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HAMILTONIAN CIRCUITS ON THE N-CUBE

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1. **Introduction.** The problem of finding bounds for the number $h(n)$ of Hamiltonian circuits on the n-cube has been studied by several authors, (1), (2), (3). The best upper bound known is due to Larman (5) who proved that $h(n)<2(n/2)^{2^n}$.

In this paper we use a result of Nijenhuis and Wilf (4) on permanents of (0, 1)-matrices to show that for $n\geq 5$

$$h(n) < \tfrac{1}{2}\{(n!)^{1/n}+\tau\}^{2^n}-\frac{4^n c^n n!\,(\sqrt[7]{18})^{2^n}}{2a}-\frac{(\sqrt[8]{81})^{2^n}}{2}$$

where τ, a and c are constants.

2. **An upper bound for the permanent of a (0, 1)-matrix.** If $A=(a_{ij})$ is an N-square matrix, then the permanent of A is defined as $p(A)=\sum_{\sigma\in S_N}\prod_{i=1}^N a_{i\sigma(i)}$ where the summation is over all permutations of the symmetric group S_N. Nijenhuis and Wilf (4) have shown that if $r_i=\sum_{j=1}^N a_{ij}$ $(i=1, 2, \ldots, N)$ then

$$p(A)\leq\prod_{i=1}^N\{(r_i!)^{1/r_i}+\tau\}$$

where $\tau=0.136708\cdots$.

If A_n denotes the adjacency matrix of the n-cube it follows that

$$p(A_n)\leq((n!)^{1/n}+\tau)^{2^n}.$$

3. **Hamiltonian circuits.** Let Q_n denote the n-cube.

Definition. By a circuit in Q_n we shall mean a *directed* closed path in Q_n which does not intersect itself. We allow two step circuits passing twice through the same edge.

Definition. By a *circuit covering* of Q_n we shall mean a set of circuits such that each vertex of Q_n is in exactly one circuit.

Denote the number of circuit coverings of Q_n by $NC(Q_n)$ and the number of *undirected* Hamiltonian circuits by $h(n)$. Then we can write

$$NC(Q_n)=2h(n)+j(n)$$

where $j(n)$ denotes the number of circuit coverings which are not Hamiltonian circuits.

Lemma 1. *Let A_n be the adjacency matrix of Q_n. Then $p(A_n)=NC(Q_n)$. (In fact if A is the adjacency matrix of any graph G, $p(A)=NC(G)$.)*

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Proof. Writing each permutation $\sigma \in S_N$ as a product of cyclic permutations, each term of $p(A_n)$ can be written

$$(a_{i_1i_2}a_{i_2i_3}a_{i_3i_4}\cdots a_{i_pi_1})(a_{i_{p+1}i_{p+2}}\cdots a_{i_{p+s}i_{p+1}})(\quad)\cdots.$$

Any term containing one or more factors a_{ij} corresponding to non-adjacent vertices v_i and v_j vanishes, since a_{ij} is zero. The remaining terms each represent a circuit covering, and each circuit covering corresponds to one non-zero term. Hence $p(A_n)=NC(Q_n)$.

Let $k(n)$ denote the number of circuit coverings of Q_n which do not contain a Hamiltonian circuit of a subgraph of Q_n isomorphic to Q_r $(r\geq 4)$. Let $g(n)$ denote the number of circuit coverings of Q_n which contain a Hamiltonian circuit of a subgraph of Q_n isomorphic to Q_r for some $r\geq 4$ but are not Hamiltonian circuits of Q_n.

Then $j(n)=k(n)+g(n)$ $(n\geq 4)$.

LEMMA 2. $k(n)\geq(\sqrt[8]{81})^{2^n}$ $(n\geq 3)$.

Proof. $k(3)=NC(3)=p(A_3)=81$ by direct calculation.

By considering the n-cube as two $(n-1)$-cubes joined by 2^{n-1} edges we have

$$k(n)\geq k(n-1)k(n-1).$$

The result follows by induction.

LEMMA 3. *$g(n)\geq(4^nc^nn!(\sqrt[7]{18})^{2^n}/a)$ $(n\geq 5)$ where c is chosen so that $h(n)>c(\sqrt[7]{18})^{2^n}$ for $n=2, 3, 4$ and $a=4096(\sqrt[7]{18})^{16}c^4/2187$.*

Proof. Considering the n-cube as two $(n-1)$-cubes joined by 2^{n-1} edges in n different ways, and counting combinations of Hamiltonian circuits in one $(n-1)$-cube and the circuit coverings counted by g in the other $(n-1)$-cube and vice versa, we get

$$\begin{aligned} g(n) &\geq 2n(2h(n-1))g(n-1) \qquad (n>5)\\ &= 4nh(n-1)g(n-1). \end{aligned}$$

It was proved by Douglas (2) that

$$h(n)\geq c(\sqrt[7]{18})^{2^n}.$$

Also

$$\begin{aligned} g(5) &\geq 5(2h(4))NC(Q_4)\\ &> 10h(4)(NC(Q_3))^2\\ &\geq 10h(4)(P(A_3))^2\\ &\geq 10c(\sqrt[7]{18})^{16}81^2. \end{aligned}$$

The result now follows by induction.

THEOREM. $h(n)\leq\frac{1}{2}\{(n!)^{1/n}+\tau\}^{2^n}-(4^nc^nn!(\sqrt[7]{18})^{2^n})/2a-(\sqrt[8]{81})^{2^n}/2$.

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Proof. $h(n)=\frac{1}{2}\{NC(Q_n)-k(n)-g(n)\}$ $(n\geq 5)$ and the result follows.

This result provides further evidence in support of the conjecture $\lim_{n\to\infty} h(n)^{2^{-n}}/n = 1/e$.

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