_\mathbb{K}$$

and

$$u \cdot (z_1, \ldots, z_r, w_1, \ldots, w_s)$$
$$= (\sigma_1(u)z_1, \ldots, \sigma_r(u)z_r, \sigma_{r+1}(u)w_1, \ldots, \sigma_{r+s}(u)w_s), \quad \text{for every } u \in \mathcal{O}_\mathbb{K}^{*,+}.$$

There always exists a free subgroup U of rank r of $\mathcal{O}_{\mathbb{K}}^{*,+}$ such that $\mathrm{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in $\mathbb{R}^r$, where $l: \mathcal{O}_{\mathbb{K}}^{*,+} \to \mathbb{R}^{r+s}$ is the logarithmic representation of units

$$l(u) = (\log \sigma_1(u), \dots, \log \sigma_r(u), 2\log|\sigma_{r+1}(u)|, \dots, 2\log|\sigma_{r+s}(u)|)$$

and $\mathrm{pr}_{\mathbb{R}^r}: \mathbb{R}^{r+s} \to \mathbb{R}^r$ is the projection on the first r coordinates. The action of $U \ltimes \mathcal{O}_{\mathbb{K}}$ on $\mathbb{H}^r \times \mathbb{C}^s$ is free, properly discontinuous, and co-compact. An *Oeljeklaus–Toma manifold* is then defined as the quotient

$$M := \frac{\mathbb{H}^r \times \mathbb{C}^s}{U \ltimes \mathcal{O}_{\mathbb{K}}},$$

and it is a compact complex manifold having complex dimension $r + s$.

The structure of torus bundle of an Oeljeklaus–Toma manifold can be seen as follows: we have

$$\frac{\mathbb{H}^r \times \mathbb{C}^s}{\mathcal{O}_{\mathbb{K}}} = \mathbb{R}_+^r \times \mathbb{T}^{r+2s},$$

and that the action of U on $\mathbb{H}^r \times \mathbb{C}^s$ induces an action on $\mathbb{R}_+^r \times \mathbb{T}^{r+2s}$ such that, for every $x \in \mathbb{R}_+^r$ and $u \in U$, the induced map

$$u: (x, \mathbb{T}^{r+2s}) \mapsto (\sigma_1(u)x_1, \dots, \sigma_r(u)x_r, \mathbb{T}^{r+2s})$$

is a diffeomorphism. Hence,

$$M = \frac{\mathbb{R}_+^r \times \mathbb{T}^{r+2s}}{U}$$

inherits the structure of a $\mathbb{T}^{r+2s}$-bundle over $\mathbb{T}^r$. We denote by π and F the projections

$$\pi: \mathbb{H}^r \times \mathbb{C}^s \to M, \quad F: M \to \mathbb{T}^r.$$

From the viewpoint of Lie groups, the universal covering of an Oeljeklaus–Toma manifold M has a natural structure of solvable Lie group G and the complex structure on M lifts to a left-invariant complex structure [13]. Therefore, Oeljeklaus–Toma manifolds can be seen as compact solvmanifolds with a left-invariant complex structure. The solvable structure on the universal covering of M can be described in terms of the existence of a left-invariant $(1,0)$-coframe $\{\omega^1, \dots, \omega^r, \gamma^1, \dots, \gamma^s\}$ such that

(1)
$$\begin{cases} d\omega^k = \frac{\sqrt{-1}}{2}\omega^k \wedge \bar{\omega}^k, & k = 1, \dots, r, \\ d\gamma^i = \sum_{k=1}^r \lambda_{ki}\omega^k \wedge \gamma^i - \sum_{k=1}^r \lambda_{ki}\bar{\omega}^k \wedge \gamma^i, & i = 1, \dots, s, \end{cases}$$

where

$$\lambda_{ki} = \frac{\sqrt{-1}}{4}b_{ki} - \frac{1}{2}c_{ki}$$

and $b_{ki}, c_{ki} \in \mathbb{R}$ depend on the embeddings σ_j as

(2)
$$\sigma_{r+i}(u) = \left(\prod_{k=1}^r (\sigma_k(u))^{\frac{b_{ki}}{2}}\right) e^{\sqrt{-1}\sum_{k=1}^r c_{ki}\log\sigma_k(u)},$$

for any $u \in U$, $k = 1, \ldots, r$ and $i = 1, \ldots, s$. Since $U \subseteq \mathcal{O}_\mathbb{K}^*$, it is easy to see that

$$l(U) \subseteq \left\{ x \in \mathbb{R}^{r+s} \; \Bigg| \; \sum_{i=1}^{r+s} x_i = 0 \right\}.$$

This fact together with (2) implies that, for every $u \in U$,

$$\sum_{i=1}^{r} \log \sigma_i(u) \left(1 + \sum_{k=1}^{s} b_{ik} \right) = 0 \,,$$

which, since $\mathrm{pr}_{\mathbb{R}^r} \circ l(U)$ is a lattice of rank r in $\mathbb{R}^r$, is equivalent to

$$(3) \qquad\qquad \sum_{k=1}^{s} b_{ik} = -1, \quad \text{for all } i = 1, \ldots, r.$$

The dual frame $\{Z_1, \ldots, Z_r, W_1, \ldots, W_s\}$ to $\{\omega^1, \ldots, \omega^r, \gamma^1, \ldots, \gamma^s\}$ satisfies the following structure equations:

$$[Z_k, \bar{Z}_k] = -\frac{\sqrt{-1}}{2}(Z_k + \bar{Z}_k), \quad [Z_k, W_i] = -\lambda_{ki} W_i, \quad [Z_k, \bar{W}_i] = \bar{\lambda}_{ki} \bar{W}_i,$$

for $k = 1, \ldots, r$ and $i = 1, \ldots, s$. Consequently, the Lie algebra $\mathfrak{g}$ of the universal covering of M splits as vector space as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{I},$$

where $\mathfrak{I}$ is an abelian ideal and $\mathfrak{h}$ is a subalgebra isomorphic to $\underbrace{\mathfrak{f} \oplus \cdots \oplus \mathfrak{f}}_{r\text{-times}}$, where $\mathfrak{f}$ is

the *filiform* Lie algebra $\mathfrak{f} = \langle e_1, e_2 \rangle$, $[e_1, e_2] = -\frac{1}{2}e_1$. The complex structure J induced on $\mathfrak{g}$ preserves both $\mathfrak{h}$ and $\mathfrak{I}$, and its restriction $J_\mathfrak{h}$ on $\mathfrak{h}$ satisfies

$$J_\mathfrak{h} = \underbrace{J_\mathfrak{f} \oplus \cdots \oplus J_\mathfrak{f}}_{r\text{-times}},$$

where $J_\mathfrak{f}$ is the complex structure on $\mathfrak{f}$ defined by $J_\mathfrak{f}(e_1) = e_2$. Moreover,

$$[\mathfrak{h}^{1,0}, \mathfrak{I}^{0,1}] \subseteq \mathfrak{I}^{0,1}.$$

3 Convergence in the Gromov–Hausdorff sense

We briefly recall Gromov–Hausdorff convergence of metric spaces. The *Gromov–Hausdorff distance* between two metric spaces (X, d_X), (Y, d_Y) is the infimum of all positive ε for which there exist two functions $F: X \to Y$, $G: Y \to X$, not necessarily continuous, satisfying the following four properties:

$$|d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| \le \varepsilon, \quad d_X(x, G(F(x))) \le \varepsilon,$$
$$|d_Y(y_1, y_2) - d_X(G(y_1), G(y_2))| \le \varepsilon, \quad d_Y(y, F(G(y))) \le \varepsilon,$$

for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$. If $\{d_t\}_{t \in [0, \infty)}$ is a one-parameter family of distances on X, (X, d_t) *converges to* (Y, d_Y) *in the Gromov–Hausdorff sense* if the Gromov–Hausdorff distance between (X, d_t) and (Y, d) tends to 0 as $t \to \infty$.

Let $\{\omega_t\}_{t\in[0,\infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus–Toma manifold, and let d_t be the induced distance on M. For a smooth curve γ on M, let $L_t(\gamma)$ be the length of γ with respect to ω_t. We further denote by $\mathcal{H}$ the foliation induced by $\mathfrak{h}$ on M.

Proposition 3.1 *Let $\{\omega_t\}_{t\in[0,\infty)}$ be a smooth curve of Hermitian metrics on an Oeljeklaus–Toma manifold such that*

$$\lim_{t\to\infty}\omega_t=\omega_\infty$$

pointwise. Assume that there exist $T\in(0,\infty)$ and $C>0$ such that:

1. *$L_t(\gamma)\le CL_0(\gamma)$, for every smooth curve γ in M.*
2. *$L_t(\gamma)\le(C/\sqrt{t})L_0(\gamma)$, for every smooth curve γ in M such that $\dot{\gamma}\in\ker\omega_\infty$.*

Assume further that:

3. *For every $\varepsilon,\ell>0$, there exists $T>0$ such that $|L_t(\gamma)-L_\infty(\gamma)|<\varepsilon$, for every $t>T$ and every curve γ in M tangent to $\mathcal{H}$ and such that $L_\infty(\gamma)<\ell$.*

Then (M,d_t) converges in the Gromov–Hausdorff sense to $(\mathbb{T}^r,d)$, where d is the distance induced by ω_∞ onto $\mathbb{T}^r$.

Proof   We follow the approach in [28, Section 5] and in [32, Proof of Theorem 1.1]. Let M be an Oeljeklaus–Toma manifold. Consider the structure of M as $\mathbb{T}^{r+2s}$-bundle over a $\mathbb{T}^r$. Let $F\colon M\to\mathbb{T}^r$ be the projection onto the base, and let $G\colon\mathbb{T}^r\to M$ be an arbitrary map such that $F\circ G=\mathrm{Id}_{\mathbb{T}^r}$. We show that, for every $\varepsilon>0$, there exists $T>0$ such that

$$(4)\qquad\qquad\qquad |d_t(p,q)-d(F(p),F(q))|\le\varepsilon,$$

$$(5)\qquad\qquad\qquad |d(a,b)-d_t(G(a),G(b))|\le\varepsilon,$$

$$(6)\qquad\qquad\qquad d_t(p,G(F(p)))\le\varepsilon,$$

$$(7)\qquad\qquad\qquad d(a,F(G(a)))\le\varepsilon,$$

for every $t\ge T$, $p,q\in M$, $a,b\in\mathbb{T}^r$, which implies the statement.

Note that (7) is trivial since

$$d(a,F(G(a)))=0,$$

for every $a\in\mathbb{T}^r$.

Then we show that (6) is satisfied. Let $p,q\in M$ be two points in the same fiber over $\mathbb{T}^r$. Assume that $p=\pi(z,w)$. We denote with $\mathcal{L}_{(z,w)}$ the leaf of the foliation $\ker\omega_\infty$ on the universal covering of M passing through (z,w). We easily see that, for all $(z,w)\in\mathbb{H}^r\times\mathbb{C}^s$, $\mathcal{L}_{(z,w)}=\{z\}\times\mathbb{C}^s$. In view of [30, Section 2], for every $z\in\mathbb{H}^r$, $\pi(\{z\}\times\mathbb{C}^s)$ is the leaf of the foliation $\ker\omega_\infty$ on M passing through p and it is dense in the fiber $F^{-1}(F(p))$. Let B_R be the standard ball in $\mathbb{C}^s$ about the origin having radius R. We can choose R so that every point in $F^{-1}(F(p))$ has distance with respect to d_0 less than $\varepsilon/2C$ to $\pi(\{z\}\times\bar{B}_R)$. On the other hand, given two points in $\pi(\{z\}\times\bar{B}_R)$, they can be joined with a curve γ in $F^{-1}(F(p))$ which is tangent to $\ker\omega_\infty$. Hence, for any such curve, condition 2 implies

$$L_t(\gamma) \le \frac{C'}{\sqrt{t}},$$

for a uniform constant C' depending only on R. Let $p_0 = \pi(z, 0)$, let γ_1 be a curve in $F^{-1}(F(p))$ connecting p with p_0 tangent to $\ker \omega_\infty$, and let γ_2 be a curve connecting p_0 with q having minimal length with respect to d_0. Hence, by using condition 1, for t sufficiently large, we have

$$d_t(p, q) \le L_t(\gamma_1) + L_t(\gamma_2) \le \frac{C'}{\sqrt{t}} + CL_0(\gamma_2) \le \frac{C'}{\sqrt{t}} + \frac{\varepsilon}{2} \le \varepsilon,$$

i.e.,

$$d_t(p, q) \le \varepsilon,$$

and (6) follows.

Next, we show (4) and (5). First of all, we denote with g the Riemannian metric on $\mathbb{T}^r$ induced by ω_∞, for an explicit expression of g (see [32, Section 2]), and we observe that

(8) $$L_g(F(\gamma)) \le L_\infty(\gamma), \text{ for every curve } \gamma \text{ in } M,$$

and the equality holds if and only if

$$\dot\gamma \in \mathcal{Y} = \mathrm{span}_{\mathbb{C}} \left\{ \frac{1}{2\sqrt{-1}} \left(Z_i - \bar{Z}_i \right) \;\middle|\; i = 1, \ldots, r \right\}.$$

Let $p, q \in M$. We can find a curve γ in M connecting p with a point $\tilde{q}$ in the $\mathbb{T}^{r+2s}$-fiber containing q which is tangent to $\mathcal{Y}$ and such that $F(\gamma)$ is a minimal geodesic on $(\mathbb{T}^r, g)$ (see, for instance, [28, Proof of Theorem 5.1] or [32, Proof of Theorem 1.1]). By applying condition 3, we have

$$d_t(p, q) \le d_t(p, \tilde{q}) + d_t(\tilde{q}, q) \le d_t(p, \tilde{q}) + \varepsilon \le L_t(\gamma) + \varepsilon \le L_\infty(\gamma) + 2\varepsilon$$
$$= L_g(F(\gamma)) + 2\varepsilon = d(F(p), F(q)) + 2\varepsilon,$$

for t big enough, i.e.,

(9) $$d_t(p, q) - d(F(p), F(q)) \le 2\varepsilon,$$

for t sufficiently large.

Next, using again (8), we obtain, for $p, q \in M$,

$$d(F(p), F(q)) \le L_g(F(\gamma)) \le L_\infty(\gamma) \le L_t(\gamma) + \varepsilon = d_t(p, q) + \varepsilon,$$

for t big enough, where γ is a curve which realizes the distance $d_t(p, q)$. Hence, we obtain

(10) $$d(F(p), F(q)) - d_t(p, q) \le \varepsilon.$$

By substituting $p = G(a)$ and $q = G(b)$ in (9) and (10), we infer

$$-\varepsilon \le d_t(G(a), G(b)) - d(a, b) \le 2\varepsilon,$$

and (4) and (5) follow. ∎

4 The left-invariant Chern–Ricci flow on Oeljeklaus–Toma manifolds

Given a Hermitian manifold (M, ω), the Chern connection of ω is the unique connection ∇ on (M, ω) preserving both ω and the complex structure such that the $(1, 1)$-component of its torsion tensor is vanishing. The *Chern–Ricci form* of ω is the real closed $(1, 1)$-form

$$\rho_C(X, Y) := \sqrt{-1} \sum_{i=1}^{n} R_C(X, Y, X_i, \bar{X}_i),$$

where R_C is the curvature tensor of ∇ and $\{X_i\}$ is a unitary frame with respect to ω. The *Chern–Ricci flow* is then defined as the geometric flow

$$\partial_t \omega_t = -\rho_C(\omega_t), \quad \omega_{|t=0} = \omega.$$

In this section, we prove the following Proposition.

Proposition 4.1 *Let ω be a left-invariant Hermitian metric on an Oeljeklaus–Toma manifold M. Then ω lifts to an expanding algebraic soliton for the Chern–Ricci flow on the universal covering of M if and only if it takes the following expression with respect to the coframe $\{\omega^1, \ldots, \omega^r, \gamma^1, \ldots, \gamma^s\}$ satisfying (1):*

$$\text{(11)} \qquad \omega = \sqrt{-1} \left(A \sum_{i=1}^{r} \omega^i \wedge \bar{\omega}^i + \sum_{i,j=1}^{s} g_{r+i\overline{r+j}} \gamma^i \wedge \bar{\gamma}^j \right).$$

Moreover, the Chern–Ricci flow starting from ω has a long-time solution $\{\omega_t\}$ such that $(M, \frac{\omega_t}{1+t})$ converges as $t \to \infty$ in the Gromov–Hausdorff sense to $(\mathbb{T}^r, d)$, where d is the distance induced by ω_∞ onto $\mathbb{T}^r$. Finally, $(\mathbb{H}^r \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger–Gromov sense to $(\mathbb{H}^r \times \mathbb{C}^s, \tilde{\omega}_\infty)$, where $\tilde{\omega}_\infty$ is an algebraic soliton.

The proof of Proposition 4.1 is based on the following theorem of Lauret.

Theorem 4.2 (Lauret [15]) *Let (G, J) be a Lie group with a left-invariant complex structure. Then the Chern–Ricci form of a left-invariant Hermitian metric ω on (G, J) does not depend on the Hermitian metric. Moreover, if $P \neq 0$ is the endomorphism associated with ρ_C with respect to ω, then the following are equivalent:*

(1) *ω is an algebraic soliton of the Chern–Ricci flow.*
(2) *$P = cI + D$, for some $D \in \mathrm{Der}(\mathfrak{g})$.*
(3) *The eigenvalues of P are either 0 or c, for some $c \in \mathbb{R}$ with $c \neq 0$, $\ker P$ is an abelian ideal of the Lie algebra of G, and $(\ker P)^\perp$ is a subalgebra.*

Proof of Proposition 4.1 Let M be an Oeljeklaus–Toma manifold. Since the Chern–Ricci form does not depend on the choice of the left-invariant Hermitian metric, it is enough to compute ρ_C for the "canonical metric"

$$\text{(12)} \qquad \omega = \sqrt{-1} \left(\sum_{i=1}^{r} \omega^i \wedge \bar{\omega}^i + \sum_{j=1}^{s} \gamma^j \wedge \bar{\gamma}^j \right).$$

We recall that the Chern–Ricci form of a left-invariant Hermitian metric $\omega = \sqrt{-1}\sum_{a=1}^{n}\alpha^a \wedge \bar{\alpha}^a$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$(13) \qquad \rho_C(X,Y) = -\sum_{a=1}^{n}\left(\omega([[X,Y]^{0,1},X_a],\bar{X}_a) + \omega([[X,Y]^{1,0},\bar{X}_a],X_a)\right),$$

for every left-invariant vector fields X, Y on G, where $\{\alpha^i\}$ is a left-invariant unitary $(1,0)$-coframe with dual frame $\{X_a\}$ (see, e.g., [31]). By applying (13) to the canonical metric (12), we have

$$\rho_C(X,Y) = -\sum_{a=1}^{r}\left\{\omega([[X,Y]^{0,1},Z_a],\bar{Z}_a) + \omega([[X,Y]^{1,0},\bar{Z}_a],Z_a)\right\}$$
$$-\sum_{b=1}^{s}\left\{\omega([[X,Y]^{0,1},W_b],\bar{W}_b) + \omega([[X,Y]^{1,0},\bar{W}_b],W_b)\right\}.$$

Clearly,

$$\rho_C(Z_i,\bar{Z}_j) = 0, \quad \text{for all } i \neq j, \quad \rho_C(W_i,\bar{W}_j) = 0, \quad \text{for every } i, j = 1,\dots,s.$$

Moreover, since $\mathfrak{J}$ is an abelian ideal and ω makes $\mathfrak{J}$ and $\mathfrak{h}$ orthogonal, we have

$$\rho_C(Z_i,\bar{W}_j) = 0, \quad \text{for all } i = 1,\dots,r, \quad j = 1,\dots,s.$$

Moreover, we have

$$\omega([[Z_i,\bar{Z}_i]^{0,1},Z_a],\bar{Z}_a) = \frac{\sqrt{-1}}{4}\delta_{ia}, \quad \omega([[Z_i,\bar{Z}_i]^{1,0},\bar{Z}_a],Z_a) = \frac{\sqrt{-1}}{4}\delta_{ia}$$

and

$$\omega([[Z_i,\bar{Z}_i]^{0,1},W_b],\bar{W}_b) = \frac{1}{2}\lambda_{ib}, \quad \omega([[Z_i,\bar{Z}_i]^{1,0},\bar{W}_b],W_b) = -\frac{1}{2}\bar{\lambda}_{ib},$$

which imply

$$\rho_C(Z_i,\bar{Z}_i) = -\sqrt{-1}\left(\frac{1}{2} + \sum_{b=1}^{s}\mathfrak{Im}(\lambda_{ib})\right) = -\frac{\sqrt{-1}}{4},$$

and, consequently,

$$\rho_C = -\omega_\infty,$$

where ω_∞ is the degenerate metric induced on M by the Poincaré metric on $\mathbb{H}^r$, namely,

$$\omega_\infty = \frac{\sqrt{-1}}{4}\sum_{i=1}^{r}\omega^i \wedge \bar{\omega}^i.$$

In general, we have that

$$P_i^j = (\rho_C)_{i\bar{k}}g^{\bar{k}j} = \begin{cases} -\frac{1}{4}g^{\bar{i}j}, & \text{if } i \in \{1,\dots,r\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then part (3) of Theorem 4.2 readily implies that any left-invariant Hermitian metrics of the form (11) lifts to an expanding algebraic soliton on the universal covering of M

with cosmological constant $c = \frac{1}{4A}$. Conversely, let ω be an algebraic soliton for the Chern–Ricci flow. Then, thanks to part (2) of Theorem 4.2, we have that

$$P - cI \in \mathrm{Der}(\mathfrak{g}).$$

On the other hand, we can easily see that, if $D \in \mathrm{Der}(\mathfrak{g})$, then $\mathfrak{h} \subseteq \ker D$ (see the proof of Corollary 5.4 for the details). This readily implies that

$$-\frac{1}{4}g^{i\bar{i}} = -\frac{1}{4}g^{\bar{j}j} = c, \quad \text{for all } i, j = 1, \dots, r, \quad g^{\bar{i}j} = 0, \quad \text{for all } i \in \{1, \dots, r\},\ j \neq i,$$

from which the claim follows.

Moreover, the Chern–Ricci flow evolves an arbitrary left-invariant Hermitian metric ω as $\omega_t = \omega + t\omega_\infty$ and $\frac{\omega_t}{1+t} \to \omega_\infty$ as $t \to \infty$. In order to obtain the claim regarding the Gromov–Hausdorff convergence, we show that $\frac{\omega_t}{1+t}$ satisfies conditions 1–3 in Proposition 3.1. Here, we denote by $|\cdot|_t$ the norm induced by ω_t.

Condition 2 is trivially satisfied since $\omega_{t|\mathfrak{I}\oplus\mathfrak{I}} = \omega_0$, for every $t \geq 0$, and

$$L_t(\gamma) = \frac{1}{\sqrt{1+t}}L_0(\gamma),$$

for every curve γ in M tangent to $\ker \omega_\infty$.

On the other hand, for a vector $v \in \mathfrak{h}$, we have

$$\frac{1}{\sqrt{1+t}}|v|_t \leq C|v|_0,$$

for a constant $C > 0$ independent on v. This, together with condition 2, guarantees condition 1.

In order to prove condition 3, let $\varepsilon, \ell > 0$ and $T > 0$ be such that

$$\left|\frac{|v|_t}{\sqrt{1+t}} - |v|_\infty\right| \leq \frac{\varepsilon}{\ell},$$

for every $v \in \mathfrak{h}$ and $t \geq T$. Let γ be a curve in M tangent to $\mathcal{H}$ which is parametrized by arclength with respect to ω_∞ and such that $L_\infty(\gamma) < \ell$. Then

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left|\frac{1}{\sqrt{1+t}}|\dot{\gamma}|_t - |\dot{\gamma}|_\infty\right| da \leq \frac{\varepsilon}{\ell}b \leq \varepsilon,$$

since $b \leq \ell$.

For the last statement, we identify ω_t with its pullback onto $\mathbb{H}^r \times \mathbb{C}^s$ and we fix as base point the identity element of $\mathbb{H}^r \times \mathbb{C}^s$. First, we observe that the endomorphism D represented with respect to the frame $\{Z_1, \dots, Z_r, W_1, \dots, W_s\}$ by the following matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\mathfrak{J}} \end{pmatrix}$$

is a derivation of $\mathfrak{g}$. Moreover, we can construct

$$\exp(s(t)D) = \begin{pmatrix} \mathrm{I}_{\mathfrak{h}} & 0 \\ 0 & e^{s(t)}\mathrm{I}_{\mathfrak{J}} \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}, J), \quad \text{for every } t \geq 0,$$

where $s(t) = \log(\sqrt{1+t})$ and define the one-parameter family $\{\varphi_t\} \subseteq \operatorname{Aut}(\mathbb{H}^r \times \mathbb{C}^s, J)$ such that

$$d\varphi_t = \exp(s(t)D), \quad \text{for every } t \geq 0.$$

Trivially, we see that

$$\varphi_t^* \frac{\omega_t}{1+t}(Z_i, \bar{Z}_j) = \sqrt{-1}\frac{1}{1+t}\left(g_{ij} + \frac{t}{4}\delta_{ij}\right) \to \frac{\sqrt{-1}}{4}\delta_{ij} \quad \text{as } t \to \infty,$$

$$\varphi_t^* \frac{\omega_t}{1+t}(Z_i, \bar{W}_j) = \sqrt{-1}\frac{e^{s(t)}}{1+t}g_{i\overline{r+j}} \to 0 \quad \text{as } t \to \infty,$$

$$\varphi_t^* \frac{\omega_t}{1+t}(W_i, \bar{W}_j) = \sqrt{-1}\frac{e^{2s(t)}}{1+t}g_{r+i\overline{r+j}} \to \sqrt{-1}g_{r+i\overline{r+j}} \quad \text{as } t \to \infty.$$

These facts guarantee that

$$\varphi_t^* \frac{\omega_t}{1+t} \to \omega_\infty + \omega_{\mathfrak{J}\oplus\mathfrak{J}} \quad \text{as } t \to \infty;$$

hence, the assertion follows. ∎

5 Proof of the main result

In this section, we prove Theorem 1.1.

The existence of pluriclosed metrics on Oeljeklaus–Toma manifolds was studied in [1, 8, 18]. In particular, from [1] it follows the following result.

Theorem 5.1 ([1, Corollary 3]) *An Oeljeklaus–Toma manifold of type (r, s) admits a pluriclosed metric if and only if $r = s$ and*

$$(14) \qquad \sigma_j(u)|\sigma_{r+j}(u)|^2 = 1, \quad \text{for every } j = 1, \dots, s \text{ and } u \in U.$$

Condition (14) in the previous theorem can be rewritten in terms of the structure constants appearing in (1). Indeed, (1) together with (14) forces $b_{ki} \in \{0, -1\}$ and $b_{ki}b_{li} = 0$, for every $i, k, l = 1, \dots, s$ with $k \neq l$. In particular, using (3), for every fixed index $k \in \{1, \dots, s\}$, there exists a unique $i_k \in \{1, \dots, s\}$ such that

$$b_{ki_k} = -1, \quad b_{ki} = 0,$$

for all $i \neq i_k$ and, if $k \neq l$, then $i_k \neq i_l$. Hence, up to a reorder of the γ_j's, we may and do assume, without loss of generality, $i_k = k$, for every $k \in \{1, \dots, s\}$, i.e.

$$(15) \qquad \lambda_{ki} = \begin{cases} -\frac{1}{2}c_{ki}, & \text{if } i \neq k, \\ -\frac{1}{2}c_{kk} - \frac{\sqrt{-1}}{4}, & \text{if } i = k. \end{cases}$$

Proposition 5.2 (Characterization of left-invariant pluriclosed metrics on Oeljeklaus—Toma manifolds). *A left-invariant metric ω on an Oeljeklaus–Toma manifold admitting pluriclosed metrics is pluriclosed if and only if it takes the following expression with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15):*

$$(16) \quad \omega = \sqrt{-1}\sum_{i=1}^{s} A_i\omega^i \wedge \bar{\omega}^i + B_i\gamma^i \wedge \bar{\gamma}^i + \sqrt{-1}\sum_{r=1}^{k}\left(C_r\omega^{p_r} \wedge \bar{\gamma}^{p_r} + \bar{C}_r\gamma^{p_r} \wedge \bar{\omega}^{p_r}\right)$$

for some $A_1, \ldots, A_s, B_1, \ldots, B_s \in \mathbb{R}_+, C_1, \ldots, C_k \in \mathbb{C}$, where $\{p_1, \ldots, p_k\} \subseteq \{1, \ldots, s\}$ are such that

$$\lambda_{j p_i} = 0, \text{ for all } j \neq p_i, \text{ for all } i = 1, \ldots, k.$$

Proof We assume $s > 1$ since the case $s = 1$ is trivial. Let

$$\omega = \sqrt{-1} \sum_{p,q=1}^{s} A_{p\bar{q}} \omega^p \wedge \bar{\omega}^q + B_{p\bar{q}} \gamma^p \wedge \bar{\gamma}^q + C_{p\bar{q}} \omega^p \wedge \bar{\gamma}^q + \bar{C}_{p\bar{q}} \gamma^q \wedge \bar{\omega}^p$$

be an arbitrary real left-invariant $(1,1)$-form on M, with $A_{p\bar{p}}, B_{p\bar{p}} \in \mathbb{R}$, for every $p = 1, \ldots, s$, $A_{p\bar{q}}, B_{p\bar{q}} \in \mathbb{C}$, for all $p, q = 1, \ldots, s$ with $p \neq q$, and $C_{p\bar{q}} \in \mathbb{C}$, for every $p, q = 1, \ldots, s$.

From the structure equations (1), it easily follows

(17)
$$\begin{cases} \partial\bar{\partial}(\omega^p \wedge \bar{\omega}^q) \in \langle \omega^p \wedge \omega^q \wedge \bar{\omega}^p \wedge \bar{\omega}^q \rangle, \\ \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) \in \langle \omega^i \wedge \omega^j \wedge \bar{\omega}^l \wedge \bar{\gamma}^m \rangle, \\ \partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) \in \langle \omega^i \wedge \bar{\omega}^j \wedge \gamma^l \wedge \bar{\gamma}^m \rangle, \end{cases}$$

and that ω is pluriclosed if and only if the following three conditions are satisfied:

(18)
$$\sum_{p,q=1}^{s} A_{p\bar{q}} \partial\bar{\partial}(\omega^p \wedge \bar{\omega}^q) = 0,$$

(19)
$$\sum_{p,q=1}^{s} B_{p\bar{q}} \partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = 0,$$

(20)
$$\sum_{p,q=1}^{s} C_{p\bar{q}} \partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = 0.$$

The first relation in (17) yields that (18) is satisfied if and only if

$$A_{p\bar{q}} = 0, \text{ for all } p \neq q.$$

Next, we focus on (19). We have

$$\partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = \partial\left(-\sum_{\delta=1}^{s} \lambda_{\delta p} \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q - \gamma^p \wedge \sum_{\delta=1}^{s} \bar{\lambda}_{\delta q} \bar{\omega}^\delta \wedge \bar{\gamma}^q \right)$$

and

$$\partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = \sum_{\delta=1}^{s} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \left(\partial\bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q - \bar{\omega}^\delta \wedge \partial\gamma^p \wedge \bar{\gamma}^q + \bar{\omega}^\delta \wedge \gamma^p \wedge \partial\bar{\gamma}^q \right),$$

which implies that

$$\partial\bar{\partial}(\gamma^p \wedge \bar{\gamma}^q) = \sum_{\delta=1}^{s} \frac{\sqrt{-1}}{2} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^\delta \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q - \sum_{\delta=1}^{s} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \bar{\omega}^\delta \wedge \left(\sum_{a=1}^{s} \lambda_{ap} \omega^a \wedge \gamma^p \right) \wedge \bar{\gamma}^q$$

$$+ \sum_{\delta=1}^{s} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \bar{\omega}^\delta \wedge \gamma^p \wedge \left(-\sum_{a=1}^{s} \bar{\lambda}_{aq} \omega^a \wedge \bar{\gamma}^q \right)$$

$$= \sum_{\delta=1}^{s} \frac{\sqrt{-1}}{2} (\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^\delta \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q + \sum_{\delta,a} (\lambda_{ap} - \bar{\lambda}_{aq})(\bar{\lambda}_{\delta q} - \lambda_{\delta p}) \omega^a \wedge \bar{\omega}^\delta \wedge \gamma^p \wedge \bar{\gamma}^q.$$

Finally, we get

$$\partial\bar\partial(\gamma^p \wedge \bar\gamma^q) = \sum_{\delta=1}^{s}(\bar\lambda_{\delta q} - \lambda_{\delta p})\left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar\lambda_{\delta q}\right)\omega^\delta \wedge \bar\omega^\delta \wedge \gamma^p \wedge \bar\gamma^q$$
$$+ \sum_{\delta \neq a}(\lambda_{ap} - \bar\lambda_{aq})(\bar\lambda_{\delta q} - \lambda_{\delta p})\omega^a \wedge \bar\omega^\delta \wedge \gamma^p \wedge \bar\gamma^q$$

and that condition (19) is equivalent to

$$B_{p\bar q}\left(\sum_{\delta=1}^{s}(\bar\lambda_{\delta q} - \lambda_{\delta p})\left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar\lambda_{\delta q}\right)\omega^\delta \wedge \bar\omega^\delta + \sum_{\delta \neq a}(\lambda_{ap} - \bar\lambda_{aq})(\bar\lambda_{\delta q} - \lambda_{\delta p})\omega^a \wedge \bar\omega^\delta\right) = 0,$$

for every $p, q = 1, \ldots, s$.

By using our conditions on the b_{ki}'s, it is easy to show that the quantity

$$\sum_{\delta=1}^{s}(\bar\lambda_{\delta q} - \lambda_{\delta p})\left(\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar\lambda_{\delta q}\right)\omega^\delta \wedge \bar\omega^\delta + \sum_{\delta \neq a}(\lambda_{ap} - \bar\lambda_{aq})(\bar\lambda_{\delta q} - \lambda_{\delta p})\omega^a \wedge \bar\omega^\delta$$

is vanishing for $p = q$ and, consequently, there are no restrictions on the $B_{q\bar q}$'s. Now, we observe that the real part of

$$(\bar\lambda_{pq} - \lambda_{pp})\left(\frac{\sqrt{-1}}{2} + \lambda_{pp} - \bar\lambda_{pq}\right)$$

is different from 0, for every p, q with $p \neq q$, which forces $B_{p\bar q} = 0$, for $p \neq q$. Indeed, we have

$$\bar\lambda_{\delta q} - \lambda_{\delta p} = \frac{1}{2}(c_{\delta p} - c_{\delta q}) - \frac{\sqrt{-1}}{4}(b_{\delta p} + b_{\delta q}),$$
$$\frac{\sqrt{-1}}{2} + \lambda_{\delta p} - \bar\lambda_{\delta q} = -\frac{1}{2}(c_{\delta p} - c_{\delta q}) + \frac{\sqrt{-1}}{2}\left(1 + \frac{b_{\delta p} + b_{\delta q}}{2}\right),$$

which implies that

(21)
$$\mathfrak{Re}\left((\bar\lambda_{\delta q} - \lambda_{\delta p})\left(\frac{\sqrt{-1}}{2} + \lambda_{\delta q} - \bar\lambda_{\delta p}\right)\right)$$
$$= -\frac{(c_{\delta p} - c_{\delta q})^2}{4} + \frac{1}{4}\left(\frac{b_{\delta p} + b_{\delta q}}{2}\right)\left(1 + \frac{b_{\delta p} + b_{\delta q}}{2}\right).$$

Since $p \neq q$, we have

$$b_{pp} = -1, \quad b_{pq} = 0,$$

and so (21) computed for $\delta = q$ gives

$$\mathfrak{Re}\left((\bar\lambda_{pq} - \lambda_{pp})\left(\frac{\sqrt{-1}}{2} + \lambda_{pq} - \bar\lambda_{pp}\right)\right) = \frac{1}{4}\left(-(c_{pp} - c_{pq})^2 - \frac{1}{4}\right) \neq 0,$$

as required. Therefore, equation (19) is satisfied if and only if

$$B_{p\bar q} = 0, \text{ for all } p \neq q.$$

Next, we focus on (20). We have

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \partial\left(\frac{\sqrt{-1}}{2}\omega^p \wedge \bar{\omega}^p \wedge \bar{\gamma}^q - \omega^p \wedge \left(\sum_{\delta=1}^{s} \bar{\lambda}_{\delta q}\bar{\omega}^\delta \wedge \bar{\gamma}^q\right)\right)$$

and

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \frac{\sqrt{-1}}{2}\left(-\frac{\sqrt{-1}}{2}\omega^p \wedge \omega^p \wedge \bar{\omega}^p \wedge \bar{\gamma}^q + \omega^p \wedge \bar{\omega}^p \wedge \left(-\sum_{\delta=1}^{s}\bar{\lambda}_{\delta q}\omega^\delta \wedge \bar{\gamma}^q\right)\right)$$
$$+ \sum_{\delta=1}^{s}\frac{\sqrt{-1}}{2}\bar{\lambda}_{\delta q}\omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\delta=1}^{s}\bar{\lambda}_{\delta q}\omega^p \wedge \bar{\omega}^\delta \wedge \left(\sum_{a=1}^{s}\bar{\lambda}_{aq}\omega^a \wedge \bar{\gamma}^q\right).$$

Hence, we get

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\frac{\sqrt{-1}}{2}\bar{\lambda}_{\delta q}\omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\frac{\sqrt{-1}}{2}\bar{\lambda}_{\delta q}\omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q$$
$$+ \sum_{\substack{\delta, a \\ a\neq p}}\bar{\lambda}_{\delta q}\bar{\lambda}_{aq}\omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q$$

and

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\frac{\sqrt{-1}}{2}\bar{\lambda}_{\delta q}\omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q + \sum_{\substack{a=1 \\ a\neq p}}^{s}\bar{\lambda}_{pq}\bar{\lambda}_{aq}\omega^p \wedge \bar{\omega}^p \wedge \omega^a \wedge \bar{\gamma}^q$$
$$+ \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\frac{\sqrt{-1}}{2}\bar{\lambda}_{\delta q}\omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta, a \\ \delta\neq p \\ a\neq p}}\bar{\lambda}_{\delta q}\bar{\lambda}_{aq}\omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q.$$

Therefore,

$$\partial\bar{\partial}(\omega^p \wedge \bar{\gamma}^q) = \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\bar{\lambda}_{\delta q}\left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{pq}\right)\omega^p \wedge \bar{\omega}^p \wedge \omega^\delta \wedge \bar{\gamma}^q$$
$$+ \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\bar{\lambda}_{\delta q}\left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{\delta q}\right)\omega^p \wedge \omega^\delta \wedge \bar{\omega}^\delta \wedge \bar{\gamma}^q + \sum_{\substack{\delta\neq a \\ \delta\neq p \\ a\neq p}}\bar{\lambda}_{\delta q}\bar{\lambda}_{aq}\omega^p \wedge \bar{\omega}^\delta \wedge \omega^a \wedge \bar{\gamma}^q$$

and (20) is equivalent to

$$C_{p\bar{q}}\left(\sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\bar{\lambda}_{\delta q}\left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{pq}\right)\bar{\omega}^p \wedge \omega^\delta + \sum_{\substack{\delta=1 \\ \delta\neq p}}^{s}\bar{\lambda}_{\delta q}\left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{\delta q}\right)\omega^\delta \wedge \bar{\omega}^\delta + \sum_{\substack{\delta\neq a \\ \delta\neq p \\ a\neq p}}\bar{\lambda}_{\delta q}\bar{\lambda}_{aq}\bar{\omega}^\delta \wedge \omega^a\right) = 0,$$

for every $p, q = 1, \ldots, s$. Since

$$\lambda_{pq} \neq \pm\frac{\sqrt{-1}}{2}, \quad \text{for all } p, q = 1, \ldots, s,$$

the quantity

$$E_{p\bar{q}} := \sum_{\substack{\delta=1 \\ \delta \neq p}}^{s} \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{pq} \right) \bar{\omega}^p \wedge \omega^\delta + \sum_{\substack{\delta=1 \\ \delta \neq p}}^{s} \bar{\lambda}_{\delta q} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{\delta q} \right) \omega^\delta \wedge \bar{\omega}^\delta + \sum_{\substack{\delta \neq a \\ \delta \neq p \\ a \neq p}} \bar{\lambda}_{\delta q} \bar{\lambda}_{aq} \bar{\omega}^\delta \wedge \omega^a$$

is vanishing if and only if

$$\lambda_{\delta q} = 0, \quad \text{for all } \delta \neq p.$$

Since $\lambda_{qq} \neq 0$, it follows

$$E_{p\bar{q}} \neq 0, \quad \text{for every } p, q \text{ with } p \neq q$$

and

$$E_{p\bar{p}} = 0 \text{ if and only if } c_{\delta p} = 0, \text{for all } \delta \neq p.$$

Hence, the claim follows. ∎

Proposition 5.3 *Let*

$$(22) \qquad \omega = \sqrt{-1} \sum_{i=1}^{s} A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i + \sqrt{-1} \sum_{r=1}^{k} \left(C_r \omega^{p_r} \wedge \bar{\gamma}^{p_r} + \bar{C}_r \gamma^{p_r} \wedge \bar{\omega}^{p_r} \right)$$

be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus–Toma manifold, where the components are with respect to a coframe $\{\omega^1, \ldots, \omega^s, \gamma^1, \ldots, \gamma^s\}$ satisfying (1) and (15) and $\{p_1, \ldots, p_k\} \subseteq \{1, \ldots, s\}$ are such that

$$\lambda_{jp_i} = 0, \text{ for all } j \neq p_i, \text{ for all } i = 1, \ldots, k.$$

Then the $(1,1)$-part of the Bismut–Ricci form of ω takes the following expression:

$$\rho_B^{1,1} = -\sqrt{-1} \sum_{r=1}^{k} \frac{3}{4} \left(1 + \frac{|C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2} \right) \omega^{p_r} \wedge \bar{\omega}^{p_r} - \sqrt{-1} \sum_{i \notin \{p_1, \ldots, p_k\}} \frac{3}{4} \omega^i \wedge \bar{\omega}^i$$

$$-\sqrt{-1} \sum_{r=1}^{k} \left(-\frac{3}{16} - \frac{c_{p_r p_r}^2}{4} - \frac{\sqrt{-1} c_{p_r p_r}}{4} \right) \frac{B_{p_r} C_r}{A_{p_r} B_{p_r} - |C_r|^2} \omega^{p_r} \wedge \bar{\gamma}^{p_r} + \quad conjugates.$$

Proof We recall that the Bismut–Ricci form of a left-invariant Hermitian metric $\omega = \sqrt{-1} \sum_{a,b=1}^{n} g_{a\bar{b}} \alpha^a \wedge \bar{\alpha}^b$ on a Lie group G^{2n} with a left-invariant complex structure takes the following algebraic expression:

$$(23) \qquad \rho_B(X, Y) = - \sum_{a,b=1}^{n} g^{a\bar{b}} \omega([[X, Y]^{1,0}, X_a], \bar{X}_b) + g^{\bar{a}b} \omega([[X, Y]^{0,1}, \bar{X}_a], X_b)$$

$$+ \sqrt{-1} \sum_{a,b=1}^{n} g^{a\bar{b}} \omega([X, Y], J[X_a, \bar{X}_b]),$$

for every left-invariant vector fields X, Y on G, where $\{\alpha^i\}$ is a left-invariant $(1,0)$-coframe with dual frame $\{X_a\}$ and $(g^{\bar{b}a})$ is the inverse matrix of $(g_{i\bar{j}})$ (see, e.g., [31]). We apply (23) to a left-invariant Hermitian metric on an Oeljeklaus–Toma manifold of the form (22).

We have

$$
g^{\bar{i}s+i} = \begin{cases} 0, & \text{if } i \notin \{p_1,\dots,p_k\}\,, \\ -\dfrac{C_i}{A_iB_i-|C_i|^2}, & \text{otherwise}\,, \end{cases} \qquad g^{\bar{i}i} = \dfrac{B_i}{A_iB_i-|C_i|^2}\,, \qquad g^{\overline{s+i}s+i} = \dfrac{A_i}{A_iB_i-|C_i|^2}\,,
$$

and taking into account that the ideal $\mathfrak{I}$ is abelian, we have

$$
\rho_B(X,Y) = -\sum_{i=1}^{4}\rho_i(X,Y),
$$

where

$$
\begin{aligned}
\rho_1(X,Y) &= \sum_{a=1}^{s} g^{a\bar{a}}\big(\omega([[X,Y]^{1,0},Z_a],\bar{Z}_a) - \frac{\sqrt{-1}}{2}\omega([X,Y],Z_a-\bar{Z}_a) \\
&\quad + \omega([[X,Y]^{0,1},\bar{Z}_a],Z_a)\big), \\
\rho_2(X,Y) &= \sum_{a=1}^{s} g^{s+a\overline{s+a}}\big(\omega([[X,Y]^{1,0},W_a],\bar{W}_a) + \omega([[X,Y]^{0,1},\bar{W}_a],W_a)\big), \\
\rho_3(X,Y) &= \sum_{r=1}^{k} g^{p_r\overline{s+p_r}}\big(\omega([[X,Y]^{1,0},Z_{p_r}],\bar{W}_{p_r}) - \omega([X,Y],[Z_{p_r},\bar{W}_{p_r}])\big) \\
&\quad + g^{\overline{p_r}s+p_r}\omega([[X,Y]^{0,1},\bar{Z}_{p_r}],W_{p_r})\,, \\
\rho_4(X,Y) &= \sum_{r=1}^{k} g^{s+p_r\bar{p}_r}\big(\omega([[X,Y]^{1,0},W_{p_r}],\bar{Z}_{p_r}) + \omega([X,Y],[W_{p_r},\bar{Z}_{p_r}])\big)\big) \\
&\quad + g^{\overline{s+p_r}p_r}\omega([[X,Y]^{0,1},\bar{W}_{p_r}],Z_{p_r})\,.
\end{aligned}
$$

Next, we focus on the computation of $\rho_B(Z_i,\bar{Z}_j)$. Thanks to (1), we easily obtain that

$$
\rho_B(Z_i,\bar{Z}_j) = 0\,, \quad \text{for every } i,j=1,\dots,s,\ i\neq j.
$$

On the other hand,

$$
\begin{aligned}
\rho_1(Z_i,\bar{Z}_i) &= -\frac{\sqrt{-1}}{2}\sum_{a=1}^{s} g^{a\bar{a}}\left(-\frac{\sqrt{-1}}{2}\omega(Z_i+\bar{Z}_i,Z_a-\bar{Z}_a)\right) \\
&= \frac{\sqrt{-1}}{2}g^{i\bar{i}}A_i = \frac{\sqrt{-1}}{2}\left(\frac{A_iB_i}{A_iB_i-|C_i|^2}\right).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\rho_2(Z_i,\bar{Z}_i) &= -\frac{\sqrt{-1}}{2}\sum_{a=1}^{s} g^{s+a\overline{s+a}}\big(\omega([Z_i,W_a],\bar{W}_a) + \omega([\bar{Z}_i,\bar{W}_a],W_a) \\
&= -\sqrt{-1}\sum_{a=1}^{s} g^{s+a\overline{s+a}}\mathfrak{Re}\,\omega([Z_i,W_a],\bar{W}_a).
\end{aligned}
$$

Using (1), we have

$$
\omega([Z_i,W_a],\bar{W}_a) = -\sqrt{-1}\lambda_{ia}B_a\,,
$$

$$
\mathfrak{Re}\,\omega([Z_i,W_a],\bar{W}_a) = \frac{B_a b_{ia}}{4} = -\frac{B_a}{4}\delta_{ia}.
$$

Then

$$\rho_2(Z_i, \bar{Z}_i) = \sqrt{-1} \frac{g^{s+\overline{is+i}} B_i}{4} = \frac{\sqrt{-1}}{4} \frac{A_i B_i}{A_i B_i - |C_i|^2}.$$

Next, we observe that

$$\rho_3(Z_i, \bar{Z}_i) + \rho_4(Z_i, \bar{Z}_i) = 0,$$

which implies that

(24)
$$\rho_B(Z_i, \bar{Z}_i) = \begin{cases} -\sqrt{-1}\frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r} B_{p_r} - |C_r|^2}\right), & \text{if there exists } r = 1, \dots, k \text{ such that } i = p_r, \\ -\sqrt{-1}\frac{3}{4}, & \text{if } i \notin \{p_1, \dots, p_k\}. \end{cases}$$

We have

$$\rho_3(Z_i, \bar{Z}_i) = \sum_{j=1}^{k} g^{p_j \overline{s+p_j}} \omega([Z_i, \bar{Z}_i], [Z_{p_j}, \bar{W}_{p_j}]) = -\frac{\sqrt{-1}}{2} \sum_{j=1}^{k} g^{p_j \overline{s+p_j}} \bar{\lambda}_{p_j p_j} \omega(Z_i + \bar{Z}_i, \bar{W}_{p_j})$$

$$= \begin{cases} 0, & \text{if } i \notin \{p_1, \dots, p_k\}, \\ \frac{1}{2} g^{\overline{is+i}} \bar{\lambda}_{ii} C_i, & \text{otherwise.} \end{cases}$$

We compute the three addends in the expression of ρ_4 separately:

$$\omega([[Z_i, \bar{Z}_i]^{1,0}, W_{p_j}], \bar{Z}_{p_j}) = -\frac{1}{2}\lambda_{ip_j}\bar{C}_{p_j} = \begin{cases} 0, & \text{if } i \notin \{p_1, \dots, p_k\} \text{ or } i \neq p_j, \\ -\frac{1}{2}\lambda_{ii}\bar{C}_i, & \text{otherwise,} \end{cases}$$

$$\omega([Z_i, \bar{Z}_i], [W_{p_j}, \bar{Z}_{p_j}]) = \frac{1}{2}\lambda_{p_j p_j} g_{\overline{is+p_j}} = \begin{cases} 0, & \text{if } i \notin \{p_1, \dots, p_k\} \text{ or } i \neq p_j, \\ \frac{1}{2}\lambda_{ii}\bar{C}_i, & \text{otherwise,} \end{cases}$$

$$\omega([[Z_i, \bar{Z}_i]^{0,1}, \bar{W}_{p_j}], Z_{p_j}) = \frac{1}{2}\bar{\lambda}_{ip_j} g_{\overline{s+p_j}p_j} = \begin{cases} 0, & \text{if } i \neq p_j, \\ \frac{1}{2}\bar{\lambda}_{ii}C_i, & \text{otherwise.} \end{cases}$$

It follows

$$\rho_3(Z_i, \bar{Z}_i) = \rho_4(Z_i, \bar{Z}_i) = 0 \quad \text{if } i \notin \{p_1, \dots, p_k\},$$

and, for $i \in \{p_1, \dots, p_k\}$,

$$\rho_3(Z_i, \bar{Z}_i) + \rho_4(Z_i, \bar{Z}_i)$$
$$= -\frac{1}{2} g^{\overline{is+i}} \bar{\lambda}_{ii} C_i - g^{s+i\bar{i}} \frac{1}{2}\lambda_{ii}\bar{C}_i + g^{s+i\bar{i}} \frac{1}{2}\lambda_{ii}\bar{C}_i + g^{\overline{s+ii}} \frac{1}{2}\bar{\lambda}_{ii} C_i = 0.$$

Now, we focus on the calculation of $\rho_B(Z_i, \bar{W}_j)$. We have

$$\rho_1(Z_i, \bar{W}_j) = \sum_{a=1}^{s} g^{a\bar{a}} \bar{\lambda}_{ij} \left(-\frac{\sqrt{-1}}{2}\omega(\bar{W}_j, Z_a - \bar{Z}_a) + \omega([\bar{W}_j, \bar{Z}_a], Z_a)\right)$$

$$= \begin{cases} 0, & \text{if } i = j \in \{p_1, \dots, p_k\}, \\ \sqrt{-1} g^{i\bar{i}} C_i \bar{\lambda}_{ii}\left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{ii}\right), & \text{otherwise,} \end{cases}$$

and since $\mathfrak{J}$ is abelian

$$\rho_2(Z_i, \bar{W}_j) = 0.$$

Furthermore,

$$\rho_3(Z_i, \bar{W}_j) = \sum_{j=1}^{k} g^{\overline{p_j}s+p_j} \omega([[Z_i, \bar{W}_j]^{0,1}, \bar{Z}_{p_j}], W_{p_j}) = -\sqrt{-1} \sum_{j=1}^{k} g^{\overline{p_j}s+p_j} \bar{\lambda}_{ij} \bar{\lambda}_{p_j p_j} g_{\overline{s+js+p_j}}$$

$$= \begin{cases} 0, & \text{if } i = j \in \{p_1, \ldots, p_k\}, \\ -\sqrt{-1} \bar{\lambda}_{jj}^2 g^{\overline{js+j}} B_j, & \text{otherwise}, \end{cases}$$

and

$$\rho_4(Z_i, \bar{W}_j) = \sum_{j=1}^{k} g^{s+p_j \bar{p}_j} \omega([Z_i, \bar{W}_j], [W_{p_j}, \bar{Z}_{p_j}]) = \sqrt{-1} \sum_{j=1}^{k} g^{s+p_j \bar{p}_j} \bar{\lambda}_{ij} \lambda_{p_j p_j} g_{\overline{s+js+p_j}}$$

$$= \begin{cases} 0, & \text{if } i = j \in \{p_1, \ldots, p_k\}, \\ \sqrt{-1} g^{s+j\bar{j}} \bar{\lambda}_{jj} \lambda_{jj} B_j, & \text{otherwise}. \end{cases}$$

It follows that $\rho_B(Z_i, \bar{W}_j) \neq 0$ if and only if $i = j \in \{p_1, \ldots, p_k\}$. In such a case, we have

$$\rho_B(Z_j, \bar{W}_j) = -\sqrt{-1} \left(g^{s+j\bar{j}} B_j \left(|\lambda_{jj}|^2 - \bar{\lambda}_{jj}^2 \right) + g^{j\bar{j}} C_j \bar{\lambda}_{jj} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{jj} \right) \right).$$

Since

$$g^{s+j\bar{j}} B_j = -\frac{B_j C_j}{A_j B_j - |C_j|^2} \quad \text{and} \quad g^{j\bar{j}} C_j = \frac{B_j C_j}{A_j B_j - |C_j|^2},$$

we infer

$$\rho_B(Z_j, \bar{W}_j) = -\sqrt{-1} \left(\bar{\lambda}_{jj} \left(\frac{\sqrt{-1}}{2} - \bar{\lambda}_{jj} \right) - \left(|\lambda_{jj}|^2 - \bar{\lambda}_{jj}^2 \right) \right) \frac{B_j C_j}{A_j B_j - |C_j|^2}.$$

Taking into account that $\lambda_{jj} = -\frac{\sqrt{-1}}{4} - \frac{c_{jj}}{2}$, we obtain

$$\rho_B(Z_j, \bar{W}_j) = -\sqrt{-1} \left(-\frac{3}{16} - \frac{c_{jj}^2}{4} - \frac{\sqrt{-1} c_{jj}}{4} \right) \frac{B_j C_j}{A_j B_j - |C_j|^2},$$

and the claim follows. ∎

Corollary 5.4 *Let ω be a left-invariant pluriclosed Hermitian metric on an Oeljeklaus–Toma manifold M. Then ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M if and only if it takes the following diagonal expression with respect to a coframe $\{\omega^1, \ldots, \omega^s, \gamma^1, \ldots, \gamma^s\}$ satisfying (1) and (15):*

(25) $$\omega = \sqrt{-1} \sum_{i=1}^{s} A\omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i.$$

Proof Let ω be a pluriclosed left-invariant metric on an Oeljeklaus–Toma manifold M. In view of [15, Section 7], ω lifts to an algebraic expanding soliton of the pluriclosed

flow on the universal covering of M if and only if

$$\rho_B^{1,1}(\cdot,\cdot) = c\omega(\cdot,\cdot) + \frac{1}{2}\left(\omega(D\cdot,\cdot) + \omega(\cdot, D\cdot)\right),$$

for some $c \in \mathbb{R}_-$ and some derivation D of $\mathfrak{g}$ such that $DJ = JD$.

Assume that ω takes the expression in formula (25). Proposition 5.3 implies that ρ_B is represented with respect to the basis $\{Z_1, \ldots, Z_s, W_1, \ldots, W_s\}$ by the matrix

$$P = -\frac{3}{4A}\begin{pmatrix} \mathrm{I}_{\mathfrak{h}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\frac{3}{4A}\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I}_{\mathfrak{J}} \end{pmatrix}$$

induces a symmetric derivation on $\mathfrak{g}$, ω lifts to an algebraic expanding soliton of the pluriclosed flow on the universal covering of M and the first part of the claim follows.

In order to prove the second part of the statement, we need some preliminary observations on derivations D of $\mathfrak{g}$ that commute with J, i.e., such that

$$D(\mathfrak{g}^{1,0}) \subseteq \mathfrak{g}^{1,0}, \quad D(\mathfrak{g}^{0,1}) \subseteq \mathfrak{g}^{0,1}.$$

We can write

$$DZ_i = \sum_{j=1}^{s} k_j^i Z_j + m_j^i W_j \quad \text{and} \quad D\bar{Z}_i = \sum_{j=1}^{s} l_j^i \bar{Z}_j + r_j^i \bar{W}_j.$$

Since D is a derivation, we have, for all $i = 1, \ldots, s$,

$$D[Z_i, \bar{Z}_i] = [DZ_i, \bar{Z}_i] + [Z_i, D\bar{Z}_i].$$

On the other hand,

$$D[Z_i, \bar{Z}_i] = -\frac{\sqrt{-1}}{2}\left(\sum_{j=1}^{s} k_j^i Z_j + l_j^i \bar{Z}_j + m_j^i W_j + r_j^i \bar{W}_j\right),$$

$$[DZ_i, \bar{Z}_i] = -\frac{\sqrt{-1}}{2}k_i^i(Z_i + \bar{Z}_i) - \sum_{j=1}^{s} m_j^i \lambda_{ij} W_j,$$

$$[Z_i, D\bar{Z}_i] = -\frac{\sqrt{-1}}{2}l_i^i(Z_i + \bar{Z}_i) + \sum_{j=1}^{s} r_j^i \bar{\lambda}_{ij}\bar{W}_j$$

and

$$0 = D[Z_i, \bar{Z}_i] - [DZ_i, \bar{Z}_i] - [Z_i, D\bar{Z}_i]$$

$$= -\frac{\sqrt{-1}}{2}\sum_{j\neq i} k_j^i Z_j + l_j^i \bar{Z}_j + \frac{\sqrt{-1}}{2}l_i^i Z_i + \frac{\sqrt{-1}}{2}k_i^i \bar{Z}_i$$

$$+ \sum_{j=1}^{s} m_j^i\left(\lambda_{ij} - \frac{\sqrt{-1}}{2}\right)W_j - r_j^i\left(\frac{\sqrt{-1}}{2} + \bar{\lambda}_{ij}\right)\bar{W}_j,$$

which forces $DZ_i, D\bar{Z}_i = 0$, for all $i = 1, \ldots, s$. It follows that $D_{|\mathfrak{h}} = 0$.

Moreover, for all $I, I' \in \mathfrak{J}$, we have

$$0 = D[I, I'] = [DI, I'] + [I, DI'],$$

which implies that

$$[DI, I'] = -[I, DI'].$$

Assume that

$$DW_i = \sum_{j=1}^{s} k_j^{s+i} Z_j + m_j^{s+i} W_j \quad \text{and} \quad D\bar{W}_i = \sum_{j=1}^{s} l_j^{s+i} \bar{Z}_j + r_j^{s+i} \bar{W}_j,$$

then

$$[DW_i, \bar{W}_i] = \sum_{j=1}^{s} k_j^{s+i} [Z_j, \bar{W}_i] \in \mathfrak{J}^{0,1} \quad \text{and} \quad [W_i, D\bar{W}_i] = \sum_{j=1}^{s} l_j^{s+i} [W_i, \bar{Z}_j] \in \mathfrak{J}^{1,0}.$$

This implies that

$$DW_i = \sum_{j=1}^{s} m_j^{s+i} W_j, \quad D\bar{W}_i = \sum_{j=1}^{s} r_j^{s+i} \bar{W}_j,$$

i.e., $D(\mathfrak{J}) \subseteq \mathfrak{J}$. Moreover, for all $i = 1, \ldots, s$, we have that

$$D[Z_i, W_i] = -\lambda_{ii} DW_i = -\sum_{j=1}^{s} \lambda_{ii} m_j^{s+i} W_j,$$

whereas $[DZ_i, W_i] = 0$ and

$$[Z_i, DW_i] = -\sum_{j=1}^{s} m_j^{s+i} \lambda_{ij} W_j.$$

Using again the fact that D is a derivation, we have

$$DW_i = \sum_{j \in J_i} m_j W_j,$$

where

$$J_i = \{ j \in \{1, \ldots, s\} \mid \lambda_{ii} = \lambda_{ij} \}.$$

With analogous computations, we infer

$$D\bar{W}_i = \sum_{j \in J_i} r_j^{s+i} \bar{W}_j.$$

Clearly, $i \in J_i$. On the other hand, for all $i = 1, \ldots, s$, we know that $\mathfrak{Im}(\lambda_{ii}) \neq 0$, whereas, for all $i \neq j$, $\lambda_{ij} \in \mathbb{R}$. This guarantees that, for all $i = 1, \ldots, s$,

$$J_i = \{i\}.$$

This allows us to write

$$DW_i = m_i^{s+i} W_i, \quad D\bar{W}_i = r_i^{s+i} \bar{W}_i.$$

From the relations above, we obtain that

$$\mathrm{Der}(\mathfrak{g})^{1,0} = \{E \in \mathrm{End}(\mathfrak{g})^{1,0} \mid \mathfrak{h} \subseteq \ker(E), \ E(\langle W_i \rangle) \subseteq \langle W_i \rangle, \quad \text{for all } i = 1, \dots, s\}.$$

First of all, we suppose that ω is a pluriclosed Hermitian metric which takes the following diagonal expression with respect to a coframe $\{\omega^1, \dots, \omega^s, \gamma^1, \dots, \gamma^s\}$ satisfying (1) and (15):

$$\omega = \sqrt{-1} \sum_{i=1}^{s} A_i \omega^i \wedge \bar{\omega}^i + B_i \gamma^i \wedge \bar{\gamma}^i,$$

such that there exist $i, j \in \{1, \dots, s\}$ such that $A_i \neq A_j$ and we suppose that ω is an algebraic soliton. Thanks to the facts regarding derivations proved before, we have that

$$-\sqrt{-1}\frac{3}{4} = \rho_B(Z_i, \bar{Z}_i) = c\omega(Z_i, \bar{Z}_i) + \frac{1}{2}\left(\omega(DZ_i, \bar{Z}_i) + \omega(Z_i, D\bar{Z}_i)\right) = \sqrt{-1}cA_i,$$

$$-\sqrt{-1}\frac{3}{4} = \rho_B(Z_j, \bar{Z}_j) = c\omega(Z_j, \bar{Z}_j) + \frac{1}{2}\left(\omega(DZ_j, \bar{Z}_j) + \omega(Z_j, D\bar{Z}_j)\right) = \sqrt{-1}cA_j,$$

which is impossible, since $A_i \neq A_j$.

Now, suppose that ω is a pluriclosed metric on M which is not diagonal. So, we suppose that there exists $\tilde{j} = 1, \dots, s$ such that $C_{\tilde{j}} \neq 0$. Then assume that there exist a constant $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{g})$ such that

$$(\rho_B)^{1,1}(\cdot, \cdot) = c\omega(\cdot, \cdot) + \frac{1}{2}\left(\omega(D\cdot, \cdot) + \omega(\cdot, D\cdot)\right), \quad DJ = JD.$$

On the other hand,

$$0 = \rho_B(W_{\tilde{j}}, \bar{W}_{\tilde{j}}) = c\omega(W_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \frac{1}{2}\left(\omega(DW_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \omega(W_{\tilde{j}}, D\bar{W}_{\tilde{j}})\right) = \sqrt{-1}cB_{\tilde{j}} + \frac{\sqrt{-1}}{2}(r_{\tilde{j}}^{s+\tilde{j}} + m_{\tilde{j}}^{s+\tilde{j}})B_{\tilde{j}},$$

$$\rho_B(Z_{\tilde{j}}, \bar{W}_{\tilde{j}}) = c\omega(Z_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \frac{1}{2}\left(\omega(DZ_{\tilde{j}}, \bar{W}_{\tilde{j}}) + \omega(Z_{\tilde{j}}, D\bar{W}_{\tilde{j}})\right) = \sqrt{-1}cC_{\tilde{j}} + \frac{\sqrt{-1}}{2}r_{\tilde{j}}^{s+\tilde{j}}C_{\tilde{j}},$$

$$\rho_B(\bar{Z}_{\tilde{j}}, W_{\tilde{j}}) = c\omega(\bar{Z}_{\tilde{j}}, W_{\tilde{j}}) + \frac{1}{2}\left(\omega(D\bar{Z}_{\tilde{j}}, W_{\tilde{j}}) + \omega(\bar{Z}_{\tilde{j}}, DW_{\tilde{j}})\right) = -\sqrt{-1}c\bar{C}_{\tilde{j}} - \frac{\sqrt{-1}}{2}m_{\tilde{j}}^{s+\tilde{j}}\bar{C}_{\tilde{j}},$$

which implies that

$$c = -\frac{1}{2}(r_{\tilde{j}}^{s+\tilde{j}} + m_{\tilde{j}}^{s+\tilde{j}}).$$

On the other hand,

$$\rho_B(Z_{\tilde{j}}, \bar{W}_{\tilde{j}}) = \sqrt{-1}KC_{\tilde{j}},$$

where

$$K = \left(\frac{3}{16} + \frac{c_{\tilde{j}\tilde{j}}^2}{4} + \frac{\sqrt{-1}c_{\tilde{j}\tilde{j}}}{4}\right)\frac{B_{\tilde{j}}}{A_{\tilde{j}}B_{\tilde{j}} - |C_{\tilde{j}}|^2}.$$

Then

$$K = c + \frac{1}{2}r_{\tilde{j}}^{s+\tilde{j}} = -\frac{1}{2}m_{\tilde{j}}^{s+\tilde{j}}$$

and

$$\bar{K} = c + \frac{1}{2}m_{\bar{j}}^{s+\bar{j}} = -\frac{1}{2}r_{\bar{j}}^{s+\bar{j}}.$$

From this, we obtain that

$$c = K + \bar{K} = 2\Re\mathfrak{e}(K) > 0.$$

On the other hand, we have

$$-\sqrt{-1}\frac{3}{4}\left(1 + \frac{|C_{\bar{j}}|^2}{A_{\bar{j}}B_{\bar{j}} - |C_{\bar{j}}|^2}\right) = \rho_B(Z_{\bar{j}}, \bar{Z}_{\bar{j}})$$

$$= c\omega(Z_{\bar{j}}, \bar{Z}_{\bar{j}}) + \frac{1}{2}\left(\omega(DZ_{\bar{j}}, \bar{Z}_{\bar{j}}) + \omega(Z_{\bar{j}}, D\bar{Z}_{\bar{j}})\right) = \sqrt{-1}cA_{\bar{j}},$$

which implies that c must be negative. From this, the claim follows. $\blacksquare$

Corollary 5.5 *Let ω be a pluriclosed Hermitian metric on an Oeljeklaus–Toma manifold which takes the form (16). Then the pluriclosed flow starting from ω is equivalent to the following system of ODEs:*

$$(26) \quad \begin{cases} A_i' = \frac{3}{4}, & \text{if } i \notin \{p_1, \ldots, p_k\}, \\ A_{p_r}' = \frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2}\right), & \text{for all } r = 1, \ldots, k, \\ B_j' = 0, & \text{for all } j = 1, \ldots, s, \\ C_r' = -\left(\frac{3}{16} + \frac{c_{p_r p_r}^2}{4} + \frac{\sqrt{-1}c_{p_r p_r}}{4}\right)\frac{B_{p_r}C_r}{A_{p_r}B_{p_r} - |C_r|^2}, & \text{for all } r = 1, \ldots, k. \end{cases}$$

Moreover, $|C_r|$ is bounded, for all $r = 1, \ldots, k$, and the solution exists for all $t \in [0, +\infty)$ and $A_i \sim \frac{3}{4}t$, as $t \to +\infty$, for all $i = 1, \ldots, s$.

In particular,

$$\frac{\omega_t}{1+t} \to 3\omega_\infty,$$

as $t \to \infty$.

Proof   Observe that, for every $r \in \{1, \ldots, k\}$,

$$(|C_r|^2)' = -\left(\frac{3}{8} + \frac{c_{p_r p_r}^2}{2}\right)\frac{B_{p_r}|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \le 0,$$

which guarantees that $|C_r|^2$ is bounded. On the other hand, denote, for all $r = 1, \ldots, k$,

$$u_r = A_{p_r}B_{p_r} - |C_r|^2.$$

We have that

$$u_r' = A_{p_r}'B_{p_r} - (|C_r|^2)' = \frac{3}{4}B_{p_r} + \left(\frac{9}{8} + \frac{c_{p_r p_r}^2}{2}\right)\frac{B_{p_r}|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2} \ge 0.$$

This guarantees

$$A_{p_r}' = \frac{3}{4}\left(1 + \frac{|C_r|^2}{A_{p_r}B_{p_r} - |C_r|^2}\right) \le \frac{3}{4}\left(1 + \frac{K}{u_r(0)}\right),$$

where $K > 0$ such that $|C_r|^2 \leq K$, for all $t \geq 0$. This implies the long-time existence. As regards the last part of the statement, it is sufficient to prove that

$$\lim_{t \to +\infty} \frac{|C_r|^2}{u_r} = 0.$$

However,

$$u_r' \geq \frac{3}{4} B_{p_r}.$$

Therefore,

$$u_r \geq \frac{3}{4} B_{p_r} t + u_r(0) \to +\infty, \quad t \to +\infty.$$

Then

$$\lim_{t \to +\infty} u_r(t) = +\infty,$$

and, since $|C_r|^2$ is bounded, the assertion follows. ∎

Proof of Theorem 1.1 Let ω be a left-invariant pluriclosed metric on an Oeljeklaus–Toma manifold. Corollary 5.5 implies that pluriclosed flow starting from ω has a long-time solution ω_t such that

$$\frac{\omega_t}{1+t} \to 3\omega_\infty \quad \text{as} \quad t \to \infty.$$

We show that $\frac{\omega_t}{1+t}$ satisfies conditions 1–3 in Proposition 3.1. Here, we denote by $|\cdot|_t$ the norm induced by ω_t.

Taking into account that

$$\omega_{t|\mathfrak{J} \oplus \mathfrak{J}} = \omega_{0|\mathfrak{J} \oplus \mathfrak{J}},$$

condition 2 follows.

Thanks to the fact that condition 2 holds,

$$\omega_{t|\mathfrak{h} \oplus \mathfrak{h}} = \sum_{i=1}^{s} A_i(t) \omega^i \wedge \bar{\omega}^i$$

with $\frac{A_i(t)}{1+t} \to \frac{3}{4}$ as $t \to \infty$, and there exist $C, T > 0$ such that, for every vector $v \in \mathfrak{h}$,

$$\frac{1}{\sqrt{1+t}} |v|_t \leq C |v|_0,$$

for every $t \geq T$, condition 1 is satisfied.

In order to prove condition 3, let $\varepsilon, \ell > 0$ and let γ be a curve in M tangent to $\mathcal{H}$ which is parameterized by arclength with respect to $3\omega_\infty$ and such that $L_\infty(\gamma) < \ell$. Let $v = \dot{\gamma}$ and $T > 0$ such that

$$\left| \frac{A_i(t)}{1+t} - \frac{3}{4} \right| \leq \frac{3\varepsilon^2}{4\ell^2},$$

for $t \geq T$. Then

$$\left| \frac{1}{1+t} |v|_t^2 - |v|_\infty^2 \right| \leq \sum_{i=1}^s \left| \frac{A_i(t)}{1+t} - \frac{3}{4} \right| |v_i|^2 \leq \frac{\varepsilon^2}{\ell^2}$$

and

$$|L_t(\gamma) - L_\infty(\gamma)| \leq \int_0^b \left| \frac{1}{\sqrt{1+t}} |\dot{\gamma}|_t - |\dot{\gamma}|_\infty \right| da \leq \frac{\varepsilon}{\ell} b \leq \varepsilon,$$

since $b \leq \ell$.

Now, we show the last part of the statement, using the same argument as in Proposition 4.1, and we prove that $(\mathbb{H}^s \times \mathbb{C}^s, \frac{\omega_t}{1+t})$ converges in the Cheeger–Gromov sense to $(\mathbb{H}^s \times \mathbb{C}^s, \tilde{\omega}_\infty)$, where $\tilde{\omega}_\infty$ is an algebraic soliton. Again, here we are identifying ω_t with its pullback onto $\mathbb{H}^s \times \mathbb{C}^s$ and we are fixing as base point the identity element of $\mathbb{H}^s \times \mathbb{C}^s$. It is enough to construct a one-parameter family of biholomorphisms $\{\varphi_t\}$ of $\mathbb{H}^s \times \mathbb{C}^s$ such that

$$\varphi_t^* \frac{\omega_t}{1+t} \to \tilde{\omega}_\infty.$$

As we already observed, since $\mathfrak{I}$ is abelian, the endomorphism represented by the matrix

$$D = \begin{pmatrix} 0 & 0 \\ 0 & I_\mathfrak{I} \end{pmatrix}$$

is a derivation of $\mathfrak{g}$ that commutes with the complex structure J. Then we can consider

$$d\varphi_t = \exp(s(t)D) = \begin{pmatrix} I_\mathfrak{h} & 0 \\ 0 & e^{s(t)} I_\mathfrak{I} \end{pmatrix} \in \mathrm{Aut}(\mathfrak{g}, J),$$

where $s(t) = \log(\sqrt{1+t})$. Using $d\varphi_t$, we can define

$$\varphi_t \in \mathrm{Aut}(\mathbb{H}^s \times \mathbb{C}^s, J).$$

For $i = 1, \ldots, s$, we have

$$\frac{1}{1+t}(\varphi_t^* \omega_t)(Z_i, \bar{Z}_i) = \frac{1}{1+t} \omega_t(Z_i, \bar{Z}_i) \to \frac{3}{4}\sqrt{-1}, \quad \text{as } t \to \infty,$$

$$\frac{1}{1+t}(\varphi_t^* \omega_t)(Z_i, \bar{W}_i) = \frac{1}{\sqrt{1+t}} \omega_t(Z_i, \bar{W}_i) \to 0, \quad \text{as } t \to \infty,$$

$$\frac{1}{1+t}(\varphi_t^* \omega_t)(W_i, \bar{W}_i) = \omega_t(W_i, \bar{W}_i) = \sqrt{-1}B_i(0).$$

Then

$$\frac{1}{1+t}\varphi_t^* \omega_t \to \tilde{\omega}_\infty, \quad \text{as } t \to \infty,$$

where

$$\tilde{\omega}_\infty = 3\,\omega_\infty + \omega_{|\mathfrak{I} \oplus \mathfrak{I}}.$$

Notice that $\tilde{\omega}_\infty$ is an algebraic soliton diagonal since $\omega_{|\mathfrak{I} \oplus \mathfrak{I}}$ is diagonal in view of Proposition 5.2. ∎

6 A generalization to semidirect product of Lie algebras

From the viewpoint of Lie groups, the algebraic structure of Oeljeklaus–Toma manifolds is quite rigid and some of the results in the previous sections can be generalized to semidirect product of Lie algebras.

In this section, we consider a Lie algebra $\mathfrak{g}$ which is a semidirect product of Lie algebras

$$\mathfrak{g} = \mathfrak{h} \ltimes_\lambda \mathfrak{I},$$

where $\lambda \colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{I})$ is a representation. We further assume that $\mathfrak{g}$ has a complex structure of the form

$$J = J_\mathfrak{h} \oplus J_\mathfrak{I},$$

where $J_\mathfrak{h}$ and $J_\mathfrak{I}$ are complex structures on $\mathfrak{h}$ and $\mathfrak{I}$, respectively.

The following assumptions are all satisfied in the case of an Oeljeklaus–Toma manifold:

 i. $\mathfrak{h}$ has $(1,0)$-frame such that $\{Z_1, \ldots, Z_r\}$ such that $[Z_k, \bar{Z}_k] = -\frac{\sqrt{-1}}{2}(Z_k + \bar{Z}_k)$, for all $k = 1, \ldots, r$, and the other brackets vanish.
 ii. $\mathfrak{I}$ is a $2s$-dimensional abelian Lie algebra, and $J_\mathfrak{I}$ is a complex structure on $\mathfrak{I}$.
 iii. $\lambda(\mathfrak{h}^{1,0}) \subseteq \mathrm{End}(\mathfrak{I})^{1,0}$.
 iv. $\mathfrak{I}$ has a $(1,0)$-frame $\{W_1, \ldots W_s\}$ such that $\lambda(Z) \cdot \bar{W}_r = \lambda_r(Z)\bar{W}_r$, for every $r = 1, \ldots, s$, where $\lambda_r \in \Lambda^{1,0}(\mathfrak{h})$.
 v. $\sum_{a=1}^{s} \mathfrak{Im}(\lambda_a(Z_i))$ is constant on i.
 vi. $\mathfrak{I}$ has a $(1,0)$-frame $\{W_1, \ldots W_s\}$ such that $\lambda(Z) \cdot W_r = \lambda'_r(Z)W_r$, for every $r = 1, \ldots, s$, where $\lambda'_r \in \Lambda^{1,0}(\mathfrak{h})$ and $\sum_{a=1}^{s} \mathfrak{Im}(\lambda'_a(Z_i))$ is constant on i.

Note that condition i is equivalent to require that $\mathfrak{h} = \underbrace{\mathfrak{f} \oplus \cdots \oplus \mathfrak{f}}_{r\text{-times}}$ equipped with the complex structure $J_\mathfrak{h} = \underbrace{J_\mathfrak{f} \oplus \cdots \oplus J_\mathfrak{f}}_{r\text{-times}}$, whereas in condition iv, the existence of $\{W_r\}$ and λ_r is equivalent to require that

$$\lambda(Z) \circ \lambda(Z') = \lambda(Z') \circ \lambda(Z),$$

for every $Z, Z' \in \mathfrak{h}^{1,0}$.

The computations in Section 5 can be used to study solutions to the flow

$$\tag{27} \partial_t \omega_t = -\rho_B^{1,1}(\omega_t)$$

in semidirect products of Lie algebras (this flow coincides with the pluriclosed flow only when the initial metric is pluriclosed). We have the following proposition.

Proposition 6.1 *Let $\mathfrak{g} = \mathfrak{h} \ltimes_\lambda \mathfrak{I}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J = J_\mathfrak{h} \oplus J_\mathfrak{I}$, and let ω be a Hermitian metric on $\mathfrak{g}$ making $\mathfrak{h}$ and $\mathfrak{I}$ orthogonal. Then the Bismut–Ricci form of ω satisfies $\rho_{B|\mathfrak{h} \oplus \mathfrak{I}}^{1,1} = \rho_{B|\mathfrak{I} \oplus \mathfrak{I}}^{1,1} = 0$.*

If conditions i–iv hold and $\omega_{|\mathfrak{h} \oplus \mathfrak{h}}$ is diagonal with respect to the frame $\{Z_i\}$, then the $(1,1)$-component of the Bismut–Ricci form of ω does not depend on ω and the solution

to the flow (27) starting from ω takes the following expression:

$$\omega_t = \omega - t\rho_B^{1,1}(\omega).$$

If conditions i–iv and vi hold and $\omega_{|\mathfrak{h}\oplus\mathfrak{h}}$ is a multiple of the canonical metric with respect to the frame $\{Z_i\}$, then ω is a soliton for flow (27) with cosmological constant $c = \frac{1}{2} + \sum_{a=1}^{s} \mathfrak{Im}(\lambda_a'(Z_i))$.

The previous proposition does not cover the case when properties i–iv are satisfied and the restriction to $\mathfrak{h} \oplus \mathfrak{h}$ of the initial Hermitian inner product

$$\omega = \sqrt{-1}\sum_{a,b=1}^{r} g_{a\bar{b}}\omega^a \wedge \bar{\omega}^b + \sqrt{-1}\sum_{a,b=1}^{s} g_{r+a\overline{r+b}}\gamma^a \wedge \bar{\gamma}^b$$

is not diagonal with respect to $\{Z_i\}$. In this case flow (27) evolves only the components $g_{i\bar{i}}$ of ω along $\omega^i \wedge \bar{\omega}^i$ via the ODE

$$\partial_t g_{i\bar{i}} = \frac{1}{4}\sum_{a=1}^{r} g^{\bar{a}a}\mathfrak{Re}\, g_{i\bar{a}} - \frac{1}{2}\sum_{c,d=1}^{s} \overline{g^{r+d\overline{r+c}}}\left\{\omega([Z_i, W_c], \bar{W}_d) + \omega([\bar{Z}_i, \bar{W}_c], W_d)\right\},$$

where $g_{i\bar{i}}$ depends on t. Note that the quantities $-\frac{1}{2}\sum_{c,d=1}^{s} \overline{g^{r+d\overline{r+c}}}\{\omega([Z_i, W_c], \bar{W}_d) + \omega([\bar{Z}_i, \bar{W}_c], W_d)\}$ appearing in the evolution of $g_{i\bar{i}}$ are independent on t.

The same computations as in Section 4 imply the following proposition.

Proposition 6.2 Let $\mathfrak{g} = \mathfrak{h} \ltimes_\lambda \mathfrak{I}$ be a semidirect product of Lie algebras equipped with a splitting complex structure $J = J_\mathfrak{h} \oplus J_\mathfrak{I}$. Assume that properties i–iii are satisfied, and let ω be a left-invariant Hermitian metric on $\mathfrak{g}$. Then

$$\rho_{C|\mathfrak{I}\oplus\mathfrak{I}} = \rho_{C|\mathfrak{h}\oplus\mathfrak{I}} = 0,$$

whereas $\rho_{C|\mathfrak{h}\oplus\mathfrak{h}}$ is diagonal with respect to $\{Z_1, \ldots, Z_r\}$.

If, in addition, property iv holds, then

$$\rho_C(Z_i, \bar{Z}_i) = -\sqrt{-1}\left(\frac{1}{2} - \sum_{a=1}^{s} \mathfrak{Im}(\lambda_a(Z_i))\right), \quad \text{for all } i = 1, \ldots, r.$$

If, in addition, property v holds, then ω is a soliton for the Chern–Ricci flow with cosmological constant $c = \frac{1}{2} - \sum_{a=1}^{s} \mathfrak{Im}(\lambda_a(Z_i))$ if and only if $\omega_{\mathfrak{h}\oplus\mathfrak{h}}$ is a multiple of the canonical metric on $\mathfrak{h}$ with respect to the frame $\{Z_i\}$ and $\omega_{\mathfrak{h}\oplus\mathfrak{I}} = 0$.

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