BULL. AUSTRAL. MATH. SOC. 11F67, 11F30
VOL. 64 (2001) [63–70]

ON ADMISSIBLE DISTRIBUTIONS ATTACHED TO CONVOLUTIONS OF HILBERT MODULAR FORMS

ANDRZEJ DĄBROWSKI

Dedicated to Jana

The aim of this note is to check the admissibility property of the distribution attached to convolution of Hilbert modular forms.

1. INTRODUCTION

Let F be a totally real number field of degree n over $\mathbb{Q}$. Let $\mathcal{O}_F$, $\vartheta \subset \mathcal{O}_F$, $d_F = \mathcal{N}(\vartheta)$ denote, respectively, the maximal order, the different and the discriminant of F.

Let $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \psi)$ be a primitive Hilbert cusp form of scalar integral weight $k = k_0 \cdot \mathbf{1}$ and central character ψ, and $g \in \mathcal{M}_l(\mathfrak{c}(g), \phi)$ a Hilbert modular form of half-integral weight $l = l_0 \cdot \mathbf{1}$ and character ϕ such that $l_0 < k_0$. The convolution series $D(s; \mathbf{f}, g)$ of $\mathbf{f}$ and g is defined in terms of Fourier coefficients $c(\mathfrak{m}, \mathbf{f})$ and $\lambda(\xi, \mathfrak{m}; g, \phi)$ by

$$(1) \quad D(s; \mathbf{f}, g) := \sum_{(\xi, \mathfrak{m})} c(\xi\mathfrak{m}^2, \mathbf{f})\overline{\lambda(\xi, \mathfrak{m}; g, \phi)}\xi^{-1/2(l-(1/2)\cdot 1)}\mathcal{N}(\xi\mathfrak{m}^2)^{-s} \quad (\mathrm{Re}\,(s) \gg 0),$$

where $(\xi, \mathfrak{m})$ runs over representatives for equivalence classes of pairs of totally positive numbers $\xi \in F$ and fractional ideals $\mathfrak{m}$ of F such that $\xi\mathfrak{m}^2 \subset \mathcal{O}_F$: $(\xi, \mathfrak{m})$ and $(\xi', \mathfrak{m}')$ are equivalent if $\xi = \eta^2\xi'$ and $\mathfrak{m} = \eta^{-1}\mathfrak{m}'$ for some $\eta \in F^\times$.

We fix a rational prime p, and embeddings

$$i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}, \qquad i_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$$

where $\mathbb{C}_p$ is the Tate field (the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$) endowed with a unique norm $|\cdot|_p$ such that $|p|_p = p^{-1}$. For an integral ideal $\mathfrak{a}$ denote

Received 13th September, 2000
The author would like to thank Professor Ehud de Shalit for inviting him to the Hebrew University in Jerusalem, in February-March 1998, and for the hospitality and support. Research supported in part by Polish KBN Grant 2 PO3A 038 12.

by $S(\mathfrak{a})$ its support $S(\mathfrak{a}) := \{\mathfrak{p} : \mathfrak{p} \text{ divides } \mathfrak{a}\}$. We also set $S = \{\mathfrak{p} : \mathfrak{p} \mid p \text{ in } F\}$, $\mathfrak{m}_0 = \prod_{\mathfrak{p}\in S} \mathfrak{p}$, $\mathfrak{f}_0 = \sum_{\mathfrak{a}\mid\mathfrak{m}_0} \mu(\mathfrak{a})\alpha'(\mathfrak{a})\mathfrak{f}|\mathfrak{a}$, and $\mathfrak{c} = \mathfrak{c}(\mathbf{f})4\mathfrak{c}(g)$. Fix once and for all totally positive numbers $c(\mathbf{f}), c(g) \in F$ with $(c(\mathbf{f})) = \mathfrak{c}(\mathbf{f})$ and $(c(g)) = \mathfrak{c}(g)$, and set $c := c(\mathbf{f})c(g)$. With the quadratic Hecke character $\omega = \varepsilon_{-1}$ corresponding to $F(\sqrt{-1})/F$ define the complex-valued function

$$\Psi(s;\mathbf{f},g) = L_{\mathfrak{c}\mathfrak{m}_0}\Big(4s-1, \big(\omega\psi\overline{\phi}\big)^2\Big)\Gamma\left(s-1+\frac{k_0+l_0}{2}\right)^n D\Big(s-\frac{3}{4}; \mathbf{f}_0, g\Big). \tag{2}$$

Put

$$1 - c(\mathfrak{p},\mathbf{f})X + \psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1}X^2 = \big(1-\alpha(\mathfrak{p})X\big)\big(1-\alpha'(\mathfrak{p})X\big) \in \mathbb{C}_p[X]$$

where $\alpha(\mathfrak{p})$, $\alpha'(\mathfrak{p})$ are the inverse roots of the Hecke $\mathfrak{p}$-polynomial; assume that $\operatorname{ord}_p \alpha(\mathfrak{p}) \leqslant \operatorname{ord}_p \alpha'(\mathfrak{p})$.

Let $\mathrm{Gal}_p = \mathrm{Gal}\big(F^{\mathrm{ab}}_{p,\infty}/F\big)$ denote the Galois group of the maximal Abelian extension of F unramified outside p and all primes above ∞ in F. Given an integral ideal $\mathfrak{m} \subset \mathcal{O}_F$, let $I(\mathfrak{m})$ denote the group of all fractional ideals in F, prime to $\mathfrak{m}$. Also let

$$P(\mathfrak{m}) := \big\{(\alpha) : \alpha \in F_+^\times,\ \alpha \equiv 1 \big(\mathrm{mod}^\times \mathfrak{m}\big)\big\}, \quad H(\mathfrak{m}) := I(\mathfrak{m})/P(\mathfrak{m}).$$

Then $\mathrm{Gal}_p = \varprojlim H(\mathfrak{m})$ (where the projective limit is over $\mathfrak{m}$ with the condition $S(\mathfrak{m}) \subset S(\mathfrak{m}_0)$). Let $\pi_\mathfrak{m} : \mathrm{Gal}_p \to H(\mathfrak{m})$ be the natural projection; put $(\mathfrak{m}) := \ker \pi_\mathfrak{m}$. Also put $h(\mathfrak{m}) := \operatorname{card} H(\mathfrak{m})$.

The domain of definition of our non-archimedean L-function is the p-adic analytic Lie group $\mathbf{X}_p = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}_p, \mathbb{C}_p^\times)$ of all continuous p-adic characters of Gal_p.

Recall that a p-adic measure on Gal_p may be regarded as a $\mathbb{C}_p$-linear form μ on the space $\mathcal{C}(\mathrm{Gal}_p)$ of all continous $\mathbb{C}_p$-valued functions, which is uniquely determined by its restriction to the subspace $\mathcal{C}^1(\mathrm{Gal}_p)$ of locally constant functions. The Mellin transform L_μ of μ is a bounded analytic function on $\mathbb{X}_p$.

Amice-Vélu [1] and Vishik [8] have introduced a more delicate notion of an h-admissible measure. Let $\mathcal{C}^h(\mathrm{Gal}_p)$ denote the space of $\mathbb{C}_p$-valued functions which can be locally represented by polynomials of degree less than a natural number h. The $\mathbb{C}_p$-linear form $\mu : \mathcal{C}^h(\mathrm{Gal}_p) \to \mathbb{C}_p$ is called an h-admissible measure if for all $r = 0, 1, \dots, h-1$ the following growth condition is satisfied:

$$\sup_{a\in\mathrm{Gal}_p} \left|\int_{a+(\mathfrak{m})} (\mathcal{N}x_p - \mathcal{N}a_p)^r\, d\mu\right|_p = o\big(|\mathfrak{m}|^{r-h}\big),$$

where $\mathcal{N}x_p \in \mathbf{X}_p$ denotes the natural norm homomorphism

$$\mathcal{N}x_p : \mathrm{Gal}_p \to \mathrm{Gal}\big(\mathbb{Q}^{\mathrm{ab}}_{p,\infty}/\mathbb{Q}\big) \simeq \mathbb{Z}_p^\times \to \mathbb{C}_p^\times.$$

The aim of this note is to check the admissibility property of the distribution constructed in [**4**, Theorem 2]. The corresponding non-archimedean Mellin transform is a $\mathbb{C}_p$-analytic function on $\mathbf{X}_p$ with the properties summarised in Theorem 1.

Let $\theta \in \{0,1\}$ be determined by $\theta \equiv k_0 - l_0 - 1/2 \bmod 2$. Put

$$K = \left\{\kappa_r := \theta - 1 + 2r \ : \ r \in \mathbb{Z},\ 0 \leqslant 2r \leqslant k_0 - l_0 - \frac{5}{2} + \theta\right\}.$$

Let $s_r := (\kappa_r + 1/2)/2$, with $\kappa_r \in K$, be critical points of $D(s;\mathbf{f},g)$ in the sense of [**4**, p.408-409]. Let $\mathfrak{q}$ be an integral ideal in $\mathcal{O}_F$; set $\mathfrak{q}_0 = \prod_{\mathfrak{p}|\mathfrak{q}} \mathfrak{p}$. Let $\mathfrak{p}(\sigma)$ denote a prime divisor of p in F attached to the real embedding σ. Let $\langle \mathbf{f},\mathbf{f}\rangle$ denote the Petersson scalar product.

Theorem 1. *Assume that F has class number one, $c(\mathfrak{c}(\mathbf{f}),\mathbf{f}) \neq 0$, and the ideals $\mathfrak{c}(\mathbf{f})$, $4\mathfrak{c}(g)$, $\mathfrak{m}_0$, $\mathfrak{q}$ are pairwise relatively prime. Assume that the Fourier coefficients of g are algebraic and p-adically bounded. Put $h = \left[\max_i(2\ \mathrm{ord}_p\,\alpha(\mathfrak{p}(\sigma_i)))\right] + 1$. Then there exists a $\mathbb{C}_p$-analytic function $L_{(p)}$ on $\mathbf{X}_p$ of type $o(\log^h)$ with the properties*

(i) *for all $m \in \mathbb{Z}$ with $0 \leqslant 2m \leqslant k_0 - l_0 + \theta - 2$, and for all characters of finite order $\chi \in \mathbf{X}_p^{\mathrm{tors}}$ the following equality holds:*

$$L_{(p)}(\chi \mathcal{N} x_p^m) = \chi_\infty(-1)\left(1 - (\overline{\psi}\phi^2\chi^4)^*(\mathfrak{q})\mathcal{N}(\mathfrak{q})^{2(1-\kappa_m)}\right)\gamma(s_m)\frac{\Psi\left(s_m;\mathbf{f}_0, g(\overline{\chi}^*_{\mathfrak{m}\mathfrak{q}_0})j_{\mathfrak{c},\mathfrak{m}'}\right)}{\langle \mathbf{f},\mathbf{f}\rangle}$$

*where $j_{\mathfrak{c},\mathfrak{m}'}$ is a certain inverter [**4**, p.401], χ_∞ is the archimedean part of χ, χ^* is the associated ideal character,*

$$(3)\quad \gamma(s) = \pi^{-2ns-n-nk_0} d_F^{2s} \mathcal{N}\left(\frac{\mathfrak{c}}{4}\right)^s i^{-n(k_0+l_0-2)}\Gamma\left(s + \frac{k_0 - l_0}{2}\right)^n \times \mathcal{N}(\mathfrak{m}')^{k_0+2(s-1)}\alpha(\mathfrak{m}')^{-2},$$

and $\mathfrak{m}$, $\mathfrak{m}'$ are arbitrary integral ideals in $\mathcal{O}_F$ satisfying $\mathrm{lcm}(\mathfrak{m}_0, \mathfrak{c}(\chi)) \mid \mathfrak{m}$, $\mathfrak{m}_0\mathfrak{q}_0^2\mathfrak{m} \mid \mathfrak{m}'$, $S(\mathfrak{m}) \subset S$, $S(\mathfrak{m}') \subset S \cup S(\mathfrak{q})$.

(ii) *If $h \leqslant (k_0 - l_0 + \theta - 2)/2 + 1$ then the function $L_{(p)}$ on $\mathbf{X}_p$ is uniquely determined by condition* (i).

(iii) *If $\mathrm{ord}_p\,\alpha(\mathfrak{p}(\sigma_i)) = 0$ $(i = 1, \dots, n)$ then the function $L_{(p)}$ is bounded on $\mathbf{X}_p$.*

Remarks.

(i) Part (ii) of the Theorem follows from part (i) and the characterisation of functions of type $o(\log^h)$ [**8**].

(ii) Part (iii) of the Theorem is the main result of [**4**].

(iii) "Motivic" interpretation and relation to Sym^2. Analytic properties of the standard L-function $L(\mathbf{f},s)$ suggest that $\mathbf{f}$ should correspond to a certain motive $M(\mathbf{f})$ over F of rank 2 and weight k_0 with coefficients in a field T containing all $c(\mathfrak{n},\mathbf{f})$. The principal work in this direction, concerning the construction of a compatible system of Galois representations, was carried out by Carayol, Taylor, Rogawski, Blasius and others. In such "motivic" context the series $D(s;\mathbf{f},g)$ corresponds to the symmetric square of $M(\mathbf{f})$ where g is a theta series of special kind, and the above Theorem agrees with the general conjecture on the existence of p-adic L-functions attached to critical pure motives over totally real number fields [**7**].

In the p-ordinary case (that is, $\operatorname{ord}_p \alpha(\mathfrak{p}(\sigma_i)) = 0$, $i = 1,\dots,n$) $L_{(p)}$ is the p-adic Mellin transform of a certain bounded p-adic distribution (measure) constructed in [**4**]. We show that this distribution is, in general, h-admissible in the sense of Amice-Vélu-Manin-Vishik. We give two proofs of this result. The first method is to carry over the construction from [**2**] to our situation; here we use, in particular, the deep result of Deligne and Ribet [**3**] on the existence of a p-adic Hecke L-function for F. In the second method we use a simple combinatorial lemma to avoid the above argument using the Deligne-Ribet construction.

We follow the notation and definitions from [**4, 5**] unless otherwise stated.

2. Complex valued distributions attached to convolutions of Hilbert modular forms

Let $s_r := (\kappa_r + 1/2)/2$, with $\kappa_r \in K$, be critical points of $D(s;\mathbf{f},g)$. We define $\mathbb{C}$-valued distributions $\mu^{\sim}_{s_r}$ on Gal_p by

$$\mu^{\sim}_{s_r,\mathfrak{m}}(\chi^{\star}_{\mathfrak{m}}) := \gamma(s_r)\cdot\frac{\Psi\big(s_r;\mathbf{f}_0, g(\overline{\chi}^{\star}_{\mathfrak{m}})j_{c,\mathfrak{m}'}\big)}{\langle \mathbf{f},\mathbf{f}\rangle_{c\mathfrak{m}_0^2}},$$

with arbitrary ideals $\mathfrak{m}$, $\mathfrak{m}'$ subject to $\operatorname{lcm}(\mathfrak{m}_0, \mathfrak{c}(\chi)) \mid \mathfrak{m}$ and $\mathfrak{m}_0\mathfrak{m} \mid \mathfrak{m}'$. Here $j_{c,\mathfrak{m}'}$ is a certain inverter

$$j_{c,\mathfrak{m}'}:\ \mathcal{M}_k(c\mathfrak{m}'^2,\psi) \to \mathcal{M}_k(c\mathfrak{m}'^2,\overline{\psi\varepsilon_c}),$$

where ε_c denotes the quadratic Hecke character of F corresponding to $F(\sqrt{c})/F$. Other notations are explained in the Introduction (see (1),(2),(3)).

These distributions are defined over some finite extension of $\mathbb{Q}$ (see [**4**, Proposition 5.1] for a precise formulation of the algebraicity result). The Rankin integral representation of the distributions combined with the holomorphic projection operator gives [4, p.420]:

$$\mu^{\sim}_{s_r,\mathfrak{m}}(\chi_{\mathfrak{m}}) = \frac{1}{\langle \mathbf{f},\mathbf{f}\rangle_{\mathfrak{cm}_0^2}} \cdot \langle \mathbf{f}_0, V_r(\chi)|_{J_{\mathfrak{cm}_0^2}}\rangle_{\mathfrak{cm}_0^2},$$

where $V_r(\chi) \in \mathcal{S}_k(\mathfrak{cm}_0^2, \psi)$ is a holomorphic cusp form. Put

$$\gamma(\mathfrak{m}') = \alpha(\mathfrak{m}')^{-2} m_0^{n(k_0-2)} 2^{2n(k_0-1)-1} v_1 v(c) d_F,$$

where $v(c) = \pm 1$ and v_1 is a fourth root of unity (independent of $\mathfrak{m}'$).

$V_r(\chi)$ has the following Fourier expansion:

$$V_r(\chi)(z) = \sum_{0 \ll \sigma \in \mathcal{O}_F} U(\sigma, r, \chi) e_F(\sigma z)$$

where

$$U(\sigma,r,\chi) = \gamma(\mathfrak{m}') \sum_{\substack{(\frac{m'}{m_0})^2\sigma = \sigma_1+\sigma_2,\\ \sigma_i \gg 0}} \overline{\chi}^{\star}_{\mathfrak{m}}((\sigma_1)) \sum_{\gamma \in \mathcal{O}_+^{\times}/\mathcal{O}_+^{\times 2}} \gamma^{-k/2} \lambda_g(\gamma\sigma_1, \mathcal{O})$$
$$\times L_{\mathfrak{cm}_0}(\kappa_r, \Omega_{\gamma\sigma_2 c})\, B(\gamma\sigma_2, \kappa_r; \mathfrak{cm}_0) \prod_{\nu=1}^{n} \left\{ \gamma_\nu^{-\beta_\nu} P_{\kappa_r,\nu}\left(\sigma_{2,\nu}, \left(\frac{m'_\nu}{m_{0,\nu}}\right)^2 \sigma_\nu\right)\right\},$$

and

$$P_{\kappa_r,\nu}(\sigma_{2,\nu},\sigma_\nu) = \sum_{j=0}^{\alpha_\nu} \binom{-\beta_\nu}{j} (-1)^j \frac{\Gamma(\alpha_\nu)}{\Gamma(\alpha_\nu - j)} \frac{\Gamma(k_0-1-j)}{\Gamma(k_0-1)} \sigma_\nu^j \sigma_{2,\nu}^{\alpha_\nu-1-j},$$

with $\alpha_\nu = \alpha_\nu(\kappa_r) = (\kappa_r + 1 + q_r)/2$, $\beta_\nu = \beta_\nu(\kappa_r) = (\kappa_r - q_\nu)/2$, and $q = k - l - (1/2)\cdot \mathbf{1}$. $m_0, m' \in F^\times$ are totally positive with $(m_0) = \mathfrak{m}_0\mathfrak{q}_0$ and $(m') = \mathfrak{m}'$. Also, $L_{\mathfrak{cm}_0}(s,\Omega)$ is the L-function associated to Ω, and $B(\sigma', \kappa_r; \mathfrak{cm}_0)$ is defined by

$$B(\sigma', \kappa_r; \mathfrak{cm}_0) := \sum \mu(\mathfrak{a}) \Omega^{\star}_{\sigma' c}(\mathfrak{a}) \Omega^{\star}(\mathfrak{b}^2) \mathcal{N}(\mathfrak{a})^{-\kappa_r} \mathcal{N}(\mathfrak{b})^{1-\kappa_r},$$

where the summation is over all ordered pairs $(\mathfrak{a},\mathfrak{b})$ of integral ideals in $\mathcal{O}_F$ prime to $\mathfrak{cm}_0$ such that $(\sigma') \subset \mathfrak{a}^2\mathfrak{b}^2$. (See [4, p.420] for details.) The quantities $\lambda_g(\gamma\sigma_1, \mathcal{O})$ do appear in the Fourier expansion for $g(\overline{\chi}^{\star}_{\mathfrak{m}})$:

$$g(\overline{\chi}^{\star}_{\mathfrak{m}})(2z) = \sum_{0 \ll \sigma_1 \in \mathcal{O}} \overline{\chi}^{\star}_{\mathfrak{m}}((\sigma_1)) \lambda_g(\sigma_1, \mathcal{O}) e_F(\sigma_1 z).$$

Let us now consider the linear functional given by

$$\mathbf{L} : \Phi \mapsto \frac{\langle \mathbf{f}_0, \Phi|_{J_{\mathfrak{cm}_0^2}} \rangle_{\mathfrak{cm}_0^2}}{\langle \mathbf{f},\mathbf{f}\rangle_{\mathfrak{cm}_0^2}}$$

on the complex linear space $\mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0^2, \psi)$. From the Atkin-Lehner theory (in Miyake's form [6]) it follows that $\mathbf{L}$ is defined over some number field $\mathbb{K}$, that is, there exist a finite number of ideals $\mathfrak{m}_i$ and fixed algebraic numbers $l(\mathfrak{m}_i) \in \mathbb{K}$ such that

$$\mathbf{L}(\Phi) = \sum_i c(\mathfrak{m}_i, \Phi)\, l(\mathfrak{m}_i).$$

Therefore the distributions $\mu_r^{\sim}$ can be written in the form

$$\mu_{r,\mathfrak{m}}^{\sim}(\chi_{\mathfrak{m}}) = \gamma(\mathfrak{m}\mathfrak{m}_0) \cdot \mathbf{L}(V_r(\chi)).$$

3. The growth conditions

Lemma 1. *For any positive integer N, for all integral ideals $\mathfrak{n}$, $\mathfrak{m}$ with $S(\mathfrak{m}) = S(\mathfrak{m}_0)$ and $r \in \mathbb{Z}$ such that $\kappa_r \in K$, we have that $c(\mathfrak{n}, V_r^{\sim}(\chi))$ is, modulo p^N, a finite linear combination with p-integral coefficients of terms of the form*

$$\chi^*(\mathfrak{a})\mathcal{N}(\mathfrak{a})^r \int_{\mathrm{Gal}_S} \chi \mathcal{N}_p^r \, d\mu^+(\mathfrak{a}),$$

for fractional ideals $\mathfrak{a} = \mathfrak{a}(N, \mathfrak{n}, \mathfrak{m})$, Hecke characters χ of finite order with $\mathfrak{c}(\chi) \mid \mathfrak{m}$, and $\mathcal{O}$-valued measures $\mu^+(\mathfrak{a})$.

Proof: It follows from Section 2 and [3] (see [4, p.425]). ∎

Lemma 2. *Let $h \geqslant q$ be positive rational integers, and $\alpha, \beta \in \mathcal{O}_F$, $\alpha \equiv \beta \bmod \mathfrak{m}$. Then*

$$\sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j} (-\beta)^j j^q$$

belongs to $\mathfrak{m}^{h-q}$.

Proof: Induction with respect to q. The case $q = 0$ is trivial. Now

$$\begin{aligned}
&\sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j} (-\beta)^j j^q \\
&\quad = \sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j} (-\beta)^j \left[j \cdot \ldots \cdot (j - q + 1) + P_{q-1}(j) \right] \\
&\quad = h \cdot \ldots \cdot (h - q + 1)(-\beta)^q (\alpha - \beta)^{h-q} + \sum_{j=0}^{h} \binom{h}{j} \alpha^{h-q} (-\beta)^j P_{q-1}(j)
\end{aligned}$$

where $P_{q-1}(j)$ is a polynomial of degree $q - 1$ in j. The assertion now follows. ∎

THEOREM 2. *Put $H = (k_0 - l_0 + \theta - 2)/2$. There exists a $\mathbb{C}_p$-linear form*

$$\mu^{\sim} : \mathcal{C}^{H+1}(\mathrm{Gal}_p) \to \mathbb{C}_p$$

such that

$$\int_{a+(\mathfrak{m})} \mathcal{N}x_p^r \, d\mu^{\sim} = (-1)^{rn} \int_{a+(\mathfrak{m})} d\mu_r^{\sim}, \quad r = 0, 1, \dots, H.$$

Here $\mu^{\sim}$ satisfies the growth condition:

$$\sup_{a \in \mathrm{Gal}_p} \left| \int_{a+(\mathfrak{m})} (\mathcal{N}x_p - \mathcal{N}a_p)^r d\mu^{\sim} \right|_p = O\big(|\mathfrak{m}|^{r-2\,\mathrm{ord}_p\,\alpha(\mathfrak{p})}\big).$$

PROOF: The existence follows from the definition of $\mu_r^{\sim}$.

To check the growth condition we can suppose that $a \in \mathcal{O}_F$. We obtain

$$\begin{aligned}
\int_{a+(\mathfrak{m})} (\mathcal{N}x - \mathcal{N}a)^r d\mu^{\sim} &= \sum_{j=0}^{r} \binom{r}{j} \big(-\mathcal{N}(a)\big)^{r-j} (-1)^{nj} \int_{a+(\mathfrak{m})} d\mu_j^{\sim} \\
&= (-1)^{rn} \sum_{j=0}^{r} \binom{r}{j} \big(-\mathcal{N}(-a)\big)^{r-j} \\
&\quad \times \frac{1}{h(\mathfrak{m})} \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) \mu_j^{\sim}(\chi) \\
&= (-1)^{rn} \gamma(\mathfrak{m}) \sum_{j=0}^{r} \binom{r}{j} \big(-\mathcal{N}(-a)\big)^{r-j} \\
&\quad \times \frac{1}{h(\mathfrak{m})} \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) \mathbf{L}\big(V_{r,\mathfrak{m}}(\chi)\big).
\end{aligned}$$

By using Lemma 1 and the property that $\mathbf{L}$ is defined over some number field, we see that it is sufficient to check the congruences in the above theorem for the following number A:

$$\begin{aligned}
A &:= \gamma(\mathfrak{m}) \sum_{j=0}^{r} \big(-\mathcal{N}(-a)\big)^{r-j} \\
&\quad \times \frac{1}{h(\mathfrak{m})} \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) \int \chi\Big(\frac{u_1}{u_2}x\Big) \Big(\frac{u_1}{u_2}x\Big)^{j+1} d\mu^{+}(\dots) \\
&= \gamma(\mathfrak{m}) \int_{x \equiv a u_2 u_1^{-1} (\mathrm{mod}\ \mathfrak{m})} \sum_{j=0}^{r} \binom{r}{j} \big(-\mathcal{N}(-a)\big)^{r-j} \Big(\frac{u_1}{u_2}x\Big)^{j+1} d\mu^{+}(\dots) \\
&= \gamma(\mathfrak{m}) \Big(\frac{u_1}{u_2}\Big)^{r+1} \int_{x \equiv a u_2 u_1^{-1} (\mathrm{mod}\ \mathfrak{m})} \big(x - a u_2 u_1^{-1}\big)^r x \, d\mu^{+}(\dots).
\end{aligned}$$

Since $\mu^{+}(\dots)$ is a bounded measure, the integral has order $O\big(|\mathfrak{m}|_p^r\big)$. Also $\gamma(\mathfrak{m}) = O\big(|\mathfrak{m}|_p^{-2\,\mathrm{ord}_p\,\alpha(\mathfrak{p})}\big)$. The assertion follows.

THE SECOND METHOD. We take into account the explicit form of the Fourier coefficients for $V_r(\chi)$. Taking summation over all χ, we obtain that the integral $\int(\mathcal{N}x - \mathcal{N}a)^r d\mu^\sim$ is a linear combination (with coefficients not depending on r) of terms of the form

$$\gamma(\mathfrak{m})\sum_{j=0}^{r}\alpha^{r-j}\beta^j\prod_{\nu=1}^{n}P_{\kappa_r,\nu}\Big(\sigma_{2,\nu},\Big(\frac{m'_\nu}{m_{0,\nu}}\Big)^2\sigma_\nu\Big),$$

with $\alpha+\beta\in\mathfrak{m}$. Now $P_{\kappa_r,\nu}(\dots)$ is homogeneous of degree $\alpha_\nu - 1$ in variables $\sigma_{2,\nu}$ and σ_ν, and $\prod_{\nu=1}^{n} P_{\kappa_j,\nu}(\dots)$ is a polynomial of degree $\sum\alpha_\nu$ in variable j. On the other hand, $\prod_\nu \sigma_\nu^{\alpha_\nu}$ is divisible by $\prod_\nu \mathfrak{m}^{2\alpha_\nu}$. If $r \geqslant 2\sum_\nu \alpha_\nu$ then the assertion follows from Lemma 2. The remaining case is trivial.

END OF THE PROOF OF THEOREM 1. We put $L_{(p)}(x) := \int_{\mathrm{Gal}_p} x\, d\mu$, where $\mu := \mu^\sim|_{\mathcal{C}^h(\mathrm{Gal}_p)}$ and $h = \big[\max_i(2\operatorname{ord}_p\alpha(\mathfrak{p}(\sigma_i)))\big]+1$. Then it is well known (due to Amice-Vélu [1] and Vishik [8]), that such non-archimedean Mellin transform is a $\mathbb{C}_p$-analytic function of type $o(\log^h)$. ∎

REFERENCES

[1] Y. Amice and J. Vélu, 'Distributions p-adiques associées aux séries de Hecke', *Astérisque* **24-25** (1975), 119–131.

[2] A. Dąbrowski and D. Delbourgo, 'S-adic L-functions attached to the symmetric square of a newform', *Proc. London Math. Soc.* **74** (1997), 559–611.

[3] P. Deligne and K.A. Ribet, 'Values of abelian L-functions at negative integers over totally real fields', *Invent. Math.* **59** (1980), 227–286.

[4] V. Dünger, 'p-adic interpolation of convolutions of Hilbert modular forms', *Ann. Inst. Fourier* **47** (1997), 365–428.

[5] J. Im, 'Special values of Dirichlet series attached to Hilbert modular forms', *Amer. J. Math.* **113** (1991), 975–1017.

[6] T. Miyake, 'On automorphic forms on GL_2 and Hecke operators', *Ann. of Math.* **94** (1971), 174–189.

[7] A.A. Panchishkin, 'Motives over totally real fields and p-adic L-functions', *Ann. Inst. Fourier* **44** (1994), 989–1023.

[8] M.M. Vishik, 'Non archimedean measures associated with Dirichlet series', (in Russian), *Mat. Sb.* **99** (1976), 248–260.

University of Szczecin
Institute of Mathematics
ul. Wielkopolska 15
70-451 Szczecin
Poland
e-mail: dabrowsk@sus.univ.szczecin.pl